Effective Actions, Extremal Black Holes, Pair Production, and All That

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Overview (work in progress: CM Chen, SP Kim, CR: 24xx.xxxxx)

1. Introduction and motivation

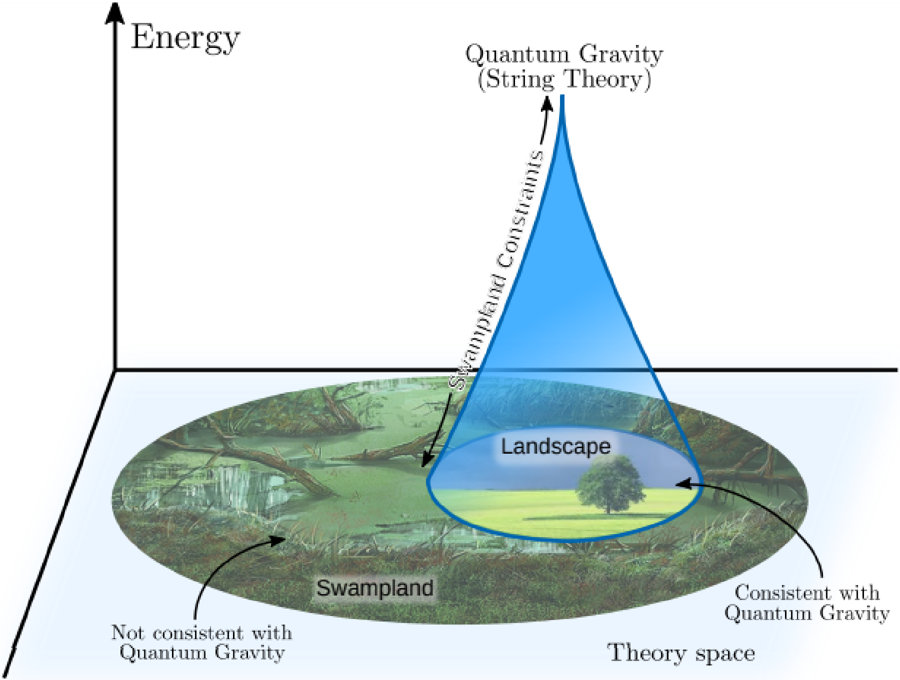
2. Computation of the effective action

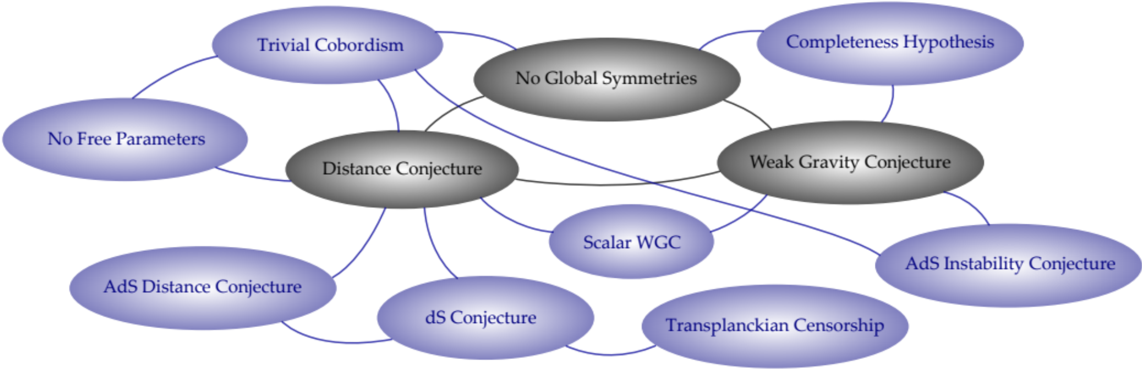
2.1 In-out formalism
2.1 Pair production in RN black holes
2.2 Strategy to obtain the effective action

3. Open questions and future directions

Introduction and motivation

- Pair production effects such as Hawking radiation and the Schwinger effect provide a natural channel for the decay of charged black holes.
- The study on decay of charged black holes has deep relations to the Weak Gravity Conjecture (WGC). (Arkani-Hamed, Motl, Nicolis, Vafa '07)
- WGC, in its simplest form, states gravity as the weakest force and requires the existence of at least one superextremal particle with charge-to-mass ratio larger that the corresponding extremality bound. What we are preventing? $\rightarrow$ Naked singularities!
- One interesting direction is to analize these bounds in black holes in dS space (because our universe is slightly dS from cosmological observations)
- Bounds in physical theories are an essential ingredient to analyze well-behaved Effective Field Theories (EFT).
- The philosophy of constraining EFTs from black hole mechanics evaporation in de Sitter space comes from the (string-inspired) Swampland program. (Vafa '05)


