Extracting the TMD soft function and CS kernel from the lattice

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with

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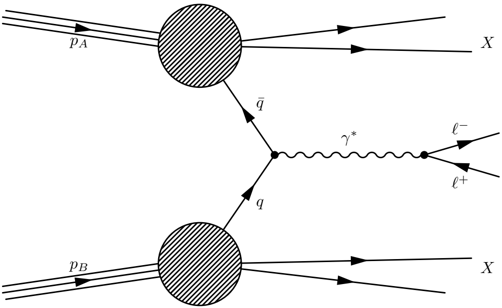
Outline

1. Motivation
2. Our approach
3. Computational setup
4. Results so far

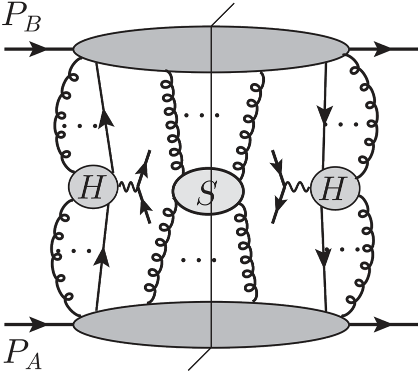
Drell-Yan process

- invariant mass: Q
- rapidity of lepton pair: Y
- transverse momentum: $\vec{q}_\perp$
- $\Lambda_{\mathrm{QCD}} \lesssim |\vec{q}_\perp| \ll Q$
- momentum fraction: x_a, x_b
- Collins-Soper (CS) scale: ζ_a, ζ_b



$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}\left(Q^2,\mu\right)\int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} \underbrace{f_i\left(x_a,\vec{b}_\perp,\mu,\zeta_a\right) f_j\left(x_b,\vec{b}_\perp,\mu,\zeta_b\right)}_{\text{TMDPDFs}}$$
$$\times\left[1+\mathcal{O}\left(\frac{q_\perp^2}{Q^2},\frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Factorization



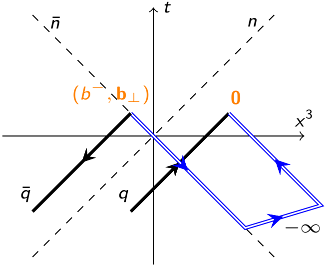
Drell-Yan leading region [Collins, 2011]

- Leading region has contribution from soft momentum states
- Need to regulate rapidity divergences present in beam and soft functions
- New scale associated with rapidity divergence: ν

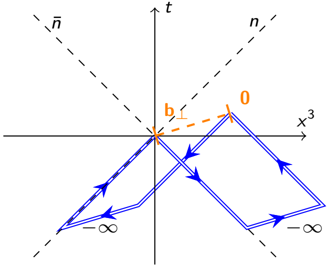
$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q,\mu) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$

$$\times S_i(b_\perp, \mu, \nu)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Beam and soft function



Staple gauge link for beam function



Wilson loop for soft function

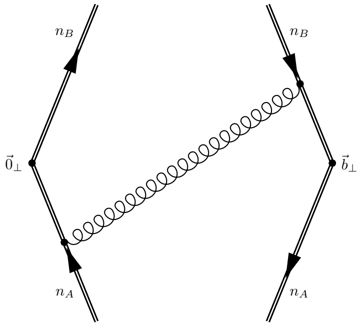
$$B_i^{0(u)}\left(x, \vec{b}_\perp, \epsilon, \tau, xP^+\right) = \int \frac{\mathrm{d}b^-}{2\pi} e^{-ib^-(xP^+)} \langle P| \bar{\psi}_i^0\left(b^-, \vec{b}_\perp\right) W_{\bar{n}}\left(b^-, \vec{b}_\perp; -\infty, 0\right)$$

$$\times \; W_\perp\left(-\infty\bar{n}; 0, b_\perp\right) W_{\bar{n}}\left(0; 0, -\infty\right) \frac{\gamma^+}{2} \psi_i^0(0) |P\rangle$$

$$S^0\left(b_\perp, \epsilon, \tau\right) = \frac{1}{N_c} \langle 0| W_n\left(b_\perp; 0, -\infty\right) W_{\bar{n}}\left(b_\perp; -\infty, 0\right) W_\perp\left(-\infty\bar{n}; 0, b_\perp\right)$$

$$\times \; W_{\bar{n}}\left(0; 0, -\infty\right) W_n\left(0; -\infty, 0\right) W_\perp\left(-\infty n; b_\perp, 0\right) |0\rangle$$

