Parton Distribution Functions from Tensor Network calculations

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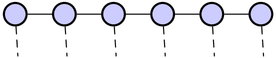


[arXiv:2409.16996]

TQCD 2nd meeting
Beimen campus of NYCU, Taipei
27 September 2024

Collaborators

Mari Carmen Bañuls

Krzysztof Cichy

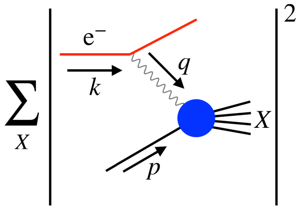
C.-J. David Lin

[Bañuls et al. 2024]

Outline

Parton Model and Deep Inelastic Scattering

- PDF: probability of finding a constituent in a hadron with momentum fraction ξ
- example: Deep Inelastic Scattering (DIS)



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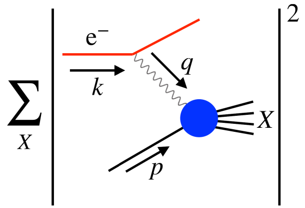
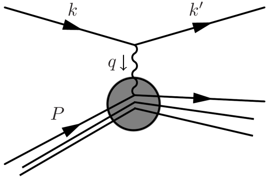
- kinematic variables: Q^2, Bjorken $\xi = \frac{Q^2}{2P\cdot q}$
- $\rightarrow$ scattering amplitude factorizes: perturbative part $\times$ PDF





[Schwartz 2014]

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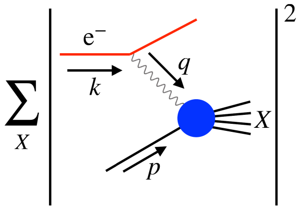
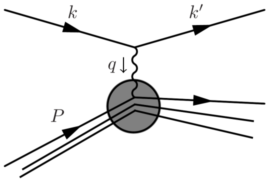
- PDF depends only on Bjorken $\xi = \frac{Q^2}{2P\cdot q}$

$$\Rightarrow \boxed{f(\xi) = \int dz^+ e^{-i\xi P^- z^+} \langle P | \bar{\psi}(z^+)\gamma^- W(z^+ \leftarrow 0)\psi(0) | P \rangle}$$

- Integration along lightcone direction z^+





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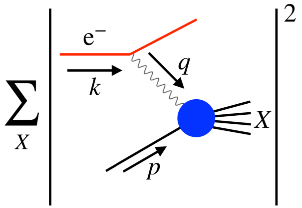
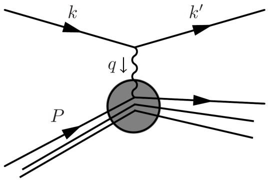
- Integration along lightcone direction z^+
- Lattice QCD in euclidean space: lightcone ~ point
- Hamiltonian formalism: lightcone in Minkowski space ✓
- → Use Tensor Network States/Quantum computing





[Schwartz 2014]

Outline

Tensor Networks

$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \Psi^{s_1 s_2 \ldots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \ldots \otimes |s_N\rangle$$

- generic state scales exponentially

Tensor Networks

- generic state scales exponentially
- Tensor Network State as ansatz
- 1d: Matrix Product State (MPS)

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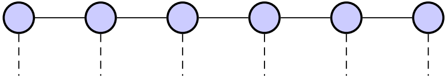
$$\Psi^{s_1 s_2 \ldots s_N} = \sum_{\{i_x\}} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \ldots A^{N,s_N}_{i_{N-1}}$$

$=$ 

Tensor Networks

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- generic state scales exponentially
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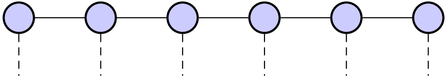
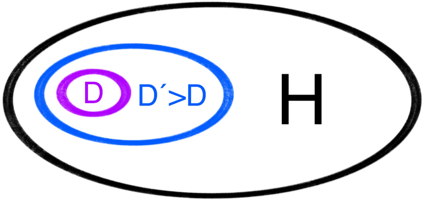
- 1d: Matrix Product State (MPS)
- truncation to bond dimenion D
- polynomial resource scaling

$$\Psi^{s_1 s_2 \ldots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \ldots A^{N,s_N}_{i_{N-1}}$$

