ANPhA
Asian Nuclear Physics Association

The 2025 Asian Nuclear Physics Association (ANPhA) board meeting
& the ANPhA symposium on nuclear physics facilities in Asia
November 27-29, 2025, Institute of Physics, Academia Sinica

# Development of Deep Learning Techniques for Hadron Spectroscopy

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# Some updates from the Philippines

**Hashimoto Prize &ANPhA 1st (gold)**

**Darwin Alexander Co (Univ. of Philippines Diliman)**

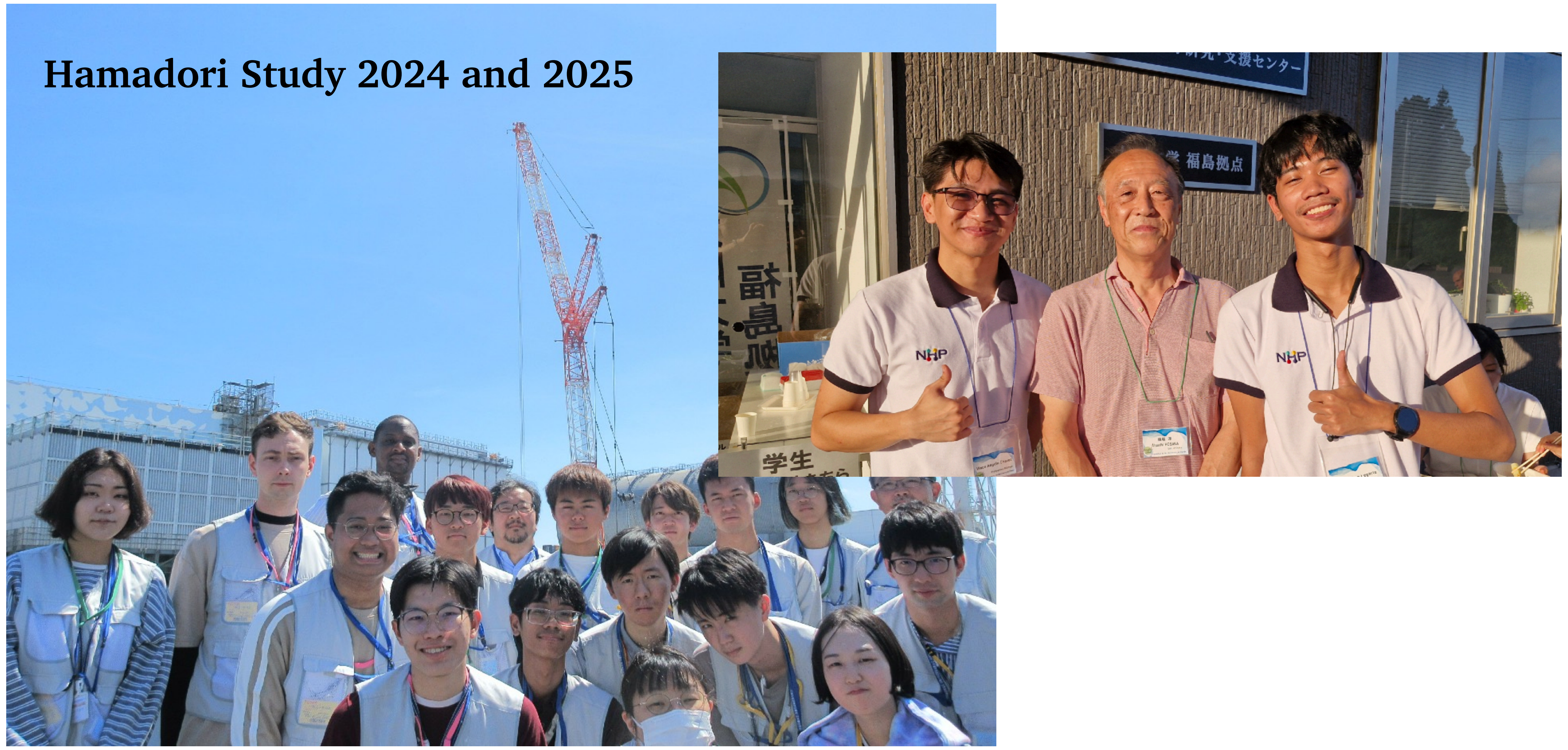
Understanding triangle singularity using deep neural network

SNP School 2023

**SNP CNS Summer School Incentive Prize**

**Vince Angelo Chavez** "Analysis of f0(980) using Deep Neural Network and Uniformized S-Matrix"

SNP CNS School 2024

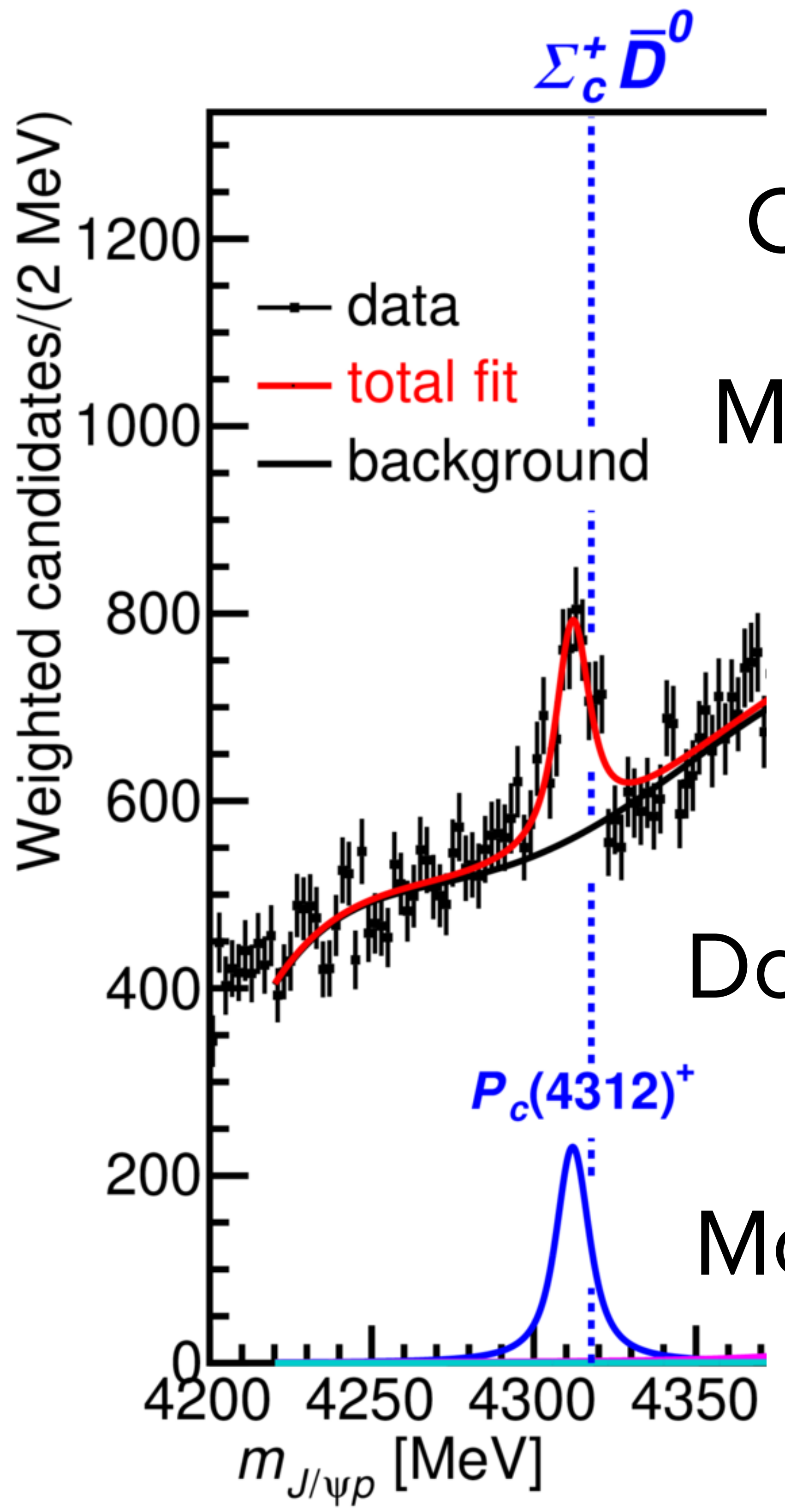



Hamadori Study 2024 and 2025

# Hadron spectroscopy - line shape interpretation

- Many observations of possible new states.
- Some are near hadron-hadron thresholds.

