Cosmology in Asymptotically Safe Gravity

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- The Einstein-Hilbert action:

$$S_{EH} = \frac{c^4}{16\pi G} \int \mathrm{d}^4x (R - 2\Lambda)$$

- The coupling constant $[G] = -2$ has negative mass dimension which suggests that general relativity is pertubatively non- renormalizable.
- However, one can argue that gravity should not be quantized at all, because Einstein gravity is an effective theory resulting from quantizing some yet unknown fundamental theory.
- To explain black hole singularity and initial singularity (at Planck scale 10^{19} GeV), we need quantum theory of gravity.
- Asymptotic safety demands:

  The RG flow of coupling constants must hit a non-trivial fixed point in the ultraviolet (UV) limit.

- Inspired by the Wilsonian path integral approach: Integrating out quantum fluctuations as a function of the RG scale k.
- The RG approach regards Einstein gravity is an effective theory (Γ_k) which is valid near a certain non-zero momentum scale k.
- This means that it arises from some fundamental theory by a "partial quantization" in which only excitations with momenta larger than k are integrated out ($p^2 > k^2$), while those with momenta smaller than k are not included ($p^2 < k^2$).
- Scale-dependent generating functional of Euclidean correlation functions

$$Z_k[J] \equiv e^{W_k[J]} = \int D\varphi \; e^{-S[\varphi]+(J,\varphi)-\Delta S_k[\varphi]} \tag{1}$$

- Regulator: quadratic functional

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \; \varphi^\star(p)\mathcal{R}_k(p^2)\varphi(p) \tag{2}$$

- **Property of $\mathcal{R}_k(p^2)$:** i. $\lim_{\frac{p^2}{k^2}\to 0} \mathcal{R}_k(p^2) = k^2$ for $p^2 \ll k^2$, the regulator screens the IR modes in a mass like fashion $m^2 \sim k^2$.

- The scale dependent modified effective action

$$\Gamma_k[\phi] = (J, \phi) - W_k[J] - \Delta S_k[\phi] \tag{3}$$

- The effective average action (Γ_k) interpolates between fundamental or an ordinary quantum effective action in the limits $k \to \infty$ and $k \to 0$ i.e $S = \Gamma_{k\to\infty}$ and $\Gamma = \Gamma_{k=0}$.
- Non-perturbative exact functional renormalization flow equation (FRGE):

$$k\partial_k\Gamma_k = \frac{1}{2} Tr\left[k\partial_k\mathcal{R}_k\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1}\right] \tag{4}$$

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- C. Wetterich, Phys. Lett. B, 301, 1 (1993) $\Rightarrow$ for scalar field
- M. Reuter, Phys. Rev. D 57, no. 10, 971 (1998) $\Rightarrow$ for gravity

- Einstein-Hilbert truncation : Truncation Up to $\sqrt{g}$ and $\sqrt{g}R$ (Ricci scalar)

$$\Gamma_k[g, \bar{g}] = (16\pi G(k))^{-1} \int d^d x\sqrt{g}\,\{-R(g) + 2\Lambda(k)\} + S_{gf}[g, \bar{g}], \tag{5}$$

- The flow equation for $G(k)$ and $\Lambda(k)$ are

$$k\partial_k G(k) = \eta_N G(k) \tag{6}$$

$$k\partial_k \Lambda(k) = \eta_N \Lambda(k) + \frac{1}{2\pi} k^4 G(k) \left[10\Phi^1_2(-2\Lambda(k)/k^2) - 8\Phi^1_2(0) - 5\eta_N \tilde{\Phi}^1_2(-2\Lambda(k)/k^2) \right] \tag{7}$$

- M. Reuter; Phys.Rev.D **57**, 971-985, (1998)

