Quantum Relativity

and

Quantum Spacetime

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OTTO C. W. KONG

— Nat'l Central U, Taiwan

Reference Frame Transformations :-

- $\longleftrightarrow$ **Relativity Symmetry**
  — spacetime symmetry/reference frame

- physical/quantum frame Vs absolute/classical frame
  — relative 'uncertainty' / entanglement

- example of quantum spatial translation

$$\hat{x}_B^{(A)} \longrightarrow -\hat{x}_A^{(B)}\,, \qquad \hat{x}_C^{(A)} \longrightarrow \hat{x}_C^{(B)} - \hat{x}_A^{(B)}\,,$$
$$\hat{p}_B^{(A)} \longrightarrow -(\hat{p}_A^{(B)} + \hat{p}_C^{(B)})\,, \quad \hat{p}_C^{(A)} \longrightarrow \hat{p}_C^{(B)}\,.$$

- quantum model of space(time) — the phase space

Unitary Q Spatial Translation :-

— generated by $\hat{p}_C$

Giacomini et.al. 19

- $e^{ix_B \hat{p}_C} \quad \rightarrow \quad e^{i\hat{x}_B^{(A)} \hat{p}_C^{(A)}} \quad \rightarrow \quad \hat{S}_x = \hat{\mathcal{P}}_{AB} e^{i\hat{x}_B^{(A)} \hat{p}_C^{(A)}}$

— $\hat{\mathcal{P}}_{AB} : |x\rangle_B \otimes |y\rangle_C \rightarrow |-x\rangle_A \otimes |y\rangle_C \;, \quad \hat{x}_B^{(A)} \rightarrow -\hat{x}_A^{(B)}$

$$\mathcal{H}_B^{(A)} \otimes \mathcal{H}_C^{(A)} \rightarrow \mathcal{H}_A^{(B)} \otimes \mathcal{H}_C^{(B)}$$

- *but* $\hat{x}_B^{(A)}$ generates $\hat{p}_B^{(A)}$ momentum translation

- Schrödinger picture : on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

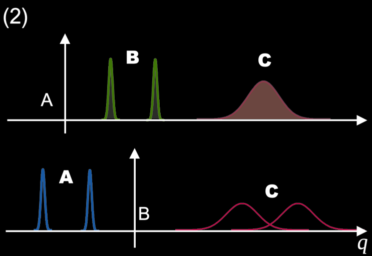
$$\hat{U}_x = \hat{\mathcal{S}}_{AB}^W \hat{I}_A \otimes \int dx' dy' \, |-x'\rangle\langle x'|_B \otimes |y' - x'\rangle\langle y'|_C$$

$$\hat{\mathcal{S}}_{AB}^W : |z\rangle_A \otimes |x\rangle_B \otimes |y\rangle_C \rightarrow |x\rangle_A \otimes |z\rangle_B \otimes |y\rangle_C$$

for $\quad |\psi\rangle = |\emptyset\rangle_A \otimes \int dx dy \; \psi(x,y) \, |x\rangle_B \otimes |y\rangle_C$

$$\hat{U}_x |\psi\rangle = \int dx dy \; \psi(x, y+x) \, |-x\rangle_A \otimes |\emptyset\rangle_B \otimes |y\rangle_C$$

The 4 scenarios: Case 2

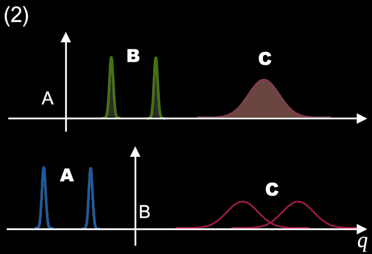


$$|\emptyset\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|x_1\rangle + |x_2\rangle\right)_B \otimes \int \mathrm{d}y\, \psi(y)|y\rangle_C$$

$$\longrightarrow \frac{1}{\sqrt{2}}\Bigg(|-x_1\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_1\rangle_C$$

$$+|-x_2\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_2\rangle_C\Bigg)$$

The 4 scenarios: Case 2

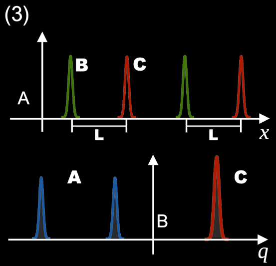


$$|\emptyset\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|x_1\rangle + |x_2\rangle\right)_B \otimes \int \mathrm{d}y\, \psi(y)|y\rangle_C$$

$$\longrightarrow \frac{1}{\sqrt{2}}\Big(|{-x_1}\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_1\rangle_C$$

$$+|{-x_2}\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_2\rangle_C\Big)$$