Energy
Quantum Gravity
(String Theory)
Swampland Constraints
Landscape
Swampland
Not consistent with
Quantum Gravity
Consistent with
Quantum Gravity
Theory space



Lectures on the Swampland Program in String Compactifications, hep-th: 2102.01111

The Swampland program

- It is characterized by a set of conjectures and inspired by developments in string theory (based on physical reasons like covariance, unitarity, causality, anomaly cancelations, positivity, UV completion, etc). *No counterexamples.*
- String theory suggest that the landscape of vacua is vast.
- It is natural to ask if this landscape is as vast as allowed by consistent-looking effective field theories, turns out that its not the case.
- There is an even more vast swampland of consistent-looking semiclassical effective field theories, which are actually inconsistent.
- EFTs emerge as low-energy limit of string theories, but not neccesarily in 10 or 11 dimensional scenarios (non-critical string theories are consistent in 4 dimensions)
- Its seems OK, but two questions appear:
  1. Why we need to care about strings when we try to constraint well-behaved EFTs?
  2. How these arguments are related to (extremal) black holes?
- **Lets take care of question 1 first:**

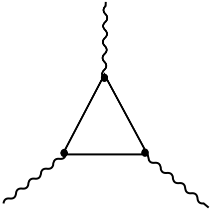
What makes a theory a good QFT (or EFT)?

- QFTs are powerful theories that allow us to compute various physical quantities with a wide range of available perturbative and non-perturbative techniques.
- In addition to the computational machinery, we are also thinking of an underlying mathematical structure (e.g. the space of operators and their algebra) that satisfy some fundamental principles such as unitarity (covariance, and so on).
- A quantum field theory usually comes with an action and their symmetries.

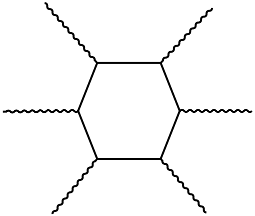
$$\langle O(\Phi)\rangle \sim \int \mathcal{D}\Phi e^{-S[\Phi]} O(\Phi).$$

- However, the symmetries do not always hold or even make sense at the quantum level. When that happens, we say the symmetry is anomalous, and if the gauge symmetry is anomalous, the theory is **inconsistent**.

- Local anomalies in four dimensions are associated with triangle Feynman diagrams.



- Six-point gravitational amplitude is related to the breakdown of general covariance. This is called gravitational anomaly and must vanish in a consistent theory of gravity.



- Another important feature of QFTs is the dependence of physical quantities on the energy scales. This dependence is often captured by the RG flow.
- There is a class of renormalizable field theories that become free theories at high energies. These theories are called asymptotically free and we can describe their **UV completion** without appealing to a cut-off or anything beyond QFT.
- For a generic QFT we are not this lucky and we usually need to define a cut-off. Depending on whether the theory is renormalizable or not, we need finite or infinite number of parameters to define the theory below the cut-off. This approach is called Effective Field Theory (EFT).
- The idea is that we can capture the physics at a certain energy scale by an effective field theory and if we are lucky enough (the theory is renormalizable) we can find the corresponding effective theory at other energy scales below the cutoff as well.
- So, its possible to define a good EFT for gravity at the UV?: NO

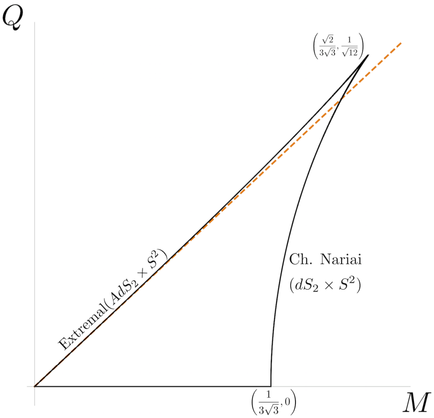
- The idea to incorporate gravity into QFT is to view the metric as a field that interacts with other fields and proceed with the quantization procedure.
- One might be tempted to take the EFT approach and input enough coupling constants from experiment to find couplings at other energy scales. However, it turns out you have infinitely many vertex operators with UV-divergent amplitudes that need to be kept track of. In other words, **our theory needs infinitely many parameters as input and is not predictive!**
- As long as we are dealing with energies below the Planck scale $E << M_P$, the changes in the couplings are small and it is reasonable to believe that we have a nice classical effective field theory. In other words, the Planck mass M_P introduces a natural scale beyond which the EFT should start to break down.
- String theory provides a route to evade these non-renormalizable feature of gravity, because it is a parameter free theory (in the sense that there is an intrinsic cut-off in the theory).
- **Now, lets take care of question 2:** How these arguments are related to (extremal) black holes?