Running Gravitational $G(k)$ and Cosmological Constant $\Lambda(k)$

- $G(k)$ and $\Lambda(k)$ can be written as for very small k:

$$G(k) = G_0 \left[1 - \omega G_0 k^2 + \omega_1 G_0^2 k^4 + \mathcal{O}(G_0^3 k^6) \right] \tag{8}$$

$$\Lambda(k) = \Lambda_0 + G_0 k^4 \left[\nu + \nu_1 G_0 k^2 + \mathcal{O}(G_0^2 k^4) \right]. \tag{9}$$

where $\Lambda_0 = 0$, $\omega = \frac{4}{\pi}\left(1 - \frac{\pi^2}{144}\right)$, $\nu = \frac{2\zeta(3)}{4\pi}$, $\omega_1 = \omega^2 - \frac{\omega}{3\pi} - \frac{13\nu}{6\pi}$, $\nu_1 = -\omega\nu + \frac{5\omega}{6\pi} + \frac{5\nu}{3\pi}$

RG improved Einstein equation:

- Einstein's field equations of general relativity with scale dependent $G(k)$ and $\Lambda(k)$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G(t)T_{\mu\nu} + \Lambda(t)g_{\mu\nu} \ . \tag{10}$$

- $G(t) \equiv G(k = k(t)), \quad \Lambda(t) \equiv \Lambda(k = k(t))$

Bianchi I universe with Ordinary continuity equation

- The anisotropic cosmology described by Bianchi I metric given as

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2 \,. \tag{11}$$

- The energy-momentum tensor of a perfect fluid :

$$T_{\mu\nu} = (p + \rho)v_\mu v_\nu + pg_{\mu\nu} \ . \tag{12}$$

- The equations for scale factors :

$$-\frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} - \frac{\dot{b}\dot{c}}{bc} = 8\pi Gp - \Lambda \quad ; \qquad -\frac{\ddot{a}}{a} - \frac{\ddot{c}}{c} - \frac{\dot{a}\dot{c}}{ac} = 8\pi Gp - \Lambda \tag{13}$$

$$-\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\dot{a}\dot{b}}{ab} = 8\pi Gp - \Lambda \quad ; \qquad \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac} = 8\pi G\rho + \Lambda \,. \tag{14}$$

Ordinary continuity equation:

- The covariant conservation of the energy-momentum tensor i.e $D^{\mu} T_{\mu\nu} = 0$ yields

$$\dot{\rho} + (p + \rho)\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) = 0\,. \tag{15}$$

- $D^{\mu}\left[-\Lambda(t) g_{\mu\nu} + 8\pi G(t) T_{\mu\nu}\right] = 0$ and $D^{\mu} T_{\mu\nu} = 0$ together leads to a consistency equation

$$8\pi\rho\dot{G} + \dot{\Lambda} = 0\,. \tag{16}$$

- The solution of scale factors:

$$a(t) = m_1 \mathcal{R}(t) \exp\left[\frac{l(2+\beta)}{3}\int \frac{dt}{\mathcal{R}^3(t)}\right] \tag{17}$$

$$b(t) = m_2 \mathcal{R}(t) \exp\left[\frac{l(\beta-1)}{3}\int \frac{dt}{\mathcal{R}^3(t)}\right] \tag{18}$$

$$c(t) = m_3 \mathcal{R}(t) \exp\left[-\frac{l(1+2\beta)}{3}\int \frac{dt}{\mathcal{R}^3(t)}\right], \tag{19}$$

where m_1, m_2, m_3 are arbitrary constants of integration satisfying $m_1 m_2 m_3 = 1$ and $\mathcal{R}^3(t) = abc$.

- The simplest behaviour of k at late times is

$$k = \sum_n \frac{\xi_n}{t^n} . \tag{20}$$

- We will keep terms up to $n = 3$ as we are interested in the behaviour of G and Λ up to $\mathcal{O}\left(\frac{t_{Pl}^4}{t^4}\right)$

$$k = \frac{\xi}{t} + \frac{\sigma}{t^2} + \frac{\delta}{t^3} . \tag{21}$$

- A. Bonanno, M. Reuter, Phys. Rev. D 65 (2002) 043508

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Energy density and average scale factor