=





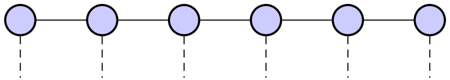
Tensor Networks

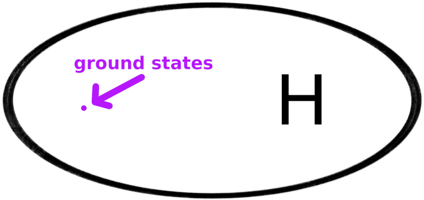
$$|\psi\rangle = \sum_{s_1, s_2, \dots, s_N} \Psi^{s_1 s_2 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle$$

- generic state scales exponentially
- Tensor Network State as ansatz
- 1d: Matrix Product State (MPS)
- truncation to bond dimenion D
- polynomial resource scaling
- good approximation for ground states and low excited states
- area laws of entanglement entropy [Hastings 2007]

$$\Psi^{s_1 s_2 \dots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \dots A^{N,s_N}_{i_{N-1}}$$

=

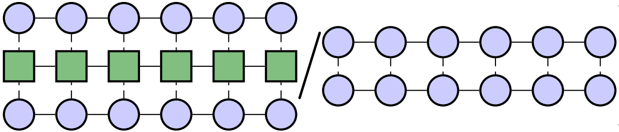




Efficient Tensor Network operations

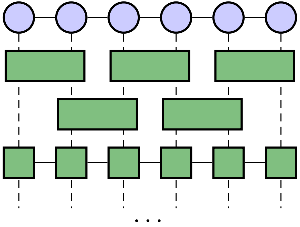
- Find groundstate and excited states

$$\min\left(E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \right)$$

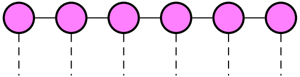


- Apply operators / time evolution

$$\hat{O} | \Psi \rangle =$$

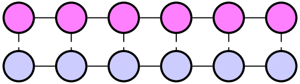


$$\longrightarrow | \Phi \rangle =$$



- Calculate overlap

$$\langle \Psi | \Phi \rangle =$$



Outline

The Schwinger Model [Hamer et al. 1997]

- Quantum electrodynamics in 1+1 dimensions, $U(1)$ symmetry
- Fermion ("parton") couples to gauge boson ("gluon")
- Bound states (hadron) [Bañuls et al. 2013]
- $\Rightarrow$ can calculate equivalent to PDF [Dai et al. 1995]
- Lagrange density:

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- For TN/QC: transform action into spin-model Hamiltonian

$$H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1 + (-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2$$

$$\left(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2} - \frac{1}{4}\right)$$

Outline

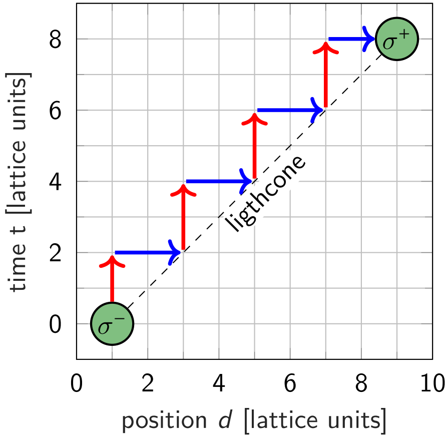
Lightcone correlators in the Schwinger Model

$$\left\langle P \middle| \bar{\Psi}(z^+)\gamma^- W(z^+ \leftarrow 0)\Psi(0) \middle| P \right\rangle$$
$$\to \mathcal{M}_{(e,e)} + \mathcal{M}_{(o,o)} - \mathcal{M}_{(o,e)} - \mathcal{M}_{(e,o)}$$
$$\to \langle P | \sigma^+(z^+) W_{z^+ \leftarrow 0} \sigma^-(0) | P \rangle + \dots$$