The 4 scenarios: Case 3



$$|\emptyset\rangle_A \otimes \left(c\,|x_1\rangle + s\,|x_2\rangle\right)_B \otimes \left(c\,|y_0 + x_1\rangle + s\,|y_0 + x_2\rangle\right)_C$$

$$\longrightarrow \left(c\,|-x_1\rangle + s\,|-x_2\rangle\right)_A \otimes |\emptyset\rangle_B \otimes |y_0\rangle_C$$

Distance Translated (A-frame to B-frame) ?

an Noncommutative Value :-

- classical $e^{i x_B \hat{p}_C}$: translation by x_B

  — say $x_B^i = 2$, $x_C^f = x_C^i - 2$, $x_A^f = -2$

- quantum : $[\hat{x}_B]_\phi^i$ as the value

  $$[\hat{x}_C]_{\phi'}^f = [\hat{x}_C]_\phi^i - [\hat{x}_B]_\phi^i \,, \quad [\hat{x}_A]_{\phi'}^f = -[\hat{x}_B]_\phi^i$$

  — $[\hat{x}_B]_\phi^i$ contains full quantum information of position

- evaluation $\hat{\mathcal{O}} \rightarrow [\hat{\mathcal{O}}]_\phi$: *algebraic homomorphism*

Quantum Relativity Principle :-

- **Penrose : Relativity Principle** $\rightarrow\!\otimes\!\leftarrow$ **Quantum**

- **Heisenberg picture – $\hat{x}$ and $\hat{p}$ as coordinates**

— **Noncommutative Geometry for Spacetime**

- **Rel. Sym. ← Quantum Ref. Frame Transformations**

— ***e.g.* translation by the NC value of $\hat{x}_A - \hat{x}_B$** (ans. Penrose)

- **Quantum Gravity as General Quantum Relativity**

Schrödinger Formulation :-

- ‘equivalent’ — only in Euclidean coordinates

- conceptually : old quantum theory thinking

- $\phi(x) \equiv \langle x|\phi\rangle$: $|\phi\rangle = \int dx\,|x\rangle\langle x|\phi\rangle = \sum_x \phi(x)|x\rangle$

— a set of coordinates on Hilbert space

— ??? configuration variables for a particle

- curvilinear $\hat{x}^i$ as x^i, no momentum vector

— metric loses its meaning in particle dynamics

Heisenberg & Dirac 1925/26 :-

— classical to quantum

only needs a new kinematic

- **H : physical quantities *not* real number variables**

- **Quine : real number as 'convenient fiction'**

- **D : q-number as the new convenient fiction**

Concept of Numbers (in history) :

- $x + 2 = 0 \quad \longrightarrow \quad$ **negative numbers**
- $2x - 1 = 0 \quad \longrightarrow \quad$ **rational numbers**
- $x^2 - 2 = 0 \quad \longrightarrow \quad$ **real numbers**
- $x^2 + 1 = 0 \quad \longrightarrow \quad$ **complex numbers**
- $xy - x - i = 0 \quad \longrightarrow \quad$ (i,2), $(\frac{1}{i-1}$,-i), ...
- $xy - yx - 1 = 0 \quad \longrightarrow \quad$ **noncommutative numbers**

★ $\hat{x}\hat{p} - \hat{p}\hat{x} - i\hbar = 0$

needs NC/q-number values for the variables

Evaluation as an Algebra Homomorphism :-

— real number is *only* an algebraic system

- classical $\quad [\phi] : f(x_i, p_i) \rightarrow \mathbb{R} \quad$ (observables have real values)

e.g. $\quad E = p^2 + x^2 = pp + xx \quad$ (1-D SHO $m = \frac{1}{2}$, $k = 2$)

$$[\phi](x) = 2,\ [\phi](p) = 3 \quad \Longrightarrow$$

$$[\phi](E) = [\phi](p^2) + [\phi](x^2) = [\phi](p)[\phi](p) + [\phi](x)[\phi](x) = 13$$

$$[\phi](x_i p_i) = [\phi](x_i)[\phi](p_i) = [\phi](p_i)[\phi](x_i) = [\phi](p_i x_i)$$