What about gravity?

- To see why the canonical method of QFT could not have worked for gravity we need to take a detour through the realm of black holes.
- Black holes are solutions to the classical equations of motion for gravity. Large 4d black holes of mass M have radius of $\sim M l_P / M_P$ and the spacetime outside their horizon is weakly curved $\mathcal{R} l_P^2 \lesssim M_P^2/M^2$ for $M \gg M_P$.
- Since the curvature is very small ($\mathcal{R} l_P^2 \ll 1$) we can think of black holes as IR backgrounds where Einstein's classical equations are **reliable**.
- Black holes in $d = 4$ are very simple in the sense that a classically stationary black hole is described by only three parameters: mass, charge, and angular momentum.
- Bekenstein speculated that there should be more degrees of freedom for black holes. Otherwise, by dropping a thermal system with entropy into a black hole we can decrease universe's entropy and violate the second law of thermodynamics.
- Black hole's entropy: $S = \frac{A}{4}$.
- This equation implies that despite the classical uniqueness of black holes, there must be a huge number of degree of freedom represented by a black hole of mass M

- But the fact that IR object such as large black hole have so much information about very UV states is very strange. It implies that extremely low energy physics (*i.e.* large black holes) and extremely high energy states somehow know about each other which is completely contrary to the EFT perspective.
- This UV-IR dependence is a remarkable failure of UV-IR decoupling used in EFT.
- The premise of EFT and renormalization is to neglect the UV physics by renormalizing IR parameters, but a quantum treatment of gravity even at large distances requires incorporating the high energy degrees of freedom.
- Another reflection of this is that scattering of gravitons at very high energies (UV) proceeds via large intermediate black holes (IR). See for example scattering amplitudes arguments based on the conformal bootstrap.(Penedones et.al. (2021))
- So, the lesson is to take seriously the results from string theory and use those arguments to delimit the space of well behaved EFT describing the physics outside of the event horizon semiclassically (and what about the interior?)

More fairy tales from the Swampland: Nariai black holes



(Montero, Van Riet, Venken '19)

Method to compute the effective action: In-out formalism

In-out formalism: scattering matrix of "out" vacuum with respect to "in" vacuum gives the one-loop effective action (Schwinger '51)

$$\mathrm{e}^{iW} = \exp\left(i \int d^D x \sqrt{-g} L_{\mathrm{eff}} \right) = \langle 0, \mathrm{out} | 0, \mathrm{in} \rangle .$$

the imaginary part of action W can be obtained from the mean number of pair production

$$|\langle 0, \mathrm{out} | 0, \mathrm{in} \rangle|^2 = \mathrm{e}^{-2\mathrm{Im}\, W}, \qquad 2\mathrm{Im}\, W = \pm VT \sum_k \ln(1 \pm \mathcal{N}_k),$$

where VT is the spacetime volume and $\mathcal{N}_k$ are the mean number of mode k. Here plus (minus) sign corresponds to a scalar(fermion) pair production.
W is "analytic", so: $W = W_{real} + iW_{imaginary}$

Bogoliubov transformation

$$\hat{a}_{k,\mathrm{out}} = \alpha_k \hat{a}_{k,\mathrm{in}} + \beta_k^* \hat{b}_{k,\mathrm{in}}^\dagger, \qquad \hat{b}_{k,\mathrm{out}} = \alpha_k \hat{b}_{k,\mathrm{in}} + \beta_k^* \hat{a}_{k,\mathrm{in}}^\dagger,$$

The out vacuum can be expressed in terms of in vacuum as

$$|0;\mathrm{out}\rangle = \prod_k \frac{1}{\alpha_k} \sum_{n_k} \left(-\frac{\beta_k^*}{\alpha_k} \right)^{n_k} |n_k, n_k; \mathrm{in}\rangle,$$

and therefore we have

$$|\langle 0, \mathrm{out}|0, \mathrm{in}\rangle|^2 = \prod_k |\alpha_k|^{-2} \qquad \Rightarrow \qquad \mathrm{Im}\, W = \frac{1}{2} \sum_k \ln |\alpha_k|^2.$$