- Experiments give us the line shape.
- How do we interpret/ classify them?




[LHCb, PRL 122 222001 (2019)](#)

Compact: [A Ali, AY Parkhomenko PLB 793 (2019)](#)

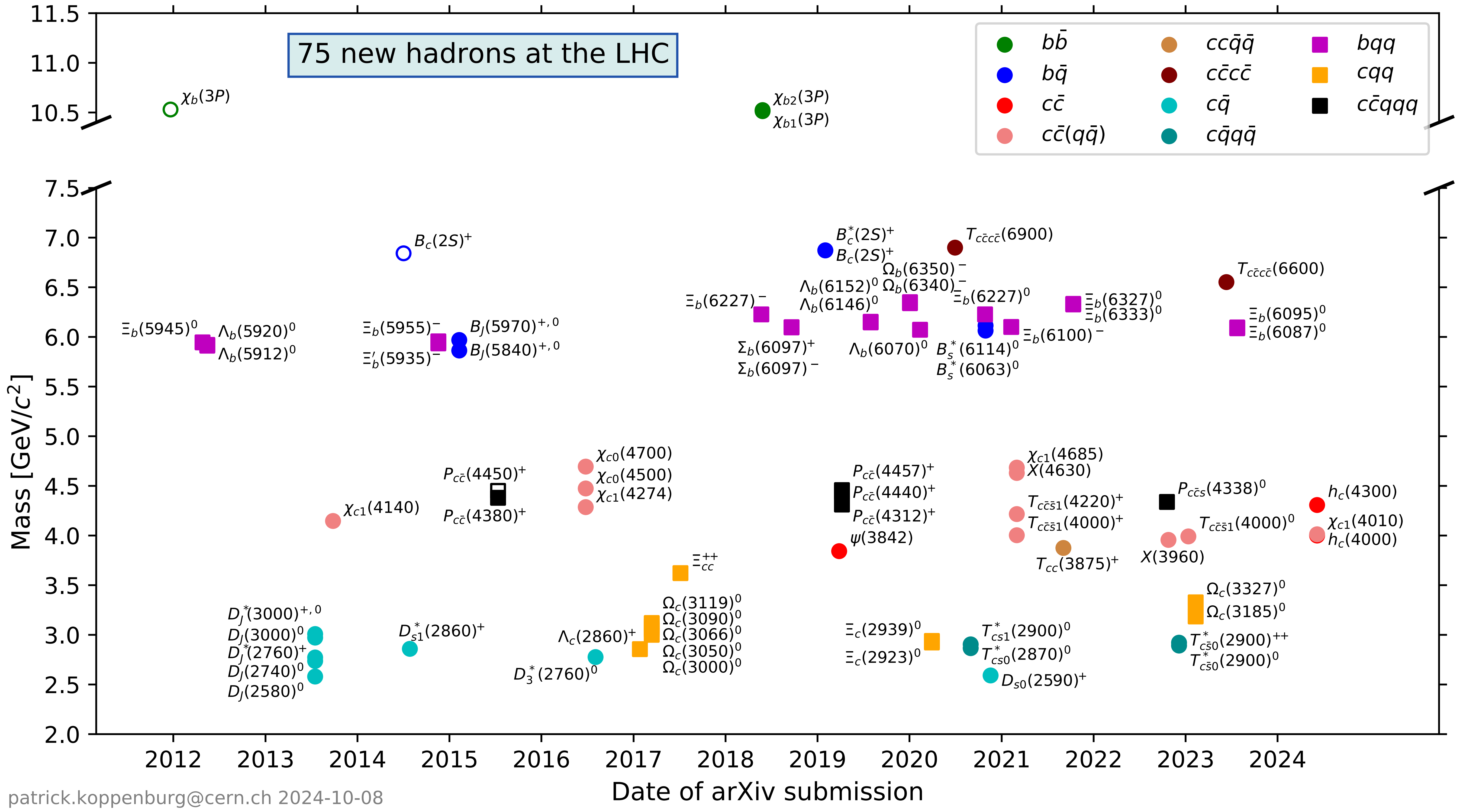
Molecular: [ML Du, V Baru, FK Guo, C Hanhart, UG Meißner, J A Oller, Q Wang PRL 124 072001 (2020)](#)

Virtual: [JPAC, PRL 123 092001 (2019)](#)

Double TS: [SX Nakamura, PRD 103 L111503 (2021)](#)

Molecule coupled to compact:

[Y Yamaguchi, H García-Tecocoatzi, A Giachino, A Hosaka, E Santopinto, S Takeuchi, M Takizawa, PRD 101 091502(R) (2020)](#)



# Hadron spectroscopy - line shape interpretation

Why use machine learning in line shape analysis?

Line shape interpretation - classification problem

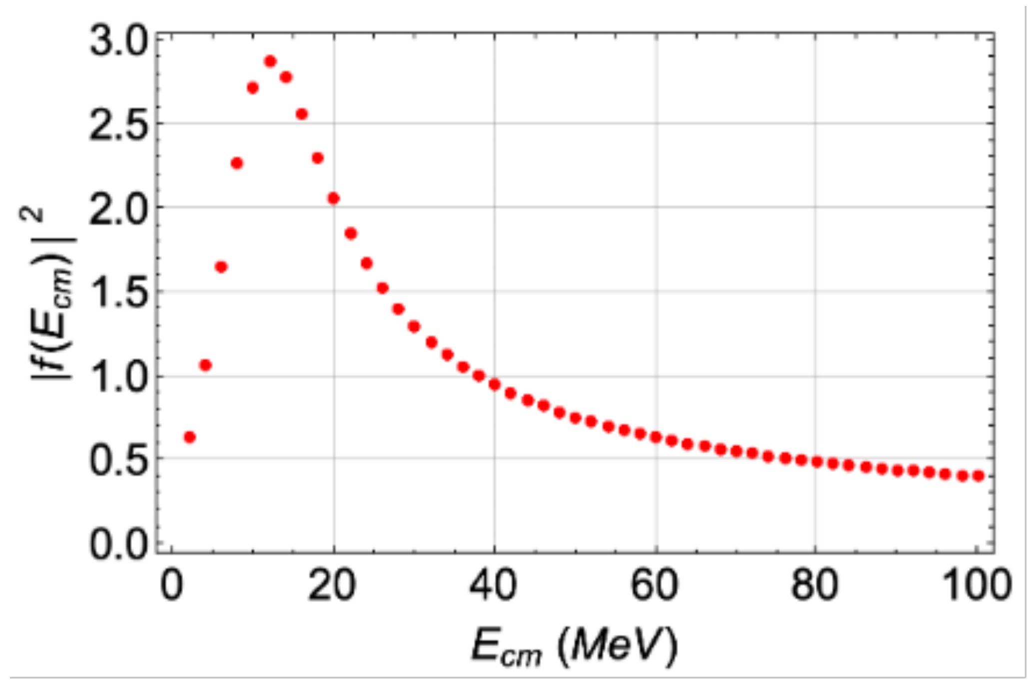
Deep neural network (DNN) as a universal approximator (universal map).