Rapidity divergence



Rapidity divergence:

$$\int \frac{\mathrm{d}k^+\mathrm{d}k^-}{(2\pi)^2} \frac{1}{k^+k^- - k_\perp^2 + i0} \frac{1}{n\cdot k - i0} \frac{1}{\bar{n}\cdot k + i0}$$
$$= \int \frac{\mathrm{d}\alpha}{(2\pi)^2} \frac{1}{\alpha - k_\perp^2 + i0} \frac{1}{\alpha - i0} \int_{-\infty}^{\infty} \mathrm{d}y$$

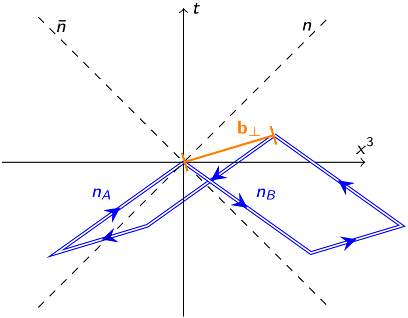
$$k^- = n\cdot k, \qquad k^+ = \bar{n}\cdot k, \qquad k^\pm = k^0 \pm k^3$$

$$\alpha = k^+k^-, \quad y = \frac{1}{2}\ln\left(\frac{k^-}{k^+}\right)$$

Divergence associated with rapidity

$$y \to \pm\infty$$

Off lightcone regulator



- Collins scheme, spacelike Wilson lines:

$$n_A \equiv n - e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B} n$$

$$f_i(x_b, \vec{b}_\perp, \mu, \zeta_b) = \lim_{\epsilon \to 0} Z^i_{\mathrm{UV}}(\mu, \epsilon, \zeta_b) \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} B_i\left(x_b, \vec{b}_\perp, \epsilon, y_B, xP^+\right) \times \sqrt{\frac{S_i(b_\perp, \epsilon, y_A - y_n)}{S_i(b_\perp, \epsilon, y_A - y_B)\, S_i(b_\perp, \epsilon, y_n - y_B)}}$$

Collins-Soper kernel

- Collins-Soper (CS) kernel governs the rapidity evolution of the TMDPDF

$$K\left(b_{\perp},\mu\right)=\frac{\mathrm{d}f_q\left(x,b_{\perp},\mu,\zeta\right)}{\mathrm{d}\log\zeta}$$

- Can be obtained from the soft function

$$S_q\left(b_{\perp},y_A,y_B,\mu\right)\overset{\substack{y_A\to+\infty\\ y_B\to-\infty}}{=}S_I\left(b_{\perp},\mu\right)e^{2K\left(b_{\perp},\mu\right)\left(y_A-y_B\right)}$$

- By direct computation:

$$K\left(b_{\perp},\mu\right)=\lim_{\substack{y_A\to+\infty\\ y_B\to-\infty}}\frac{1}{2}\frac{\partial}{\partial y_n}\log\left(\frac{S_q\left(b_{\perp},y_n,y_B,\mu\right)}{S_q\left(b_{\perp},y_A,y_n,\mu\right)}\right)$$

Lattice extraction of TMDPDFs

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = C_q\left(x\tilde{P}^z, \mu\right)\exp\left[\frac{1}{2}\gamma_\zeta^q\left(\mu, b_\perp\right)\log\frac{\tilde{\zeta}}{\zeta}\right] f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)$$
$$\times\left\{1 + \mathcal{O}\left(\frac{1}{(x\tilde{P}^z b_\perp)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(x\tilde{P}^z)^2}\right)\right\}$$

[Ebert, *et. al.*, 2019], [Ebert, *et. al*, 2022]

- C_q is a perturbatively calculable matching kernel
- $\tilde{\zeta} = x^2 m_h^2 e^{2(y_{\tilde{P}} + y_B - y_n)}$

quasi-TMDPDF

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = \tilde{f}_q^{\mathrm{naive}}\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right)\sqrt{\frac{\tilde{S}_q^{\mathrm{naive}}\left(b_\perp, \mu\right)}{S_q\left(b_\perp, \mu, 2y_n, 2y_B\right)}}$$

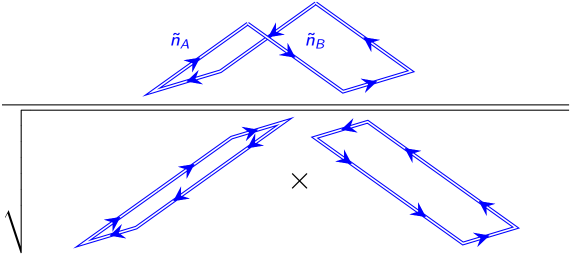
- $\tilde{f}_q^{\mathrm{naive}}$ and $\tilde{S}_q^{\mathrm{naive}}$ are lattice calculable objects
- S_q is the Collins soft function