- The energy density and the average scale factor take form in terms of t as:

$$\rho(t) = \frac{1}{4\pi}\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right)\frac{1}{G_0 t^2}\left\{1 + \frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} + \frac{\tilde{\sigma}^2}{t^2} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)\frac{G_0}{t^2} + \mathcal{O}\left(\frac{G_0^2}{t^4}\right)\right\}, \tag{22}$$

$$\mathcal{R}(t) = \left[\frac{\mathcal{M}G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{\frac{1}{(3+3\Omega)}} t^{\frac{2}{(3+3\Omega)}}\left\{1 - \frac{1}{(3+3\Omega)}\left(\frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} - \frac{(3\Omega+5)}{(3+3\Omega)}\frac{\tilde{\sigma}^2}{t^2} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)\frac{G_0}{t^2}\right) + \mathcal{O}\left(\frac{G_0^2}{t^4}\right)\right\}. \tag{23}$$

Quantum corrected Scale Factors

- For $\Omega = 0$ ($p = \Omega\rho$) and $\Omega = \frac{1}{3}$ but not $\Omega \neq 1$, the scale factors are :

$$
\begin{aligned}
a(t) &= m_1 \mathcal{R}(t) \exp\left[\frac{l(2+\beta)\alpha}{3}\mathcal{N}(t)\right] , \\
b(t) &= m_2 \mathcal{R}(t) \exp\left[\frac{l(\beta-1)\alpha}{3}\mathcal{N}(t)\right] , \\
c(t) &= m_3 \mathcal{R}(t) \exp\left[-\frac{l(1+2\beta)\alpha}{3}\mathcal{N}(t)\right] ,
\end{aligned} \tag{24}
$$

where we have written

$$
\mathcal{N}(t) = \frac{(\Omega+1)}{(\Omega-1)} t^{\frac{(\Omega-1)}{(\Omega+1)}} - \tilde{\sigma} t^{-\frac{2}{(\Omega+1)}} - \frac{1}{(\Omega+3)}\left(2\tilde{\delta} - \frac{(\Omega-1)\tilde{\sigma}^2}{(\Omega+1)} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right) t^{-\frac{(\Omega+3}{(\Omega+1}} \tag{25}
$$

$$
\alpha = \left[\frac{\mathcal{M} G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{-\frac{1}{1+\Omega}} .
$$

- We can compute the directional Hubble parameters

$$\begin{aligned}\frac{\dot a}{a} &= H(t) + \tfrac{l(2+\beta)\alpha}{3} H_1(t)\,,\\ \frac{\dot b}{b} &= H(t) + \tfrac{l(\beta-1)\alpha}{3} H_1(t)\,,\\ \frac{\dot c}{c} &= H(t) - \tfrac{l(1+2\beta)\alpha}{3} H_1(t)\,.\end{aligned} \tag{26}$$

- where "average Hubble parameter" $H(t)$ includes isotropic quantum corrections,

$$H(t) = \frac{2}{(3+3\Omega)}\frac{1}{t}\left[1 + \frac{\tilde\sigma}{t} + \left(2\tilde\delta - \tilde\sigma^2 + \left(\frac{2\tilde\omega_1}{\tilde\omega} + \frac{3\tilde\nu_1}{2\tilde\nu}\right) G_0\right)\frac{1}{t^2} + \mathcal{O}\left(\frac{t_{Pl}^3}{t^3}\right)\right], \tag{27}$$

- while the effects of anisotropy are included in the coefficients of $H_1(t)$

$$H_1(t) = t^{-\frac{2}{1+\Omega}} + \frac{2\tilde\sigma}{(1+\Omega)} t^{-\frac{(\Omega+3)}{(1+\Omega)}} + \frac{1}{(\Omega+1)}\left(2\tilde\delta - \frac{(\Omega-1)\tilde\sigma^2}{(\Omega+1)} + \left(\frac{2\tilde\omega_1}{\tilde\omega} + \frac{3\tilde\nu_1}{2\tilde\nu}\right) G_0\right) t^{-\frac{2(2+\Omega)}{1+\Omega}}\,. \tag{28}$$

$\Omega = 0$ (Dust)

- Extra Einstein equation for scale factor:

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac} = 8\pi G\rho + \Lambda \tag{29}$$

- The consistency conditions comparing different inverse powers of t :

$$\frac{\tilde{\omega}}{\tilde{\nu}} = \frac{3}{2}\,, \tag{30}$$

$$\frac{4}{3}\left[4\tilde{\delta} - \tilde{\sigma}^2 + 2\left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right] - \frac{l^2(\beta^2+\beta+1)\alpha^2}{3} = -\tilde{\nu}G_0 + 2\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right)\left[2\tilde{\delta} + \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right], \tag{31}$$

- If we take terms upto $\mathcal{O}\left(\frac{1}{t^2}\right)$ terms, we would have obtained conditions corresponding to FLRW cosmology.