- **quantum** $\quad [\boldsymbol{\phi}] : \boldsymbol{\beta}(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{p}}_i) \rightarrow$ **?**

$$[\boldsymbol{\phi}](\hat{\boldsymbol{x}}_i)[\boldsymbol{\phi}](\hat{\boldsymbol{p}}_i) = [\boldsymbol{\phi}](\hat{\boldsymbol{p}}_i)[\boldsymbol{\phi}](\hat{\boldsymbol{x}}_i) + [\boldsymbol{\phi}](\boldsymbol{i\hbar\hat{I}})$$

$\Longrightarrow [\boldsymbol{\phi}](\boldsymbol{\beta}(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{p}}_i))$ **has to be a noncommutative algebra**

The Symplectic Geometry — NC Vs C :-

- **Heisenberg** — $\frac{d}{ds}\alpha(\hat{P}_\mu, \hat{X}_\mu) = \frac{1}{i\hbar}[\alpha(\hat{P}_\mu, \hat{X}_\mu), \hat{H}_s]$

- **Schrödinger** — $\frac{d}{ds} f_\alpha(z^n, \bar{z}^n) = \{f_\alpha(z^n), f_{H_s}\}$

- $f_\alpha(z^n, \bar{z}^n) \equiv \frac{{}_\eta\langle\phi|\alpha(\hat{P}_\mu, \hat{X}_\mu)|\phi\rangle}{{}_\eta\langle\phi|\phi\rangle}$ $\qquad \left(|\phi\rangle = \sum_n z^n\, |n\rangle \right)$

— as the pull-back of $\hat{\alpha}$ under $(z^n, \bar{z}^n) \longrightarrow (\hat{P}_\mu, \hat{X}_\mu)$

- $\rightarrow$ **bijective homomorphism between NC Poisson algebras**

— **NC Kähler product** $\qquad f_\alpha \star_\kappa f_{\alpha'} = f_{\alpha\alpha'}$ $\qquad$ *Cirelli et.al 90*

NC values of NC coordinates :-

— NC number as the new convenient fiction

- **state as evaluative homorphism**

— mapping observable algebra to algebra of their NC values

$[\hat{\alpha}]_\phi = \{f_\alpha|_\phi, V_{\alpha_n}|_\phi\}$ $\quad (V_{\alpha_n} = \frac{\partial f_\alpha}{\partial z^n} = -f_\beta \bar{z}^n + \sum_m \bar{z}^m \langle m|\hat{\alpha}|n\rangle)$

- **Kähler product** — $\quad [\hat{\alpha}\hat{\alpha}']_\phi = [\hat{\alpha}]_\phi \star_\kappa [\hat{\alpha}']_\phi$

$f_{\alpha\alpha'} = f_\alpha f_{\alpha'} + \sum_n V_{\alpha_n} V_{\alpha'_{\bar{n}}}$, $\quad V_{\alpha\alpha'_n} = -f_{\alpha\alpha'} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\hat{\alpha}|l\rangle\langle l|\hat{\alpha}'|n\rangle$

Deutsch & Hayden 00 ; Kong 23

- **locality of quantum information** (**Heisenberg picture**)

Galvão & Hardy 03

- Substituting a **Qubit** for an Arbitrarily Large Number of Classical Bits'

Noncommutative Number Systems :-

- **observables are dynamical *variables***

— **a state is an evaluative homomorphism**

- **matrices as Dirac's q-numbers**

- **convention : representation of observables as**

— **DH-matrix value for a reference state, *e.g.***

$$[\hat{s}^i]_\phi = U_\phi^\dagger \sigma^i U_\phi \ , \quad |\phi\rangle = U_\phi \, |0\rangle \ , \quad U_\phi = \begin{pmatrix} c & -\bar{s} \\ s & \bar{c} \end{pmatrix}$$

- **need to fix a 'reference frame' for the states**

Quantum Mechanics

can and should be seen as

Particle Dynamics on the Quantum Space

rather than

Quantized Dynamics on the Newtonian space

Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. $\Leftrightarrow$ Quantum Gravity

Non-Euclidean G. $\Leftrightarrow$ Classical Gravity

Lorentz Covariant Quantum Physics :-

- Schrödinger wavefunction $\phi(x^\mu)$

— basic operators x_μ and $-i\hbar\partial_\mu$

- abstract operators as Minkowski four-vectors

— $\hat{X}_i \longrightarrow \hat{X}_\mu$ and $\hat{P}_i \longrightarrow \hat{P}_\mu$