Outline

Our approach

$$\tilde{n}_A = \left(in^0_A, 0, 0, n^3_A\right), \quad \tilde{n}_B = \left(in^0_B, 0, 0, n^3_B\right)$$

$$S_{\text{ratio}}\left(\vec{b}_\perp, \tilde{n}_A, \tilde{n}_B, a, \tau\right) =$$



- Ratio gives correct dependence on $b_\perp$
- Removes linear divergences associated with finite length Wilson lines
- Ensures power counting in $b^4_\perp/\tau^4$
- Approaches lattice time τ independent result for large τ

Connection to Minkowski space

- At large lattice time:

$$S_{\mathrm{ratio}}\left(b_\perp, \tilde{n}_A, \tilde{n}_B, a, \tau\right) = S_{\mathrm{lat}}\left(b_\perp, r_a, r_b, a\right) + \mathcal{O}\left(\frac{b_\perp^4}{\tau^4}\right)$$

- Direct mapping to rapidity variables in Collins' scheme:

$$r_a = \frac{n_A^3}{n_A^0} = \frac{1+e^{-2y_A}}{1-e^{-2y_A}}, \quad r_b = \frac{n_B^3}{n_B^0} = \frac{1+e^{2y_B}}{1-e^{2y_B}}$$

$$|r_a|, |r_b| > 1,\ n_A^0 n_B^0 (r_a r_b + 1) > 0$$

- Construct matching between lattice and continuum renormalization schemes

$$S\left(b_\perp, y_A, y_B, \mu\right) = C(r_a, r_b, \mu, a) \times S_{\mathrm{lat}}\left(b_\perp, r_a, r_b, a\right)$$

- Obtain CS kernel from:

$$S\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{2K(b_\perp,\mu)(y_A - y_B)}$$

Outline

Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional 'fermions' that live along the path:

$$P \exp\left\{-ig \int_{s_i}^{s_f} \mathrm{d}s n^\mu A_\mu(y(s)) \right\}$$
$$= Z_\psi^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \psi\bar{\psi} \exp\left\{ig \int_{s_i}^{s_f} \mathrm{d}s \bar{\psi} i\partial_s \psi - \bar{\psi} n \cdot A\psi \right\}$$

[Gervais, Nevau 1980], [Aref'eva 1980]

Auxiliary field propagator:

$$in \cdot DH_n(y) = \delta(y) \xrightarrow{\text{Euclidean space}} -i\tilde{n} \cdot D_E H_{\tilde{n}}(y) = \delta(y), \qquad \tilde{n} = (in_0, \vec{n})$$

Meaningful solution only obtained with a UV cutoff due to negative energy states [Aglietti, *et. al.* 1992], [Agglietti, 1994]

Auxiliary field propagator in Euclidean space

$$K(\tau) = \left(1 - \frac{H_0|_\tau}{2n}\right)^n U_4^\dagger(\tau-1)\left(1 - \frac{H_0|_{\tau-1}}{2n}\right)^n$$

$$H_0\psi(x) = -iv\cdot\Delta^\pm\psi(x) = -\sum_\mu \frac{iv_\mu}{2}\left[U_\mu(x)\,\psi(x+\hat{\mu}) - U_{-\mu}(x)\,\psi(x-\hat{\mu})\right]$$

$$G(\mathbf{x},\tau,\mathbf{x}',\tau') = K(\tau)G(\mathbf{x},\tau-1,\mathbf{x}',\tau')$$

For $n\to\infty$ and $a\to 0$, this will produced the same continuum equation as before.

We get correction terms of $\mathcal{O}\left(a^2, \frac{a^2}{n}\right)$
[Horgan, *et. al.*, 2009]

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Details of lattice calculation

- Using $N_f = 2 + 1$ flavor PACS-CS configurations
- non-perturbatively $\mathcal{O}(a)$-improved Wilson quark action and Iwasaki gauge action
- $32^3 \times 64$ lattice with $a = 0.0907(13)\ \mathrm{fm}$
- 400 configurations
- thyp2 smearing
- Up to 64 sources per configuration
- Using GPT/GRID

[PACS-CS '09, '11]