Lightcone correlators in the Schwinger Model

$$\langle P | \bar{\Psi}(z^+)\gamma^- W(z^+ \leftarrow 0)\Psi(0) | P \rangle$$
$$\to \mathcal{M}_{(\mathrm{e,e})} + \mathcal{M}_{(\mathrm{o,o})} - \mathcal{M}_{(\mathrm{o,e})} - \mathcal{M}_{(\mathrm{e,o})}$$
$$\to \langle P | \sigma^+(z^+) W_{z^+\leftarrow 0} \sigma^-(0) | P \rangle + \dots$$

- Evolution along light cone
  → small time- and space-like steps
- Time evolution:
  $e^{-i\tau H} \approx \left(e^{-i\delta\tau H_{eo}} e^{-i\delta\tau H_{oe}} e^{-i\delta\tau H_L}\right)^{\frac{\tau}{\delta\tau}}$
- Spatial evolution:
  Insert static charge and move stepwise

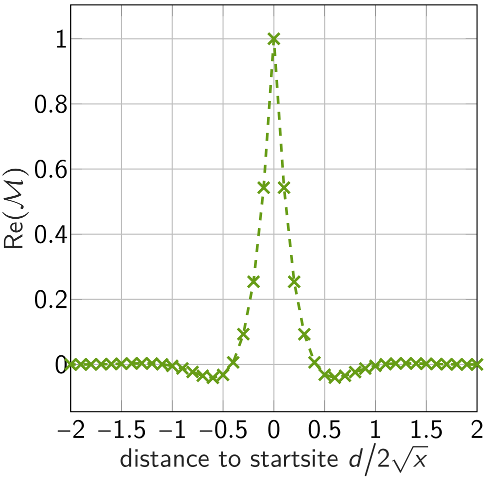


$$\mathcal{M}_{(\mathrm{e,e})}(2n,2m) = \langle P | e^{it_n H} \prod_{d<2n} (-i\sigma^z_d)\sigma^+_{2n} Q_{2n+1} e^{-i\tau H_q} Q^\dagger_{2n+1} Q_{2n+3} e^{-i\tau H_q} \dots$$
$$Q^\dagger_{2m-3} Q_{2m-1} e^{-i\tau H_q} Q^\dagger_{2m-1} \prod_{d'<2m} (i\sigma^z_{d'})\sigma^-_{2m} | P \rangle$$

Outline

Results - Matrix elements

$m/g = 5.6419$; $x = 100$; $D = 80$; $\Delta t/\Delta\tau = 100$; $\mu = 112.588$; **excit**= 1; $N/\sqrt{x} = 10$



$$\mathrm{Real}(\mathcal{M}) = \mathrm{Real}(\mathcal{M}_{ee} + \mathcal{M}_{oo} - \mathcal{M}_{eo} - \mathcal{M}_{oe})$$