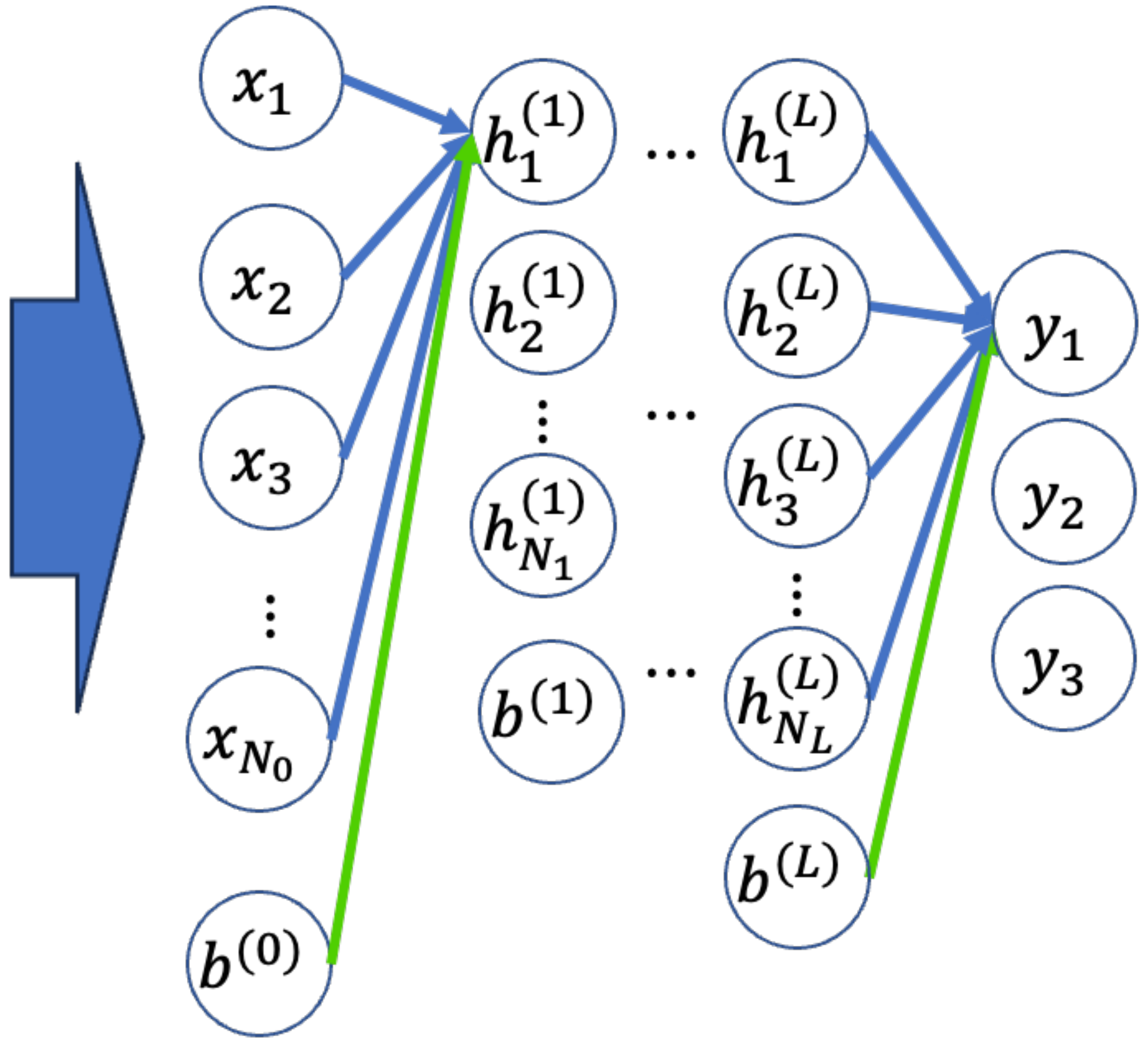
K Hornik, M Stinchcomber, H White, Neural Net., 2 5 (1989)

Linear transformation + non-linear activations and squashing

$$z_j^{(0)} = W_{ij}^{(0)} x_i + b_j^{(0)}$$
$$h_j^{(1)} = \mathrm{ReLU}(z_j^{(1)})$$
$$\vdots$$
$$z_j^{(n)} = W_{ij}^{(n)} h_i^{(n)} + b_j^{(n)}$$
$$h_j^{(n+1)} = \mathrm{ReLU}(z_j^{(n)})$$
$$\vdots$$
$$z_j^{(L)} = W_{ij}^{(L)} h_i^{(L)} + b_j^{(L)}$$
$$y_j = \frac{\exp(z_j^{(L)})}{\sum_i \exp(z_i^{(L)})}$$

DNN can be trained to map input line shape space into output interpretation space.




interpretation 1

interpretation 2

interpretation 3

Backpropagation to minimize the cost:

$$C\left(\hat{W}, \vec{b}\right) = \frac{1}{X} \sum_{\vec{x}}^{X} \vec{a}\left(\vec{x}\right) \cdot \log\left[\vec{y}_{W,b}\left(\vec{x}\right)\right]$$

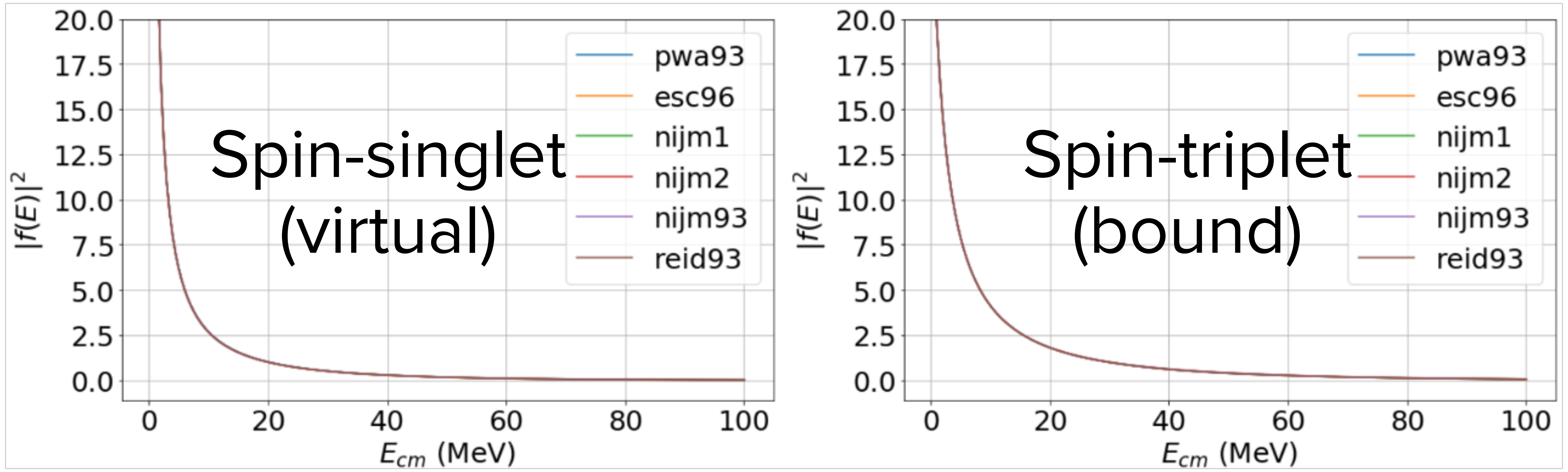
DE Rumelhart, GE Hinton, RJ Williams, Nature 323, 533-536 (1986)

# Deep learning: proof of principle

DLBS, YI, TS, AH PRD 102 016024 (2020)
DLBS, YI, TS, AH Few-Body Syst. 62, 52 (2021)

**Benchmarked on the known nucleon-nucleon bound state**

Given only the s-wave cross section, the origin of enhancement can be unambiguously identified.