Numerator and denominator factors

- At large τ, we expect

$$S_{\rm num} \overset{\tau\to\infty}{\sim} e^{2\tau(r_a+r_b)/a}/\tau^4$$

$$S_A \overset{\tau\to\infty}{\sim} e^{4(\tau r_a - iz)/a}/\tau^4$$

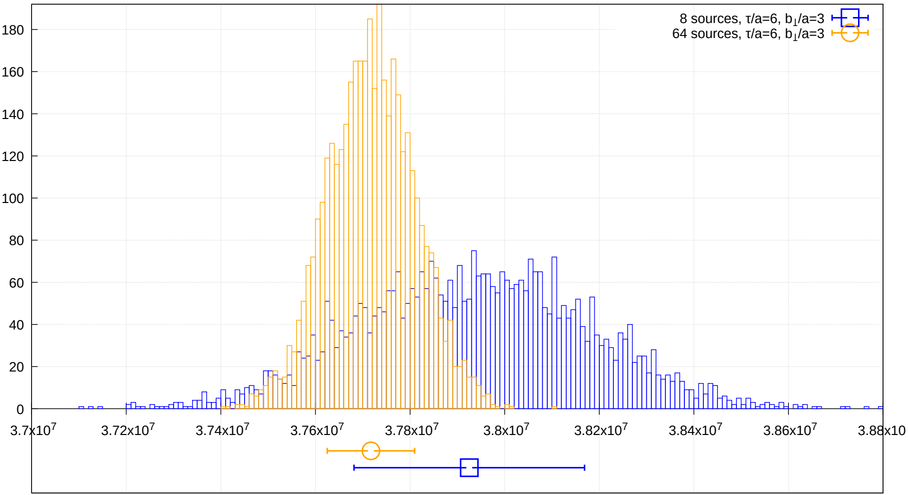
$$S_B \overset{\tau\to\infty}{\sim} e^{4(\tau r_b + iz)/a}/\tau^4$$

  exponential increase related to negative energy states

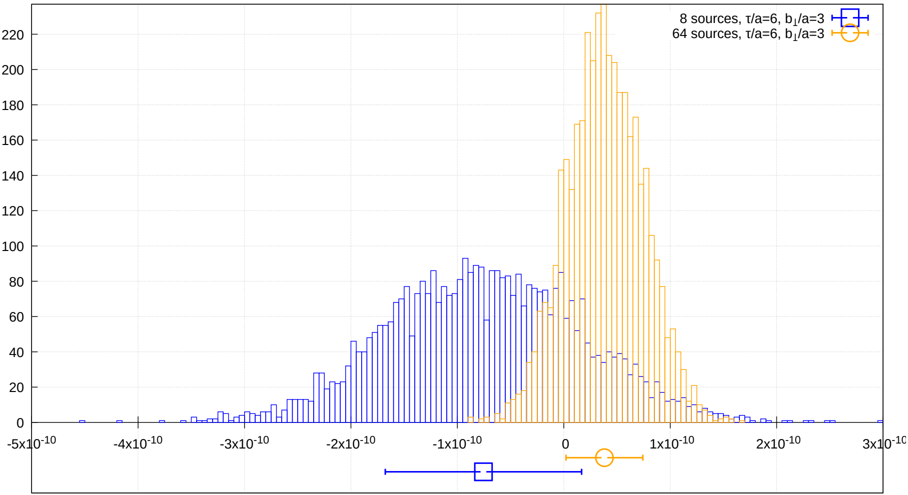
- We expect to see real and imaginary contributions to the denominator factors
- Combined denominator factor $\sqrt{S_A S_B}$, should be purely real
- Exponential τ dependence should cancel in ratio, giving plateau

$$S_{\rm ratio} = \frac{S_{\rm num}}{\sqrt{S_A S_B}} \overset{\tau\to\infty}{\sim} 1$$