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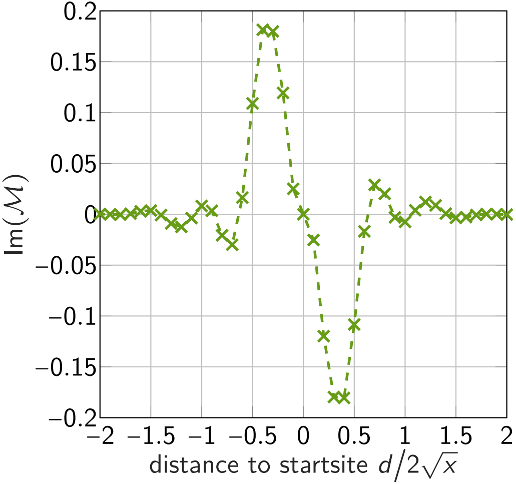
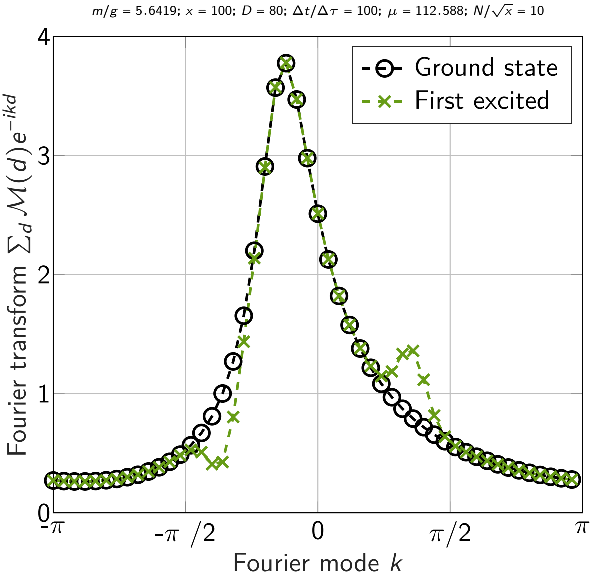


$$\mathrm{Imag}(\mathcal{M}) = \mathrm{Imag}(\mathcal{M}_{ee} + \mathcal{M}_{oo} - \mathcal{M}_{eo} - \mathcal{M}_{oe})$$

Results - Fourier transform of matrix elements



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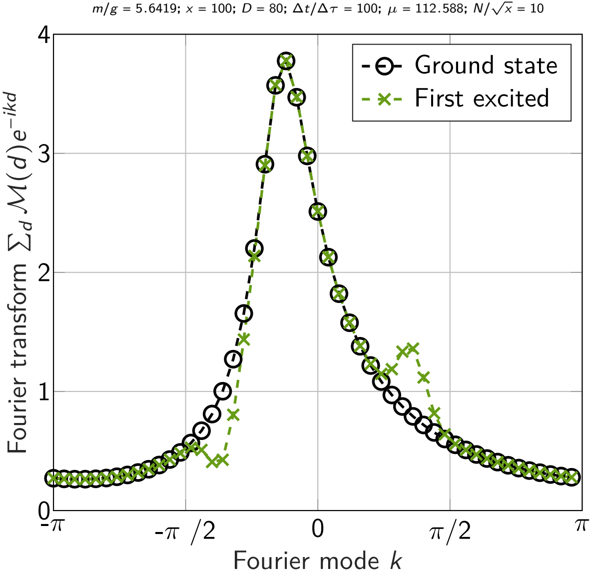
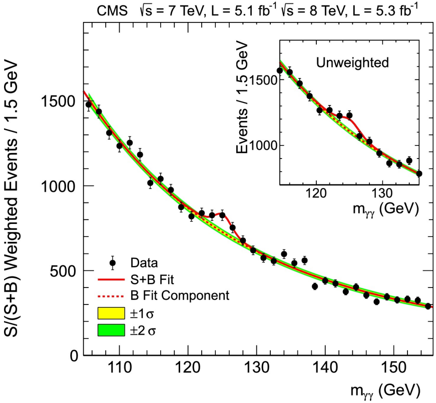
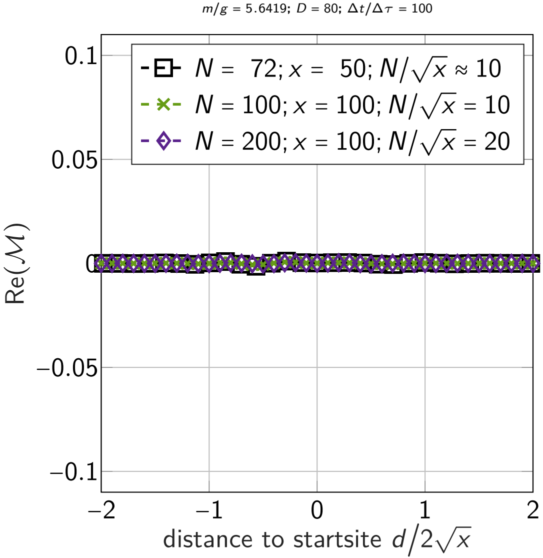