- For higher order, we have got anisotropic signature as $l \neq 0$.

- Thus we see that for $\Omega = 0$, the anisotropic Bianchi-I cosmology does not necessarily flow to the FLRW solution when quantum corrections are included.

- By comparing inverse powers of t in the consistency condition:

$$\frac{\tilde{\omega}}{\tilde{\nu}} = \frac{8}{3}\,, \tag{32}$$

$$4\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right)\tilde{\sigma} = \frac{3}{2}\tilde{\sigma} - \frac{l^2\alpha^2\left(\beta^2+\beta+1\right)}{3}\,, \tag{33}$$

- It immediately follows that

$$l^2\alpha^2\left(\beta^2+\beta+1\right) = 0\,. \tag{34}$$

- If $l \neq 0$, we must have $\beta^2+\beta+1=0$. This implies that the two roots of β are complex.
- Since the scale factors must be real, it follows that $l = 0$
- Hence we see that all the directional Hubble parameters are equal, i.e. the universe must be FLRW, in the presence of radiation.

$\Omega = 1$ (Stiff Matter)

- We can write l in terms of other constant β for $\Omega = 1$

$$l = \frac{1}{\alpha}\frac{\sqrt{1-6(\frac{\tilde{\nu}}{\tilde{\omega}})}}{\sqrt{(\beta^2+\beta+1)}} \quad . \tag{35}$$

- As $\frac{\dot{a}}{a}$, $\frac{\dot{b}}{b}$ and $\frac{\dot{c}}{c}$ are real, we get the following condition from the above equation :

$$1 - 6\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right) \geqslant 0 \qquad \Rightarrow \qquad \frac{\tilde{\nu}}{\tilde{\omega}} \leqslant \frac{1}{6} \tag{36}$$

- When $\frac{\tilde{\nu}}{\tilde{\omega}} = \frac{1}{6}$, $l^2(\beta^2+\beta+1) = 0$, which implies that $l = 0$ since β must be real.
- we observe that for large β, we get a Kasner type solution, i.e. there are expanding and contracting directions.

$$\begin{aligned} \frac{\dot{a}}{a} &= \frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{l(2+\beta)\alpha}{3}\bar{H}(t) \\ \frac{\dot{b}}{b} &= \frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{l(\beta-1)\alpha}{3}\bar{H}(t) \\ \frac{\dot{c}}{c} &= \frac{\dot{\mathcal{R}}}{\mathcal{R}} - \frac{l(1+2\beta)\alpha}{3}\bar{H}(t)\,, \end{aligned} \tag{37}$$

- For large positive β the expanding directions would involve the scale factors a, b and contracting direction would involve c.

FLRW Cosmology

- FLRW cosmology described by a metric which is given as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] . \tag{38}$$

- The equation for scale factors $a(t)$:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G(t)\rho + \frac{\Lambda(t)}{3}. \tag{39}$$

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- **Ordinary continuity equation:** $D^\mu\left[-\Lambda(t)g_{\mu\nu} + 8\pi G(t)T_{\mu\nu}\right] = 0$ and $D^\mu T_{\mu\nu} = 0$ together yields

$$\dot{\rho} + 3H(p+\rho) = 0 \quad , \qquad 8\pi\rho\dot{G} + \dot{\Lambda} = 0 \tag{40}$$

- **Modified continuity equation:** $D^\mu\left[-\Lambda(t)g_{\mu\nu} + 8\pi G(t)T_{\mu\nu}\right] = 0$ leads

$$\dot{\rho} + 3H(p+\rho) = -\frac{8\pi\rho\dot{G} + \dot{\Lambda}}{8\pi G(t)} \tag{41}$$