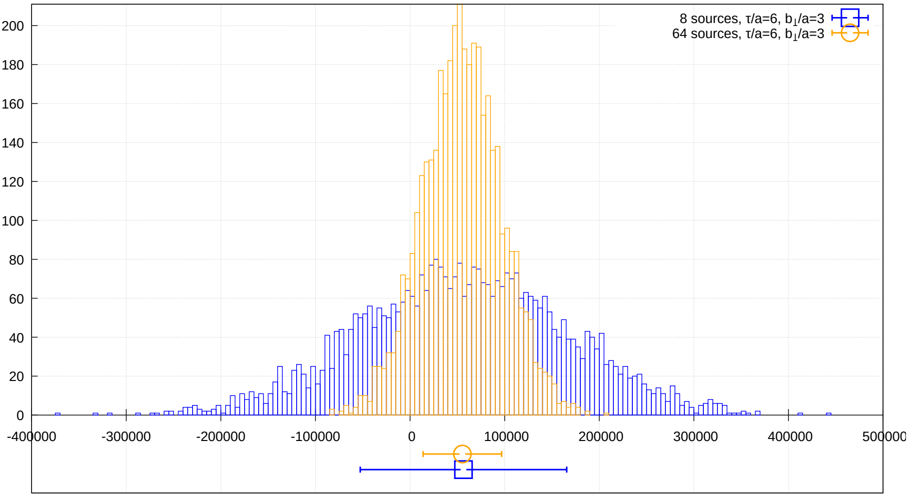
Numerator, real part



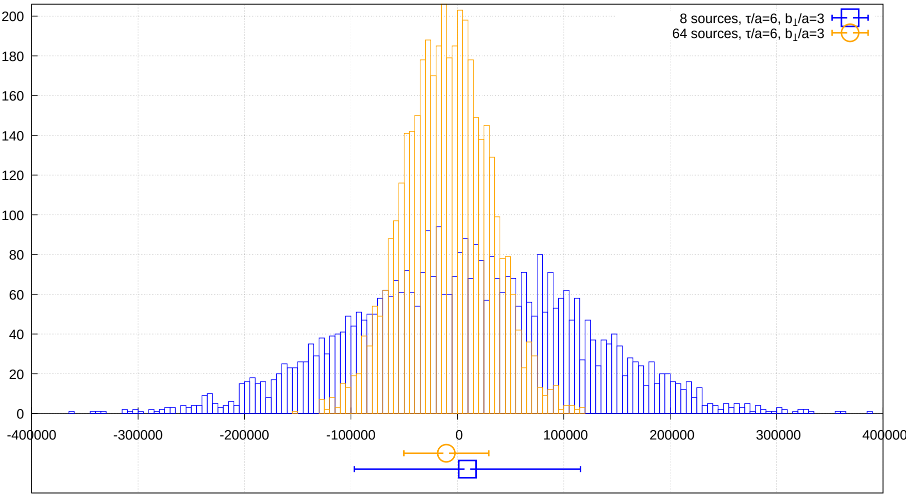
Numerator, imaginary part



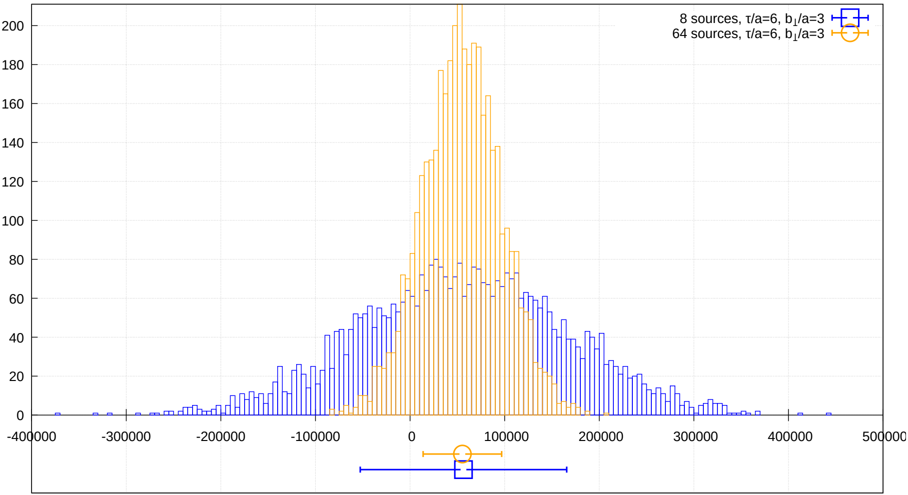
Denominator A, real part



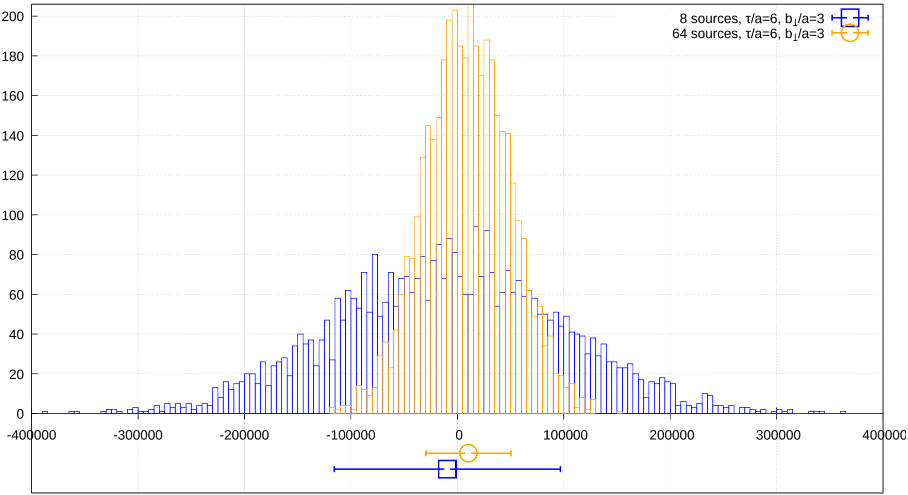
Denominator A, imaginary part



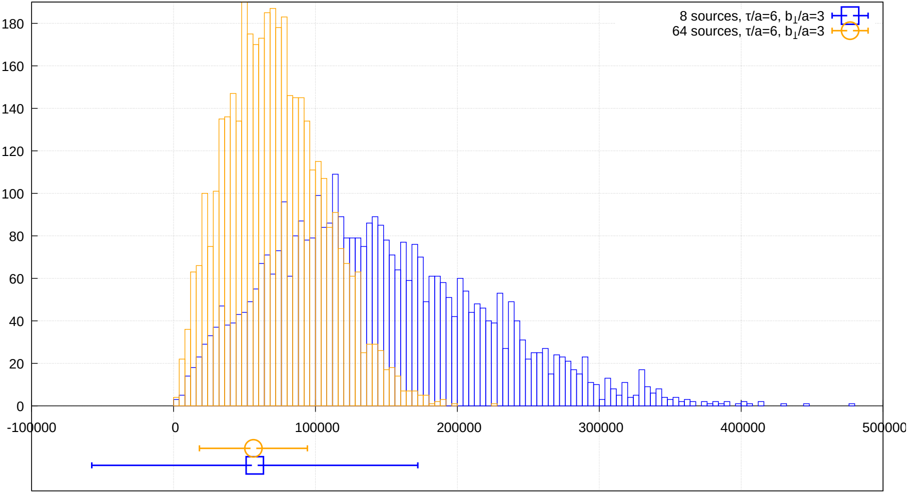
Denominator B, real part



Denominator B, imaginary part

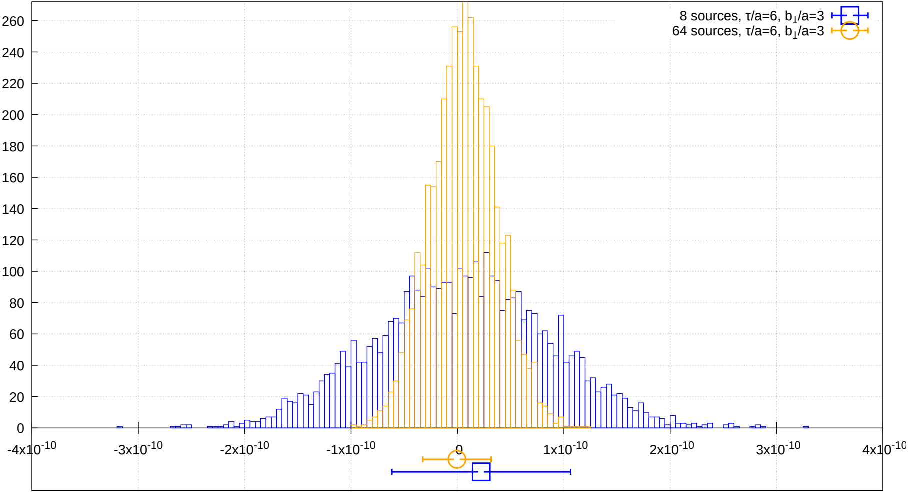


Full denominator, real part



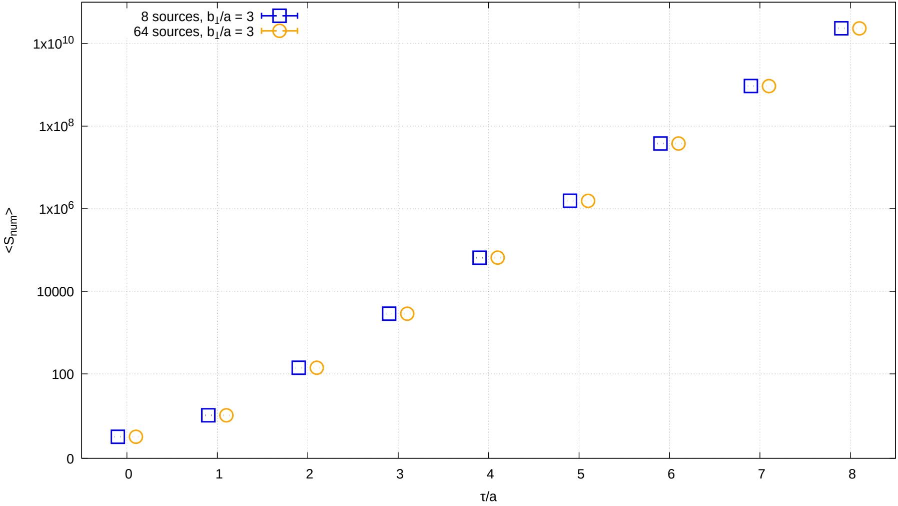
$$\Re\left(\sqrt{S_A S_B}\right)$$

Full denominator, imaginary part

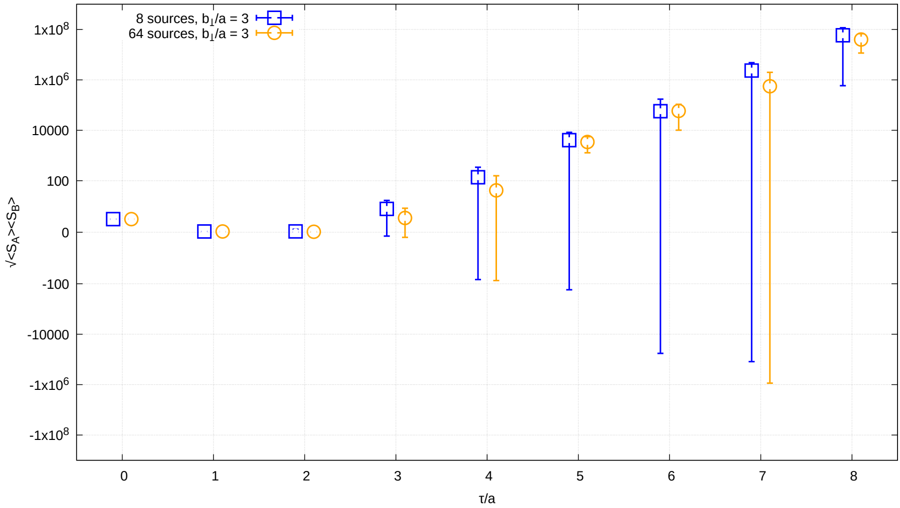


$$\Im\left(\sqrt{S_A S_B}\right)$$

Numerator, τ dependence



Full denominator, τ dependence