For near-threshold pole:

$k\cot\delta \sim -1/a$ (constant)

$$|f(k)|^{-2} = |k\cot\delta - ik|^2 \sim \frac{1}{a^2} + k^2$$

Not possible to distinguish bound vs virtual pole enhancements.

S-matrix can have distant singularities on the unphysical sheet.

Use different (unitary, analytic) backgrounds to help DNN distinguish bound and virtual enhancements.

$$S(k) = \exp\left[2i\delta_{bg}(k)\right]\frac{k+i\gamma}{k-i\gamma}; \quad \delta_{bg} = \alpha\tan^{-1}\left(\frac{k}{\beta}\right)$$

# Deep learning: proof of principle

DLBS, YI, TS, AH PRD 102 016024 (2020)
DLBS, YI, TS, AH Few-Body Syst. 62, 52 (2021)

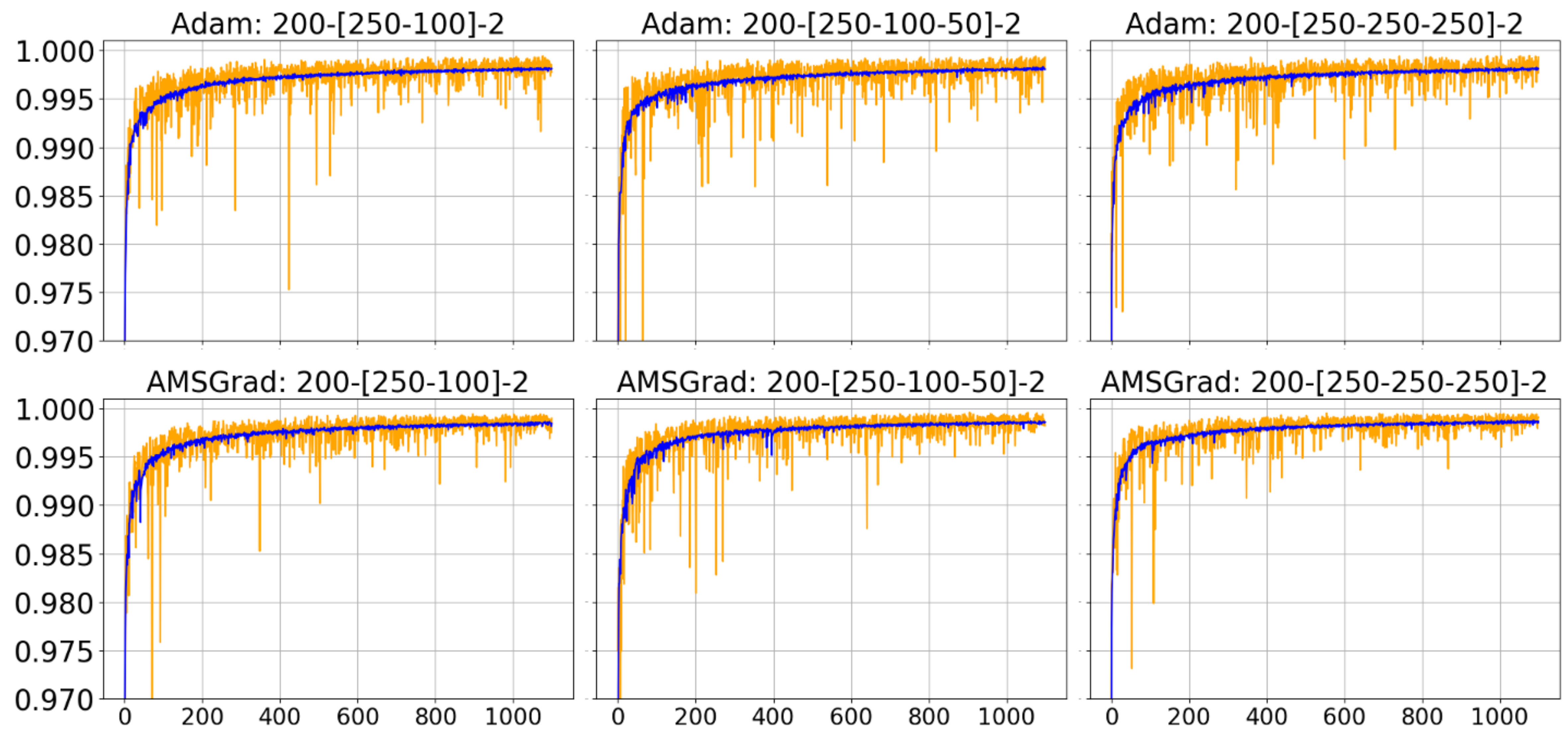
Optimize parameters of DNN using mock amplitudes:

$$S(k) = \exp\left[2i\delta_{bg}(k)\right]\frac{k+i\gamma}{k-i\gamma}$$

3,200,000 training data

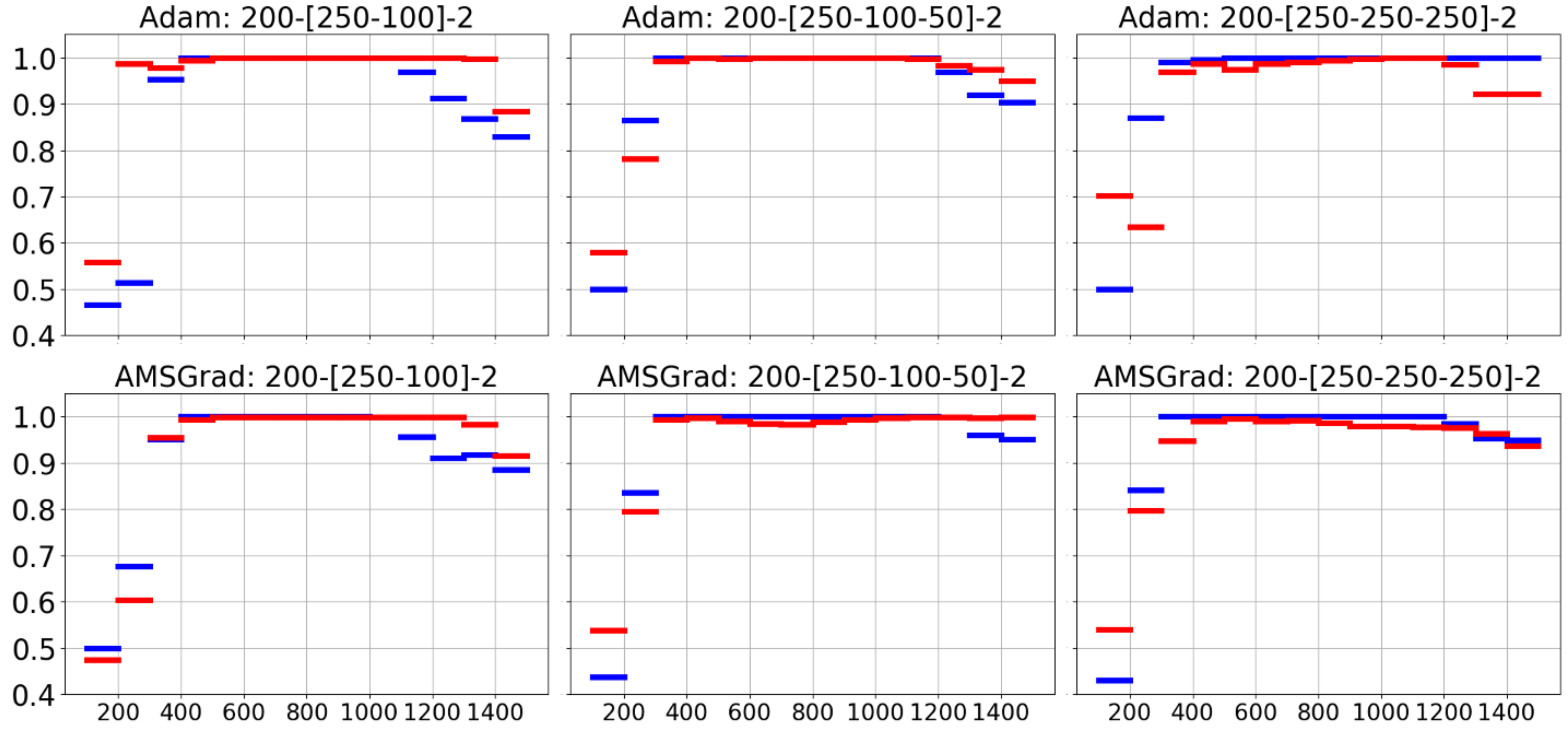
800,000 testing data

Accuracy vs training epoch

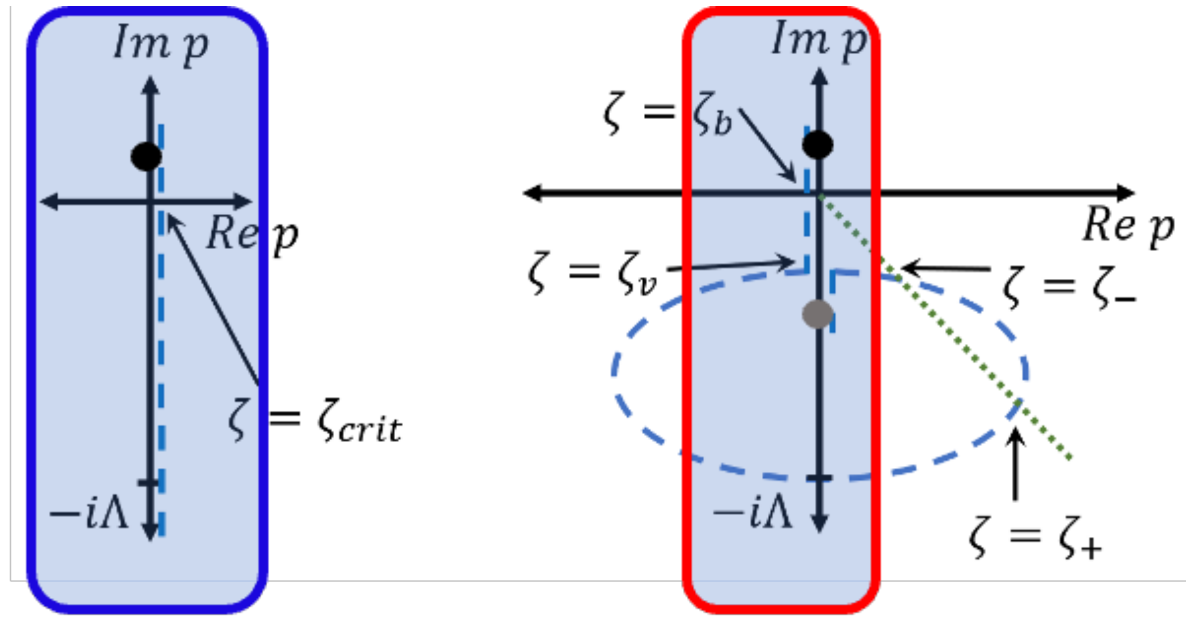



Validation using separable potential model

$$v(p,p') = \zeta\left(\frac{\Lambda^2}{p^2+\Lambda^2}\right)\left(\frac{\Lambda^2}{p'^2+\Lambda^2}\right)$$




cut-off parameter $\Lambda$




Energy independent coupling
$\zeta \rightarrow$ constant

Energy independent coupling
$\zeta \rightarrow (E - M_{sep})\zeta$

Trained and validated DNN deployed to probe the nucleon-nucleon

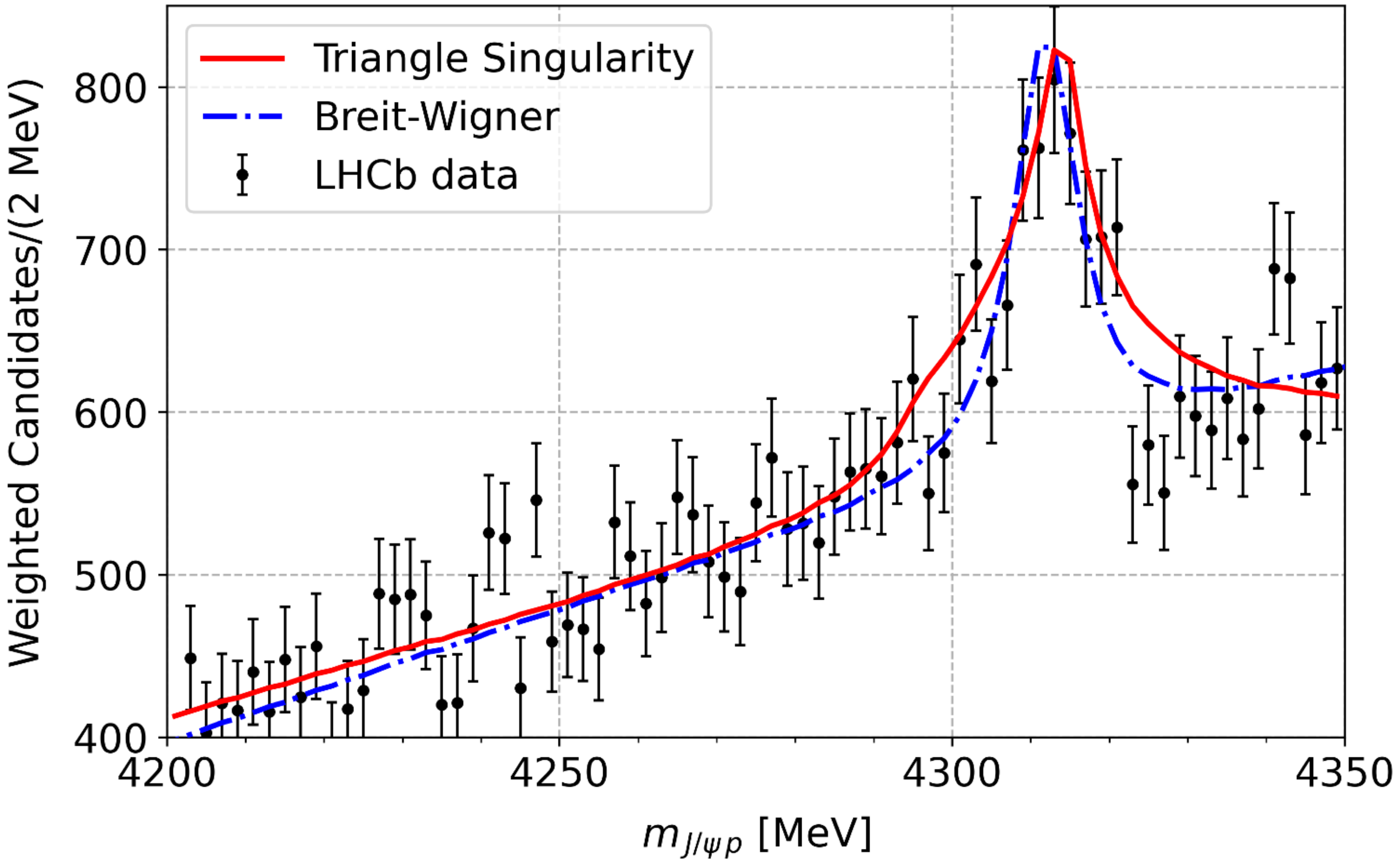
| | PWA93 | ECS96 | NijmI | NijmII | Nijm93 | Reid93 |
|---|---|---|---|---|---|---|
| ${}^1S_0$ | virtual | virtual | virtual | virtual | virtual | virtual |
| ${}^3S_1$ | bound | bound | bound | bound | bound | bound |