— $[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\eta_{\mu\nu}$

- Heisenberg-Weyl symmetry — $[Y_\mu, E_\nu] = i\hbar\, c\, \eta_{\mu\nu}\, M$

— **M is an effective Casimir element** → **Newtonian mass m**

— $m\hat{X}_\mu \longleftarrow Y_\mu$, different m for different irr. representations

— $\hat{P}_\mu \longleftarrow \frac{1}{c}E_\mu$, constant c (… $c \rightarrow \infty$ limit)

Minkowski Metric Operator $\hat{\eta}$ on Krein Space :-

• **Minkowski** nature of proper invariant **inner product**

— effectively, bra as ${}_{\eta}\langle\cdot| = \langle\cdot|\,\hat{\eta}$

— naive $|\phi(x^\mu)|^2$ **integral** cannot avoid **divergence**

• **observables (pseudo-)Hermitian,** $\quad {}_{\eta}\Big\langle \cdot \Big| \hat{A}^{\dagger^\eta} \cdot \Big\rangle = {}_{\eta}\Big\langle \hat{A} \cdot \Big| \cdot \Big\rangle$

— $\hat{X}_\mu = \hat{\eta}\hat{X}^\mu\hat{\eta}^{-1}$ and $\hat{P}_\mu = \hat{\eta}\hat{P}^\mu\hat{\eta}^{-1}$

• **noncommutative geometric picture**

— $\hat{X}^\mu$ **and** $\hat{P}^\mu$ **as coordinates**

Special Quantum Relativity :-

- **QRFT as relative position :** $\mathcal{P}_{ON} e^{i\hat{x}_N^\mu \hat{p}_\mu}$,

- **quantum observables never take zero value**

— **translation in single coordinate only as approximation**

zero eigenstate of $\hat{x}_N$ keeps $\psi(x)$, but $\hat{p}_o = \ldots$

- **otherwise :** $\mathcal{P}_{N'N} e^{i\hat{x}_N \hat{p}_{\hat{x}}} \mathcal{P}_{ON'} e^{i\hat{y}_{N'} \hat{p}_{\hat{y}}} \neq \mathcal{P}_{NN'} e^{i\hat{y}_{N'} \hat{p}_{\hat{y}}} \mathcal{P}_{ON} e^{i\hat{x}_N \hat{p}_{\hat{x}}}$

- **boost and rotation similarly approx.**

Quantum Lorentz Boost :-

$$\hat{z}' = c\hat{\tau}\sinh(\hat{\beta} - \hat{\beta}_N) = \hat{z}\cosh\hat{\beta}_N - c\hat{t}\sinh\hat{\beta}_N \ ,$$
$$c\hat{t}' = c\hat{\tau}\cosh(\hat{\beta} - \hat{\beta}_N) = -\hat{z}\sinh\hat{\beta}_N + c\hat{t}\ \cosh\hat{\beta}_N \ ,$$
$$\hat{p}'_{\hat{z}} = \cosh\hat{\beta}_N \hat{p}_{\hat{z}} + \sinh\hat{\beta}_N \hat{p}_{c\hat{t}} \ ,$$
$$\hat{p}'_{c\hat{t}} = \sinh\hat{\beta}_N \hat{p}_{\hat{z}} + \cosh\hat{\beta}_N \hat{p}_{c\hat{t}} \ .$$

- 'unitary' operator $\mathcal{P}_{ON} e^{i\hat{\beta}_N \hat{p}_{\hat{\beta}}}$, $\quad \hat{p}_{\hat{\beta}} = \frac{\partial\hat{z}}{\partial\hat{\beta}}\hat{p}_{\hat{z}} + \frac{\partial c\hat{t}}{\partial\hat{\beta}}\hat{p}_{c\hat{t}}$

- approx. by $\mathcal{P}_{ON} e^{i\hat{x}^\mu_N \hat{p}_\mu}$ with $\hat{x}^\mu = (c\hat{\tau}, \hat{z}, \hat{x}, \hat{y})$

— metric not preserved

Towards Gravity : on a particle -

- **quantum geodesic equation** (Heisenberg)

— *e.g.* instantaneous frame of free-fall

$$\frac{d^2\hat{x}^\mu}{ds^2} + \frac{d\hat{x}^\nu}{ds}\Gamma^\mu_{\nu\sigma}(\hat{x})\frac{d\hat{x}^\sigma}{ds} = 0$$