Conclusion and outlook

- Important to have multiple methods for computing the soft function
- Euclidean space calculation of soft function has a direct mapping to Minkowski space result
- Preliminary numerical investigation shows a decrease in error with increased statistics
- Expect to see a signal with ~ 1000 sources

Thank you!

Group members

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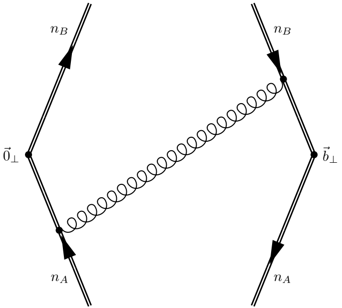
WM (NYCU), Yong Zhao (Argonne)

Euclidean space calculation



Define Euclidean space Wilson line directions as:

$$\tilde{n}_A = (in_A^0, \vec{0}_\perp, n_A^3), \qquad \tilde{n}_B = (in_B^0, \vec{0}_\perp, -n_B^3)$$

$$r_a \equiv \frac{n_A^3}{n_A^0}, \qquad r_b \equiv \frac{n_B^3}{n_B^0}$$

$$S_A^E(b_\perp, \epsilon, r_a, r_b) = g^2 C_F (\tilde{n}_A \cdot \tilde{n}_B) \int_{-\infty}^{0} \mathrm{d}s \int_{-\infty}^{0} \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b + s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2}$$

Coordinate space

Perform integration in coordinate space:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A-t\tilde{n}_B)} \frac{1}{k^2} = \int_0^\infty \mathrm{d}u \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-uk^2} e^{-(b+s\tilde{n}_A-t\tilde{n}_B)^2/4u}$$

$$= \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}} \frac{1}{((b+s\tilde{n}_A-t\tilde{n}_B)^2/4)^{d/2-1}}$$