With Cutoff scale $k = \frac{\xi}{t}$

- Quantum corrected Hubble parameter:

$$H(t) = \frac{1}{\alpha t}\left[\left(1 + \frac{c}{t} + \frac{c^2}{t^2} + \frac{c^3}{t^3} + \frac{c^4}{t^4} + \cdots\right) - \frac{\alpha^2 \tilde{\nu} G_0}{3t^2}\left(1 + \frac{c}{t} + \frac{c^2}{t^2} + \frac{c^2}{3t^2} - \frac{\alpha^2 \tilde{\nu} G_0}{9t^2} + \cdots\right)\right]$$

where $\alpha = \frac{(3+3\Omega)}{2}$

-

$$H(t) = \frac{1}{\alpha t}\left[1 - \frac{\alpha^2 \tilde{\nu} G_0}{3t_c t}\left(\frac{t_c}{t} - 1\right) - \left(\frac{\alpha^2 \tilde{\nu} G_0}{3t_c t}\right)^2\left(\frac{t_c}{t} - 1\right) - \left(\frac{\alpha^2 \tilde{\nu} G_0}{3t_c t}\right)^3\left(\frac{t_c}{t} - 1\right) + \mathcal{O}\left(t^{-4}\right)\right] \tag{43}$$

the critical time scale $t_c = \alpha^2 \tilde{\nu} G_0 / 3c$

- Attractive Phase: The expansion rate of the FLRW universe slows down than the speed of classical expansion, $H(t) < H_{\rm cl}(t) = 1/\alpha t$, for $t < t_c$
- Repulsive phase: The expansion rate enhances than the classical expansion rate when $t > t_c$
- The energy density takes the form

$$\rho(t) = \frac{3}{8\pi\alpha^2 G_0 t^2}\left[1 + \frac{2c}{t} + \frac{3c^2}{t^2} + \frac{\tilde{\omega} G_0 - \alpha^2 \tilde{\nu} G_0}{t^2} + \mathcal{O}\left(t^{-3}\right)\right]. \tag{44}$$

More on cutoff choices

- **Cutoff** $k(H) = \zeta G_0^{(\beta-1)/2} H^\beta$**:**
- Three specific values of β, namely, $\beta = 1/4, 3/4, 1$, were considered in R. Mandal, S. Gangopadhyay and A. Lahiri, Eur. Phys. J. Plus 137 (2022) 10, 1110.
- The entropy generation diverges at late times for $\beta = 1/4$ while it converges to a constant for $\beta = 3/4$ and $\beta = 1$.
- The quantum corrected Hubble parameter $H(t)$:

$$H = H_0 a^{-\alpha} \left[1 + \beta_1 H_0^{4\beta-2} \left(a^{-\alpha(4\beta-2)} - 1\right)\right]^{-1/(4\beta-2)} . \quad (45)$$

- In the limit of infinitely large time, the scale factor $a \to \infty$, we observe two distinct behaviours of $H(t)$ depending on the value of β:

$$H \approx \begin{cases} \tilde{H}_0 a^{-\alpha}, & \beta > 1/2, \\ \tilde{\beta}_1, & \beta < 1/2, \end{cases} \quad \Rightarrow \quad a \approx \begin{cases} (\alpha \tilde{H}_0)^{1/\alpha} t^{1/\alpha}, & \beta > 1/2, \\ \mathrm{e}^{\tilde{\beta}_1 t}, & \beta < 1/2, \end{cases} . \quad (46)$$

- $\beta = 1/2$ is a critical value which changes the scale factor behaviour from a power law ($\beta > 1/2$) to exponential ($\beta < 1/2$)

References

- **R. Mandal**, S. Gangopadhyay and A. Lahiri, *"Cosmology of Bianchi type-I metric using renormalization group approach for quantum gravity"*, Class. Quant. Grav. 37 (2020) 6, 065012.
- **R. Mandal**, S. Gangopadhyay and A. Lahiri, *"Cosmology with modified continuity equation in asymptotically safe gravity"*, Eur. Phys. J. Plus 137 (2022) 10, 1110.
- S. Gangopadhyay, **R. Mandal** and A. Lahiri, *"Bianchi-I Cosmology in Quantum Gravity"*, Springer Proc. Phys. 277 (2022) 925-929, Contribution to: 24th DAE-BRNS High Energy Physics Symposium, 925-929.
- C.-M. Chen, **R. Mandal** and N. Ohta, Work in Progress