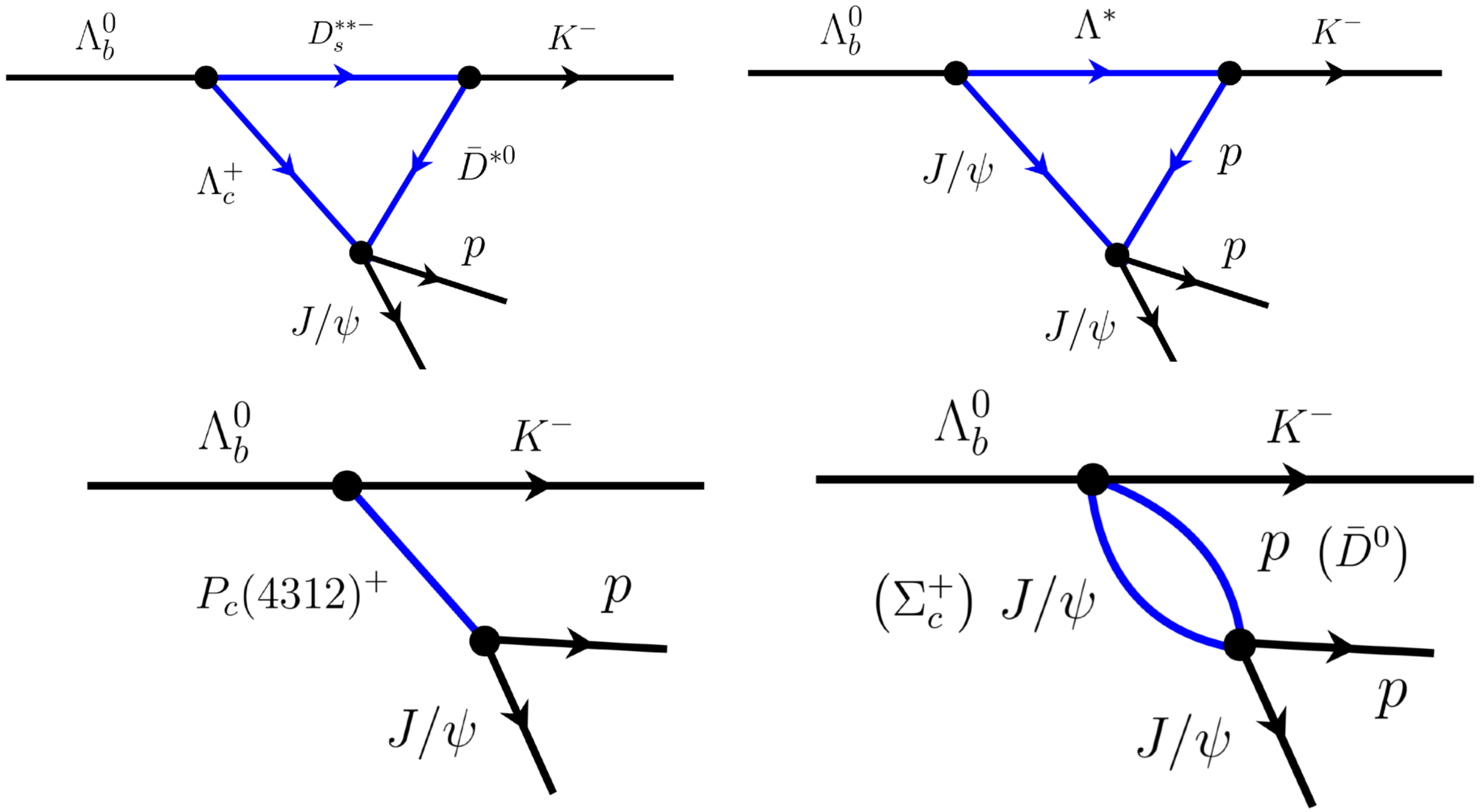
# DL as a Model-selection framework

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034

Can we train a DNN to discriminate different origin of enhancements?

- Triangle mechanism
- Coupled-channel effects
- True resonance




If yes, then DL can complement our analysis.

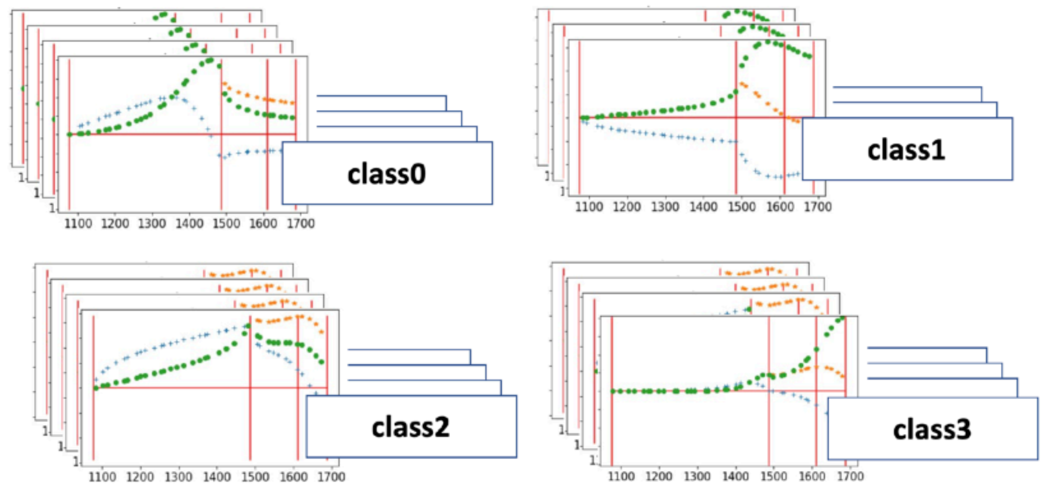
If no, then TS and pole enhancements are inherently ambiguous.