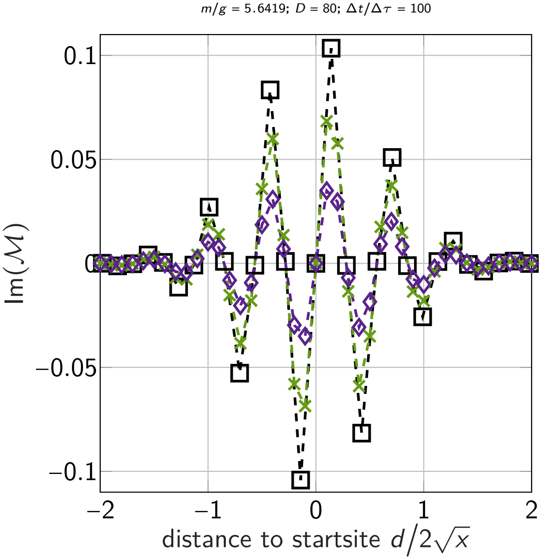
[CMS collaboration 2012]

Do we observe a particle here?

Results - Subtracted matrix elements [Collins 2011]

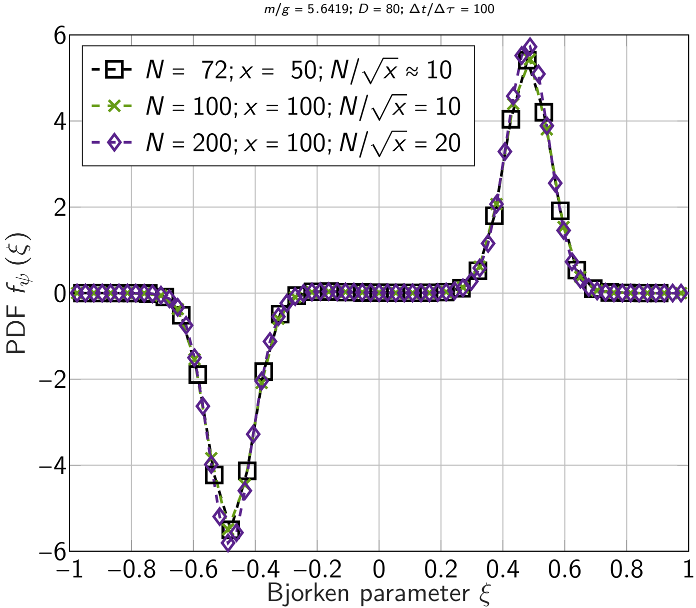


$\mathrm{Real}(\mathcal{M}_{\mathrm{excitation}} - \mathcal{M}_{\mathrm{vacuum}})$



$\mathrm{Imag}(\mathcal{M}_{\mathrm{excitation}} - \mathcal{M}_{\mathrm{vacuum}})$

Results - PDF



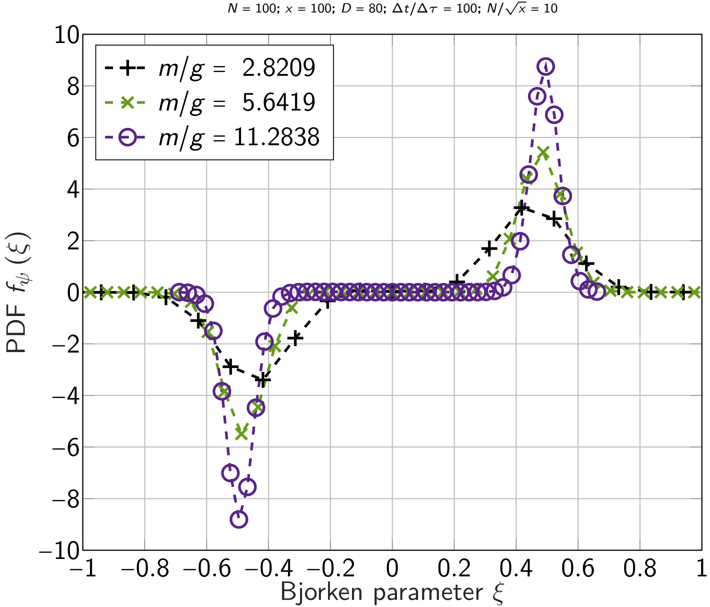
Observations:

- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- Antiparticle PDF from negative ξ:

$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

- Observed symmetry
  $\rightarrow f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓

Results - PDF



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  $\Rightarrow$ meson ✓
- Peak broadens with decreasing fermion mass ✓

Outline

Summary

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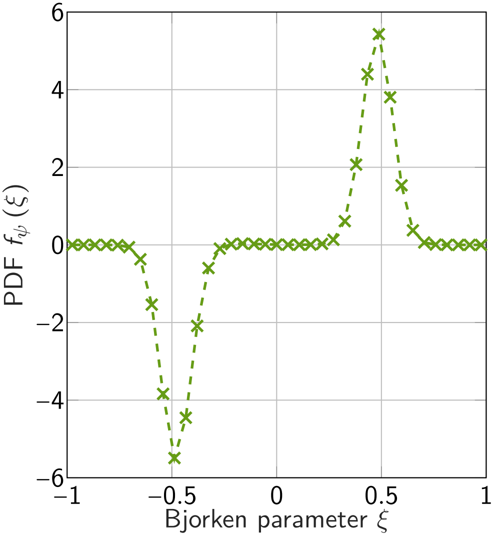
- PDFs characterize the structure of hadrons
- Euclidean space: lightcone → point
- Tensor Network States: direct evaluation of lightcone correlators
- Schwinger model: PDF with standard TN tools: MPS, time evolution
- Obtained fermion- and anti-fermion-PDF for the vector meson ✓

Outlook:

- Control errors, continuum and infinite volume limits
- Extend mass range
- Same analysis for QCD in 3+1 dimensions ☺



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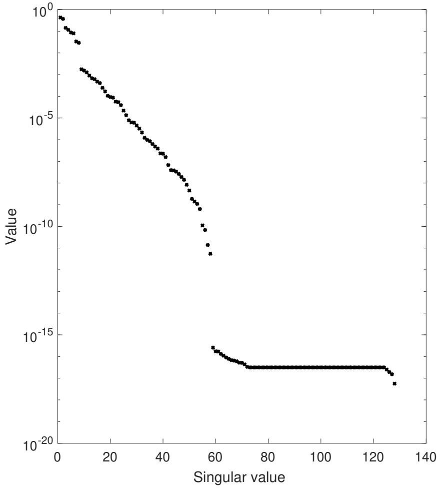
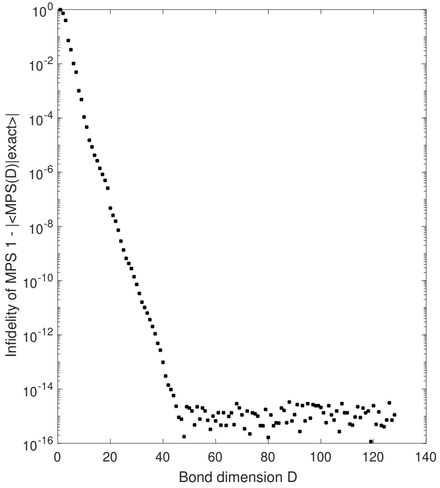
Outline