'u' integral only valid for

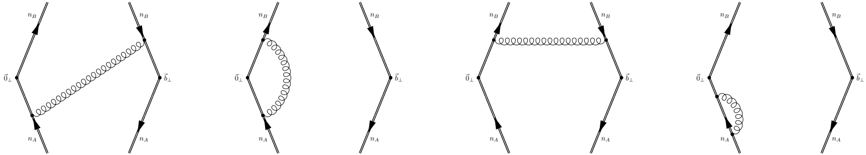
$$(s\tilde{n}_A - t\tilde{n}_B)^2 = s^2((n_A^3)^2 - (n_A^0)^2) + t^2((n_B^3)^2 - (n_B^0)^2) + st(n_A^3 n_B^3 + n_A^0 n_B^0) > 0$$

Euclidean space integral only finite when:

$$|n_A^3| > |n_A^0|, \quad |n_B^3| > |n_B^0|, \quad n_A^3 n_B^3 + n_A^0 n_B^0 > 0$$

$$\rightarrow \quad |r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

Soft function in Euclidean space at one loop



A B C D

$$S^{(1)}\left(b_\perp, \epsilon, r_a, r_b\right)$$
$$= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 + \log\left|\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right| \frac{r_a r_b + 1}{r_a + r_b}\right\}$$

$$r_a = \frac{1 + e^{-2y_A}}{1 - e^{-2y_A}}, \quad r_b = \frac{1 + e^{2y_B}}{1 - e^{2y_B}}$$

Rapidity regulator scheme

Time-like: $n_A = \left(1 + e^{-2y_A}, \vec{0}_\perp, 1 - e^{-2y_A}\right), \quad n_B = \left(1 + e^{2y_B}, \vec{0}_\perp, -1 + e^{2y_B}\right)$

$$r_a = \frac{1 - e^{-2y_A}}{1 + e^{-2y_A}}, \quad r_b = \frac{1 - e^{2y_B}}{1 + e^{2y_B}}, \qquad |r_a|, |r_b| < 1 \qquad \text{fails}$$

Space-like: $n_A = \left(1 - e^{-2y_A}, \vec{0}_\perp, 1 + e^{-2y_A}\right), \quad n_B = \left(1 - e^{2y_B}, \vec{0}_\perp, -1 - e^{2y_B}\right)$

$$r_a = \frac{1 + e^{-2y_A}}{1 - e^{-2y_A}}, \quad r_b = \frac{1 + e^{2y_B}}{1 - e^{2y_B}}, \qquad |r_a|, |r_b| > 1 \qquad \text{succeeds}$$

$$S^{(1)}\left(b_\perp, \epsilon, r_a, r_b\right) = \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 - 2|y_A - y_B|\frac{1 + e^{2(y_B - y_A)}}{1 - e^{2(y_B - y_A)}}\right\}$$

Finite L Wilson lines

For $L \to \infty$ and $r_a, r_b \to 1$:

$$S(b_\perp, a, r_a, r_b, L) = 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)}\log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right)\right)\log\left(\frac{b_\perp^2}{a^2}\right)$$
$$+ \frac{\alpha_s C_F}{2\pi}\left\{-4\log\left(\frac{b_\perp^2}{a^2}\right) + 2\frac{\pi b_\perp}{a} + 2\frac{\pi(|n_A| + |n_B|)L}{b_\perp}\right.$$
$$-2\frac{\pi(|n_A| + |n_B|)L}{a}$$
$$\left. + 2\frac{b_\perp^2}{L^2}\left(C_1 - \frac{1}{3}\right)\right\} + \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2\right)$$

$$C_1 = 1 - \frac{1}{2}\frac{1}{b_0^2(r_b^2 - 1)} - \frac{1}{2}\frac{1}{a_0^2(r_a^2 - 1)} \implies \frac{b_\perp^2}{L^2} \ll r_{a,b} - 1$$

- Incorrect $b_\perp$ dependence
- Linear divergence in L
- Power corrections are limited by $r_{a,b}$

Ratio

$$
\begin{aligned}
& S_{\text{ratio}}\left(b_\perp, a, r_a, r_b, L\right) \\
& \quad = \frac{S\left(b_\perp, a, r_a, r_b, L\right)}{\sqrt{S\left(b_\perp, a, r_a, -r_a, L\right) S\left(b_\perp, a, -r_b, r_b, L\right)}} \qquad \text{[Ji, Liu, Liu 2020]} \\
& \quad = 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)} \log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right)\right) \log\left(\frac{b_\perp^2}{a^2}\right) \\
& \qquad + \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2\right)
\end{aligned}
$$

- Recover correct $b_\perp$ dependence
- Linear divergence removed
- Power corrections now start are no longer limited by $r_{a,b}$
- Approaches the soft function at $L \to \infty$