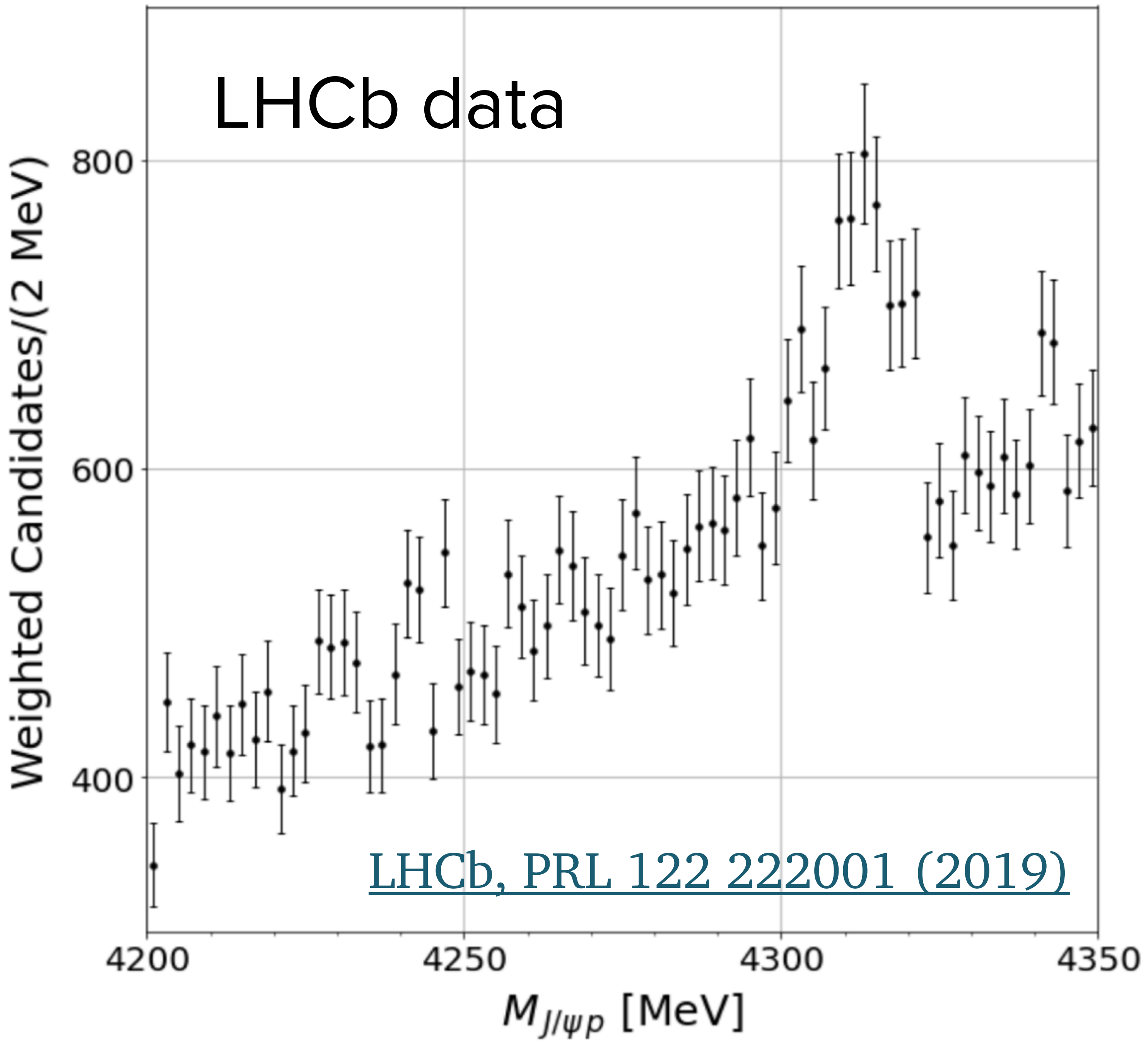
# DL as a Model-selection framework

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034

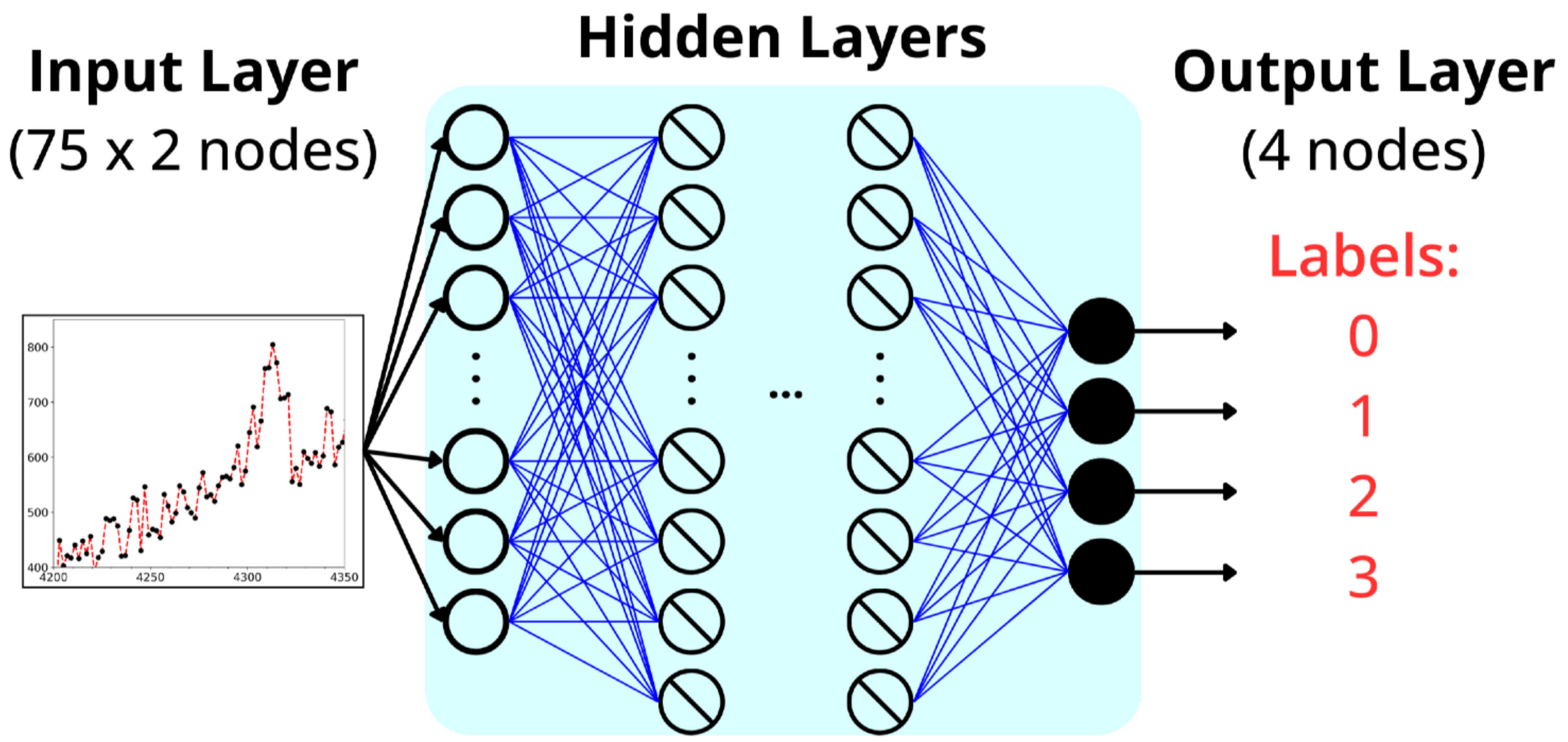
1. Choose the region of interest: [4200 MeV - 4350 MeV]

2. Generate the training dataset
- Triangle singularity
- Pole of S-matrix
  - 1 pole in 2nd RS
  - 1 pole in 4th RS
  - 1 pole each in 2nd and 3rd RS