Scalar field in the RN black hole (based on: CM Chen, SP Kim, IC Lin, JR Sun, and MF Wu; 1202.3224)

The action for a probe charged scalar field Φ with the mass m and the charge q is

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}D_\alpha\Phi^* D^\alpha\Phi - \frac{1}{2}m^2\Phi^*\Phi\right),$$

where $D_\alpha \equiv \nabla_\alpha - iqA_\alpha$ with ∇_α being the covariant derivative in curved spacetime. Eom is the Klein-Gordon (KG) equation

$$(\nabla_\alpha - iqA_\alpha)(\nabla^\alpha - iqA^\alpha)\Phi - m^2\Phi = 0$$

and the flux of a probe charged scalar field

$$D = i\sqrt{-g}g^{\rho\rho}(\Phi D_\rho\Phi^* - \Phi^* D_\rho\Phi),$$

which is positive for an outgoing mode and is negative for an ingoing mode.

Using the ansatz,

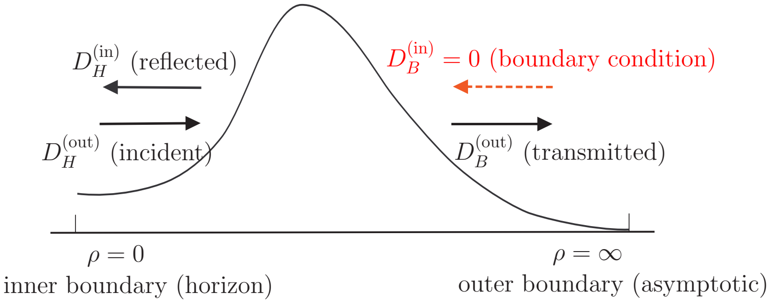
$$\Phi(\tau, \rho, \theta, \phi) = \mathrm{e}^{-i\omega\tau + in\phi} R(\rho) S(\theta),$$

KG equation is separated as

$$\begin{aligned}
\partial_\rho \left[(\rho^2 - B^2)\partial_\rho R\right] + \left[\frac{(q\rho - \omega Q)^2 Q^2}{\rho^2 - B^2} - m^2 Q^2 - \lambda_l\right] R &= 0, \\
\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta S) - \left(\frac{n^2}{\sin^2\theta} - \lambda_l\right) S &= 0,
\end{aligned}$$

where λ_l is a separation constant. The solution for $S(\theta)$ is the standard spherical harmonics with the eigenvalue $\lambda_l = l(l+1)$.
*Remark: Spontaneous pair production makes the radial equation violate the BF bound, generating an instability mapped to dual operators in the dual CFT

Boundary condition for pair production (outer bc)



$$|D_{\mathrm{incident}}| = |D_{\mathrm{reflected}}| + |D_{\mathrm{transmitted}}|,$$

$$|\alpha|^2 - |\beta|^2 = 1,$$

$$|\alpha|^2 \equiv \frac{D_{\mathrm{incident}}}{D_{\mathrm{reflected}}}, \qquad |\beta|^2 \equiv \frac{D_{\mathrm{transmitted}}}{D_{\mathrm{reflected}}}.$$

where

- α, β: Bogoliubov coefficients
- $|\alpha|^2$: vacuum persistence amplitude
- $|\beta|^2$: mean number of produced pairs

Pair production in RN black holes

Lets recall the geometry of the near horizon geometry of near extremal Reissner-Nordström black hole

$$ds^2 = -\frac{r^2 - B^2}{Q^2}dt^2 + \frac{Q^2}{r^2 - B^2}dr^2 + Q^2 d\Omega_2^2, \qquad A = -\frac{r}{Q}dt.$$

The pair production of charged scalar field in the background has been already analyzed and the corresponding Bogoliubov coefficients ($|\alpha|^2 - |\beta|^2 = 1$) are

$$|\alpha|^2 = \frac{\cosh(\pi\kappa - \pi\mu)\cosh(\pi\tilde{\kappa} + \pi\mu)}{\cosh(\pi\kappa + \pi\mu)\cosh(\pi\tilde{\kappa} - \pi\mu)}, \qquad |\beta|^2 = \frac{\sinh(2\pi\mu)\sinh(\pi\tilde{\kappa} - \pi\mu)}{\cosh(\pi\kappa + \pi\mu)\cosh(\pi\tilde{\kappa} - \pi\mu)}.$$

where

$$\tilde{\kappa} = \frac{\omega Q^2}{B}, \qquad \kappa = qQ, \qquad \mu = \sqrt{(q^2 - m^2)Q^2 - l(l+1) - 1/4} = \sqrt{\lambda^2 - l(l+1)}.$$