Singular values and cutoff

Schwinger model, $L = 14$, $\mu = 0.125$, $x = 10$, 2nd excitation



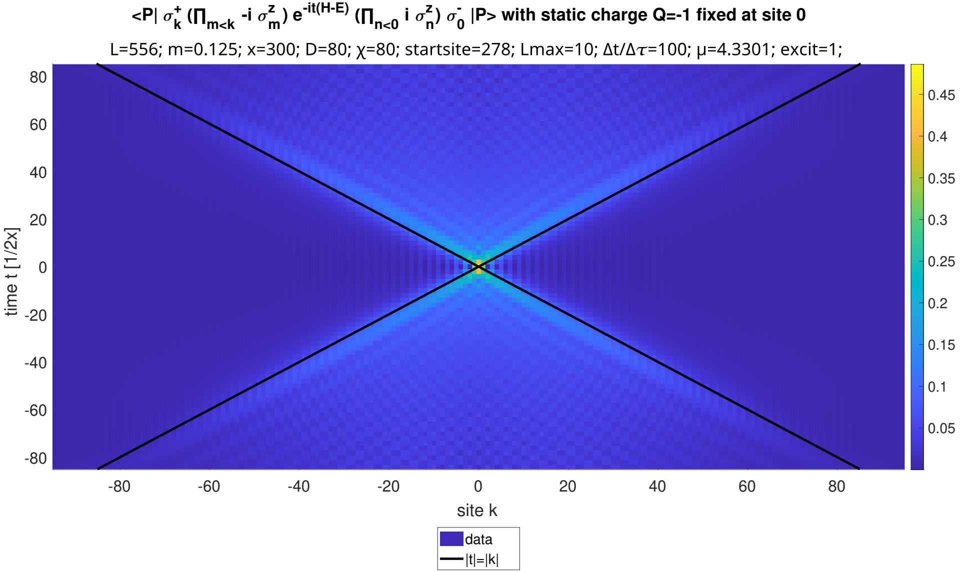
Singular values for cut in the middle



Infidelity of MPS with exact state

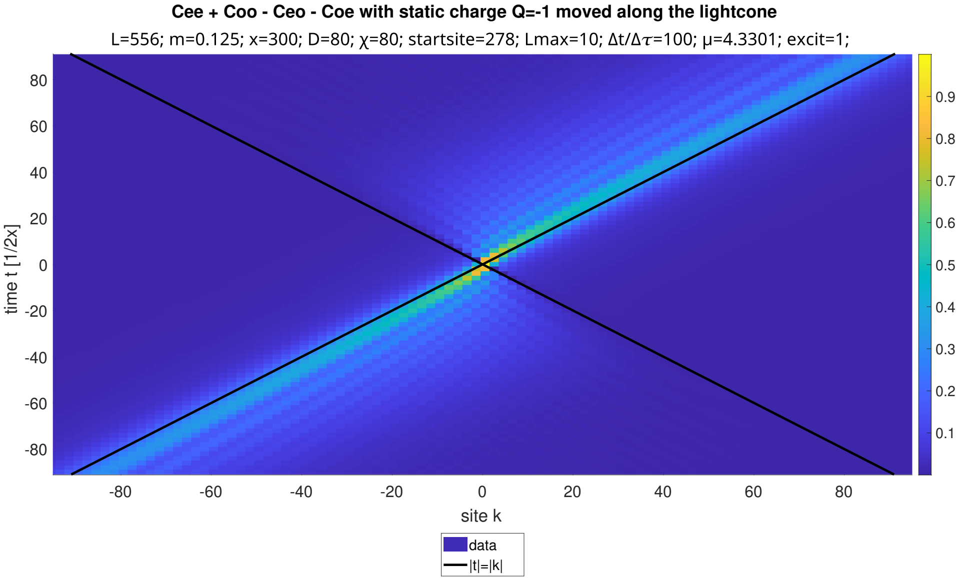
$1 - |\langle \Psi(D) | \Psi_{\text{exact}} \rangle|$

Checks - lightcone



Time evolution of correlator with fixed static charge

Checks - lightcone



Time evolution of correlator with charge moved along the lightcone

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - g\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Legendre transformation $\rightarrow \mathcal{H}$ (temporal gauge $A^0 = 0$)