$\rightarrow$ maintaining (Weak) **Equivalence Principle**

— $\{\hat{x}^a, \hat{p}_b\} = \delta^a_b, \quad \{\hat{x}^a, \hat{x}^b\} = 0 = \{\hat{p}_a, \hat{p}_b\}$

$$\frac{\partial}{\partial \hat{x}^a} \equiv \{\cdot, \hat{p}_a\}, \qquad \frac{\partial}{\partial \hat{p}_a} \equiv -\{\cdot, \hat{x}^a\}$$

Quantum Rindler Frame :-

$$\hat{x} = \hat{\rho}\cosh\frac{\hat{a}_N\hat{\tau}}{c}, \qquad c\hat{t} = \hat{\rho}\sinh\frac{\hat{a}_N\hat{\tau}}{c}$$

- **eigenstates :** $\hat{x}$ & $\hat{t}$ $\quad\longrightarrow\quad$ $\hat{\rho}$ & $\hat{a}_N\hat{\tau}$

  entanglement between $\hat{\tau}$ and $\hat{a}_N$
- **metric :** $\hat{g}_{c\hat{\tau},c\hat{\tau}} = \frac{\hat{a}_N^2\hat{\rho}^2}{c^4}$
- **quantum geodesic equations :**

$$\frac{d^2c\hat{\tau}}{ds^2} + \frac{dc\hat{\tau}}{ds}\frac{1}{\hat{\rho}}\frac{d\hat{\rho}}{ds} + \frac{d\hat{\rho}}{ds}\frac{1}{\hat{\rho}}\frac{dc\hat{\tau}}{ds} = 0\ ,$$

$$\frac{d^2\hat{\rho}}{ds^2} + \frac{dc\hat{\tau}}{ds}\frac{\hat{a}_N^2\hat{\rho}}{c^4}\frac{dc\hat{\tau}}{ds} = 0\ .$$

Quantum Mech. in Curved Spacetime :-

- $\hat{H}_{\mathrm{free}} = \frac{1}{2m}\,\hat{p}_A\, g^{Ab}(\hat{x})\,\hat{p}_b$, $\quad \hat{p}^a = g^{ab}(\hat{x})\hat{p}_b$

— **Schrödinger representation fails**

- **Hamilton's Eqs. → mass-indep. E.O.M.**

- $\hat{x}^a$ **and** $\hat{p}^a$ **as** $\hat{g}$**-Hermitian within the ref. frame**

⋆ Relativity Principle

. . .

- all positions coordinates, and their functions, Hermitian

- four vectors : $V'^a = V^i \frac{\partial x'^a}{\partial x^i}$, $W'_a = \frac{\partial x^i}{\partial x'^a} W_i$,
  $V'^A \equiv V'^{a\dagger} = \left(\frac{\partial x'^a}{\partial x^i}\right)^\dagger V^{i\dagger} \equiv \left(\frac{\partial x^A}{\partial x'^I}\right) V^I$, $W'_A = W_I \left(\frac{\partial x^I}{\partial x'^A}\right)$.

- p^i and $\frac{dx^i}{ds}$ (not $p_i = g_{iJ} p^J$) $g(x^j)$-Hermitian : RF -dep.

- quantum geodesic equation : $\frac{d^2 x^i}{dt^2} =$

$$\frac{dx^h}{ds} \frac{\partial_J g_{hK}}{2} \frac{dx^K}{dt} g^{Ji} - \frac{dx^h}{ds} \frac{\partial_K g_{hJ}}{2} g^{Ji} \frac{dx^K}{ds} - \frac{dx^k}{ds} g_{kM} \frac{dx^h}{ds} g^{Ml} \frac{\partial_h g_{lJ}}{2} g^{Ji}$$

— s a Hamiltonian evolution parameter

— proper time a quantum observable

Future (H & D $\rightarrow \ldots$) :-

- **q-number physics – Q gravity as GQR**

  more NC physics : $[\hat{x}^i, \hat{x}^j] \neq 0$?

- **q-number geometry – NC Geo. as symplectic**

  coordinate picture : 'Euclidean NC Geo.'

- **q-number theory – algebra beyond algebra**

- **q-number technology – q-information**

  Quine: To be is to be the (q-number) value of a (physical) variable.

THANK YOU !