From the in-out formalism, the imaginary part of action (Kim '08, Kim '09) is related to the mean number $\mathcal{N} = |\beta|^2$ and for the pair production in the near horizon of RN black holes it is

$$\begin{aligned} & \frac{1}{2}\ln(1+\mathcal{N}) = \frac{1}{2}\ln(|\alpha|^2) \\ = \; & \frac{1}{2}\ln\left(1+\mathrm{e}^{-2\pi(\kappa-\mu)x}\right) + \frac{1}{2}\ln\left(1+\mathrm{e}^{-2\pi(\tilde{\kappa}+\mu)x}\right) - \frac{1}{2}\ln\left(1+\mathrm{e}^{-2\pi(\kappa+\mu)x}\right) \\ & -\frac{1}{2}\ln\left(1+\mathrm{e}^{-2\pi(\tilde{\kappa}-\mu)x}\right). \end{aligned}$$

There are four terms of the form $\ln(1+\mathrm{e}^{-S})$ which can be computed by a contour integration.

Strategy to obtain the effective action

$$\ln\left(1+\mathrm{e}^{-S}\right)=\mathrm{e}^{-S}-\frac{1}{2}\mathrm{e}^{-2S}+\frac{1}{3}\mathrm{e}^{-3S}-\frac{1}{4}\mathrm{e}^{-4S}+\cdots\,,$$

as a sum of residues in the Cauchy's residue theorem. Also:

$$\csc z=\frac{1}{\sin z}=\frac{1}{z}-2z\left(\frac{1}{z^2-\pi^2}-\frac{1}{z^2-(2\pi)^2}+\frac{1}{z^2-(3\pi)^2}-\cdots\right),$$

which leads to construct the following function, with real and positive parameter S,

$$-\mathrm{e}^{-Sz/\pi}\left(\frac{1}{\sin z}-\frac{1}{z}\right)\frac{1}{z},$$

containing simple poles at $z=\pi,2\pi,3\pi,\cdots$ with residues $\mathrm{e}^{-S}/\pi,-\mathrm{e}^{-2S}/2\pi,\mathrm{e}^{-3S}/3\pi,\cdots$. Therefore the contour integral, of this function gives

$$-\oint_C \mathrm{e}^{-Sz/\pi}\left(\frac{1}{\sin z}-\frac{1}{z}\right)\frac{dz}{z}=i\ln\left(1+\mathrm{e}^{-S}\right).$$

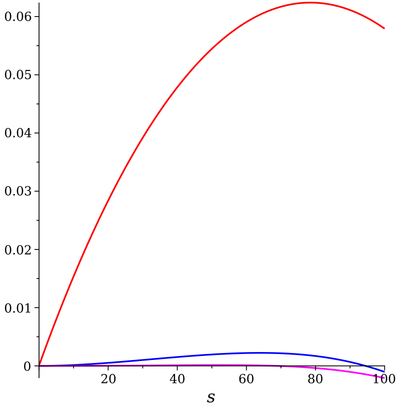


Figure: Error of approximation for the sum. l_{max}=50

The one-loop effective action is ($\Delta(s)$, complicated function of λ and s):

$$L_{\mathrm{eff}} = \lim_{\omega_0 \to \infty} \frac{1}{4\pi^2 Q^4} P \int_0^\infty \frac{ds}{s} \Delta(s) \left(\omega_0 \, \mathrm{e}^{-\kappa s} + \frac{B}{Q^2} \right) \left(\frac{1}{\sin s/2} - \frac{2}{s} \right).$$

Pair production in (near-extremal) Nariai black holes (based on C-M Chen, C-C Huang, SP Kim, C-Y Wei '23)

The near inner horizon geometry of near extremal Nariai black holes and the associated gauge field are

$$ds^2 = r_{\mathrm{ds}}^2\left[-(B^2-\rho^2)d\tau^2 + \frac{d\rho^2}{B^2-\rho^2}\right] + r_{\mathrm{n}}^2 d\Omega_2^2, \qquad A = -\frac{r_{\mathrm{ds}}^2 Q_{\mathrm{n}}}{r_{\mathrm{n}}^2}\rho d\tau.$$