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$

$$E = F^{1,0}$$

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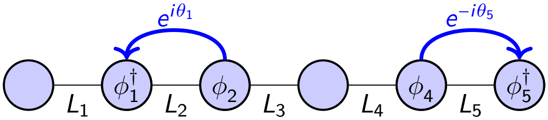
$$E = F^{1,0}$$

Lattice formulation: staggered fermions

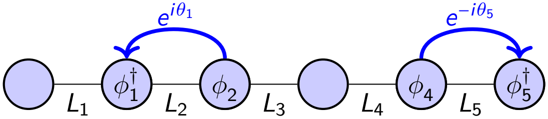
$$\phi_n \sim \begin{cases} \Psi_{\text{upper}}(x) \text{ if } n \text{ even} \\ \Psi_{\text{lower}}(x) \text{ if } n \text{ odd,} \end{cases}$$



$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger e^{i\theta_n}\phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

Spin formulation of the Schwinger Model (2)

$$H = -\frac{i}{2a}\sum_n \left(\phi_n^\dagger e^{i\theta_n}\phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n}\phi_n\right) + m\sum_n (-1)^n \phi_n^\dagger \phi_n + \frac{ag^2}{2}\sum_n L_n^2$$



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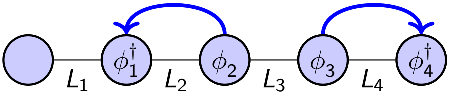
Decoupling:

$$\phi_n \to \prod_{k<n}\left(e^{-i\theta_k}\right)\phi_n.$$

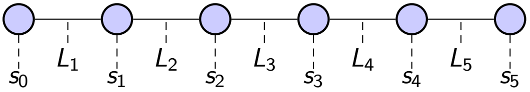


Spin formulation of the Schwinger Model (2)

$$H = -\frac{i}{2a}\sum_n \left(\phi_n^\dagger e^{i\theta_n}\phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n}\phi_n\right) + m\sum_n (-1)^n \phi_n^\dagger \phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

Decoupling:

$$\phi_n \to \prod_{k<n}\left(e^{-i\theta_k}\right)\phi_n.$$



Jordan-Wigner transformation → spin model: $\hat{\phi}_n = \prod_{k<n}\left(i\sigma_k^z\right)\sigma_n^-$:

$$H = \frac{1}{2a}\sum_n \left(\sigma_n^+\sigma_{n+1}^- + \sigma_{n+1}^-\sigma_n^+\right) + \frac{m}{2}\sum_n \left[1 + (-1)^n \sigma_n^z\right] + \frac{ag^2}{2}\sum_n L_n^2.$$

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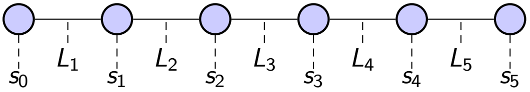

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Gauss's law:

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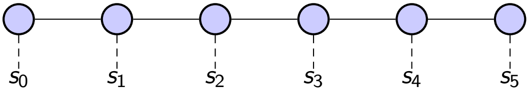

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→ Eliminate gauge degrees of freedom from H, rescaling:

$$\boxed{H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1 + (-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2} \left(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2}\right)$$

Factorization

Cross section:

$$\sigma \propto L^{\mu\nu}(k,q)\, W_{\mu\nu}(q,P)$$

Hadronic Tensor:

$$W_{\mu\nu}(\xi,P) = \sum_i \int_x^1 \frac{dz}{z} f_i(z)\, \hat{W}_{\mu\nu}\left(\frac{\xi}{z}, Q\right)$$

Leading order with $\hat{W} \propto \delta\left(1 - \frac{\xi}{z}\right)$:

$$W_{\mu\nu}(q,P) = 4\pi\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)F_1 + \frac{8\pi x}{Q^2}\left(P_\mu - \frac{P\cdot q}{q^2}q_\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)F_2$$

Factorization (leading order):

$$F_1(\xi) = \frac{1}{2}\sum_i e_i^2 f_i(\xi)$$
$$F_2(\xi) = 2xF_1(\xi)$$