The corresponding Bogoliubov coefficients are: for $\tilde{\kappa} \geq \kappa$

$$|\alpha|^2 = \frac{\cosh(\pi\tilde{\kappa}+\pi\mu)\cosh(\pi\tilde{\kappa}-\pi\mu)}{\cosh(\pi\kappa+\pi\mu)\cosh(\pi\kappa-\pi\mu)}, \qquad |\beta|^2 = \frac{\sinh(\pi\tilde{\kappa}+\pi\kappa)\sinh(\pi\tilde{\kappa}-\pi\kappa)}{\cosh(\pi\kappa+\pi\mu)\cosh(\pi\kappa-\pi\mu)},$$

$$|\alpha|^2 - |\beta|^2 = 1,$$

and then

$$\begin{aligned}\delta(\mathrm{Re}\,W) &= \frac{\mathcal{D}}{2}P\int_0^\infty \left(\mathrm{e}^{-2(\mu+\tilde\kappa)x}+\mathrm{e}^{-2(\mu-\tilde\kappa)x}-\mathrm{e}^{-2(\mu+\kappa)x}-\mathrm{e}^{-2(\mu-\kappa)x}\right)\left(\frac{1}{\sin x}-\frac{1}{x}\right)\frac{dx}{x}\\ &= \mathcal{D}P\int_0^\infty \mathrm{e}^{-2\mu x}\Big(\cosh(2\tilde\kappa x)-\cosh(2\kappa x)\Big)\left(\frac{1}{\sin x}-\frac{1}{x}\right)\frac{dx}{x}.\end{aligned}$$

Here $\mathcal{D}$ is the density of states, and for $\mathrm{AdS}_2\times S^2$ it is (here $L_{\mathrm{AdS}}=R_{\mathrm{S}}=Q$)

$$\mathcal{D}=\left(\frac{\mu}{\pi L_{\mathrm{AdS}}^2}\right)\left(\frac{1}{4\pi R_S^2}\right)=\frac{\mu}{4\pi^2 Q^4}.$$

For the total effect action, one should integrate δL_{eff} over ω from 0 to ∞ and also sum over l from 0 to ∞,

$$\begin{aligned}\mathrm{Re}\,W &= \int_0^\infty d\omega\sum_{l=0}^\infty(2l+1)\mathcal{D}P\int_0^\infty\frac{ds}{s}\mathrm{e}^{-\mu s}\Big(\cosh(\tilde\kappa s)-\cosh(\kappa s)\Big)\left(\frac{1}{\sin s/2}-\frac{2}{s}\right)\\ &= \lim_{\omega_0\to\infty}\sum_{l=0}^\infty(2l+1)\mathcal{D}P\int_0^\infty\frac{ds}{s}\mathrm{e}^{-\mu s}\left(\frac{B}{s}\sinh\left(\frac{\omega_0 s}{B}\right)-\omega_0\cosh(\kappa s)\right)\left(\frac{1}{\sin s/2}-\frac{2}{s}\right)\end{aligned}$$

Open questions and future directions

- It is unclear how to generalize the prescription for the one-loop effective action to rotating black hole (sum of separation constant $l(l+1)$, with l not an integer)
- Pair production for RN black hole full solution is hard, because of the lack of connections with analytic solutions of the (confluent) Heun-type differential equations.
- Backreaction is important in the analysis. A renormalized energy-momentum tensor (and current density) is needed.
- We need to check the validity of in-out formalism, heat kernels and monodromy with finite temperature scalar (fermion) QED at one-loop–$>$ Thermofield dynamics at the near extremal limit.
- Some recent experimental efforts will be made possible in the future to observe the Schwinger effect. (graphene 1D experiments, high-energy lasers, and acoustic analogues systems), but we are yet a couple of orders of magnitude away of the threshold to observe something...

Open questions and future directions

- Effective action is related with calculations of logarithmic corrections of entropy of black holes (Sen '10)
- Another route is to calculate the Page curve, in terms of a semiclassical approach (Page '05) or in holographic setups (Wang, Li, Wang '21)
- Overspinning or overcharging corresponds to be outside of the "shark fin". It is possible to calculate the backreaction in $T_{\mu\nu}$ directly from mean number? In-in formalism?
- It would interesting to explore recent results (Lin, Shiu '24) to attempt to calculate the full profile of the Schwinger effect. Worldline instantons.

Thanks!