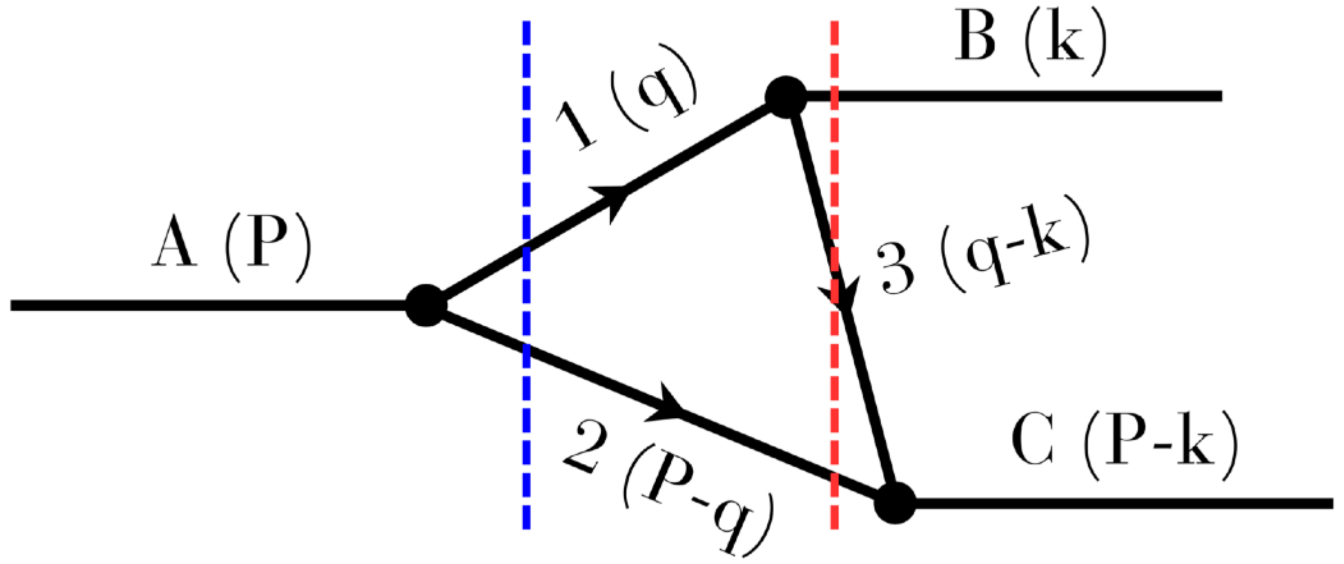

3. Design a set of DNN to solve the classification problem




4. Train, test, and validate the DNN
5. Use the trained DNN to interpret the experimental data

# TS vs Pole-based enhancements

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034




The amplitude has no singularity (no pole)

No need to introduce new hadronic state to explain the enhancement.

$$I(k) \propto \int_0^\infty \frac{q^2 f(q)\, dq}{P^0 - \sqrt{m_1^2 + q^2} - \sqrt{m_2^2 + q^2} + i\epsilon}$$

M Bayar, F Aceti, FK Guo, E Oset, PRD 126 074039 (2016)

FK Guo, XH Liu, S Sakai, PPNP 122 103757 (2020)

$$f(q) = \int_{-1}^{1} \frac{dz}{E_C - \omega(q) - \sqrt{m_3^2 + q^2 + k^2 - 2qkz} + i\epsilon}$$

$$m_1^2 \in \left[\frac{M_A^2 m_3 + M_B^2 m_2}{m_2 + m_3} - m_2 m_3, (M_A - m_2)^2\right] \qquad m_C^2 \in \left[(m_2 + m_3)^2, \frac{M_A m_3^2 - M_B^2 m_2}{M_A - m_2} + M_A m_2\right]$$

Mass condition - crucial in generating mock dataset

$$S_{11}(p_1, p_2) = \prod_m \frac{D_m(-p_1, p_2)}{D_m(p_1, p_2)}$$

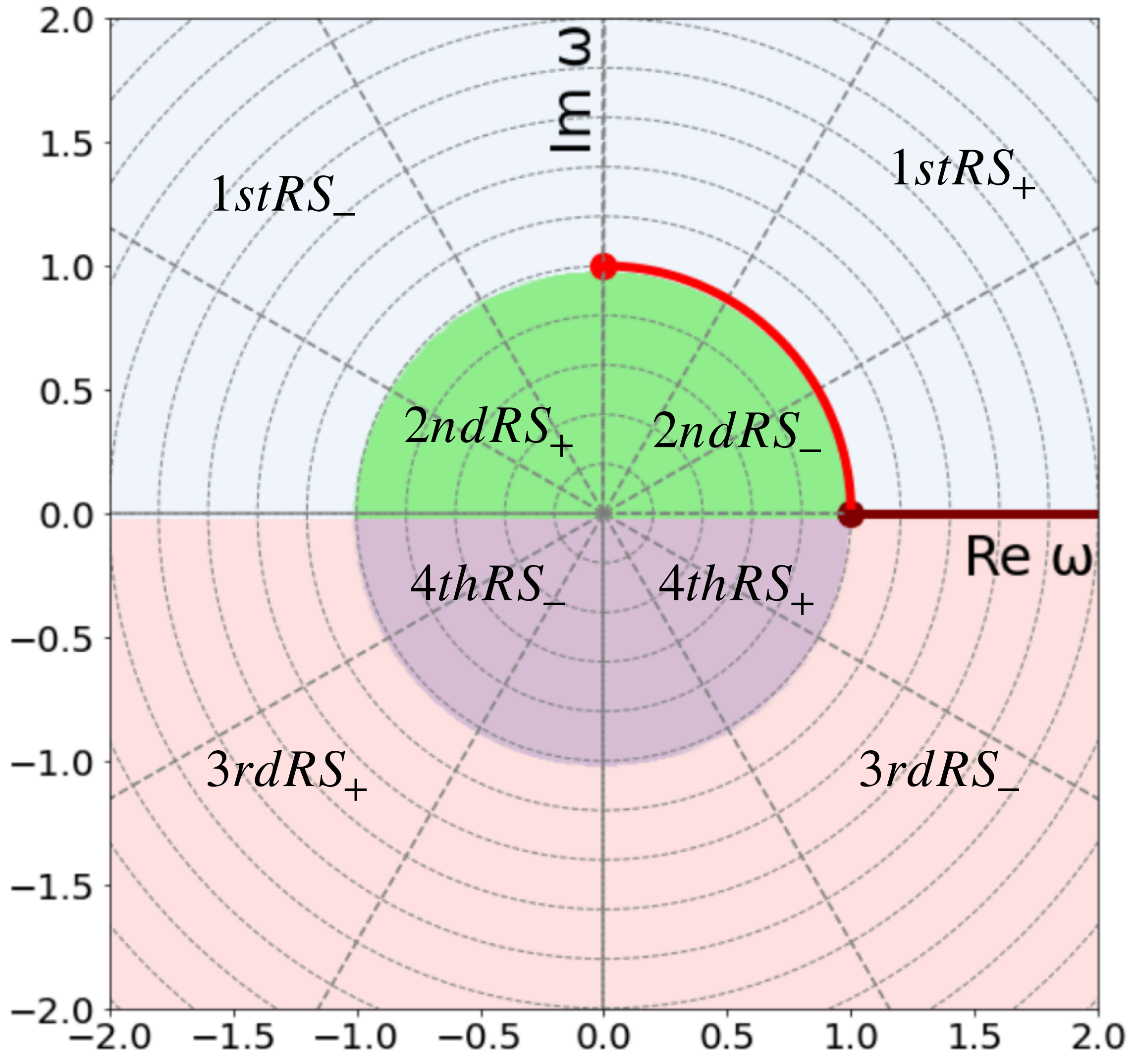
KJ Le Couteur, Proc. Roy. Soc (London) A256 (1960)
RG Newton J. Math. Phys. 2, 188 (1961)

$$p_k \to q_k; \quad s = q_k^2 + \epsilon_k^2; \quad \omega = \frac{q_1 + q_2}{\sqrt{\epsilon_2^2 - \epsilon_1^2}} \quad \frac{1}{\omega} = \frac{q_1 - q_2}{\sqrt{\epsilon_2^2 - \epsilon_1^2}}$$

M Kato, Ann. Phys., 31 1 (1965)

One pole per $D_m(q_1, q_2)$ : $\omega_m$

$$D_m(q_1, q_2) = D_m(\omega)$$
$$= \frac{1}{\omega^2} (\omega - \omega_m)(\omega + \omega_m^*)(\omega - \omega_{\bar{m}})\left(\omega + \omega_{\bar{m}}^*\right)$$

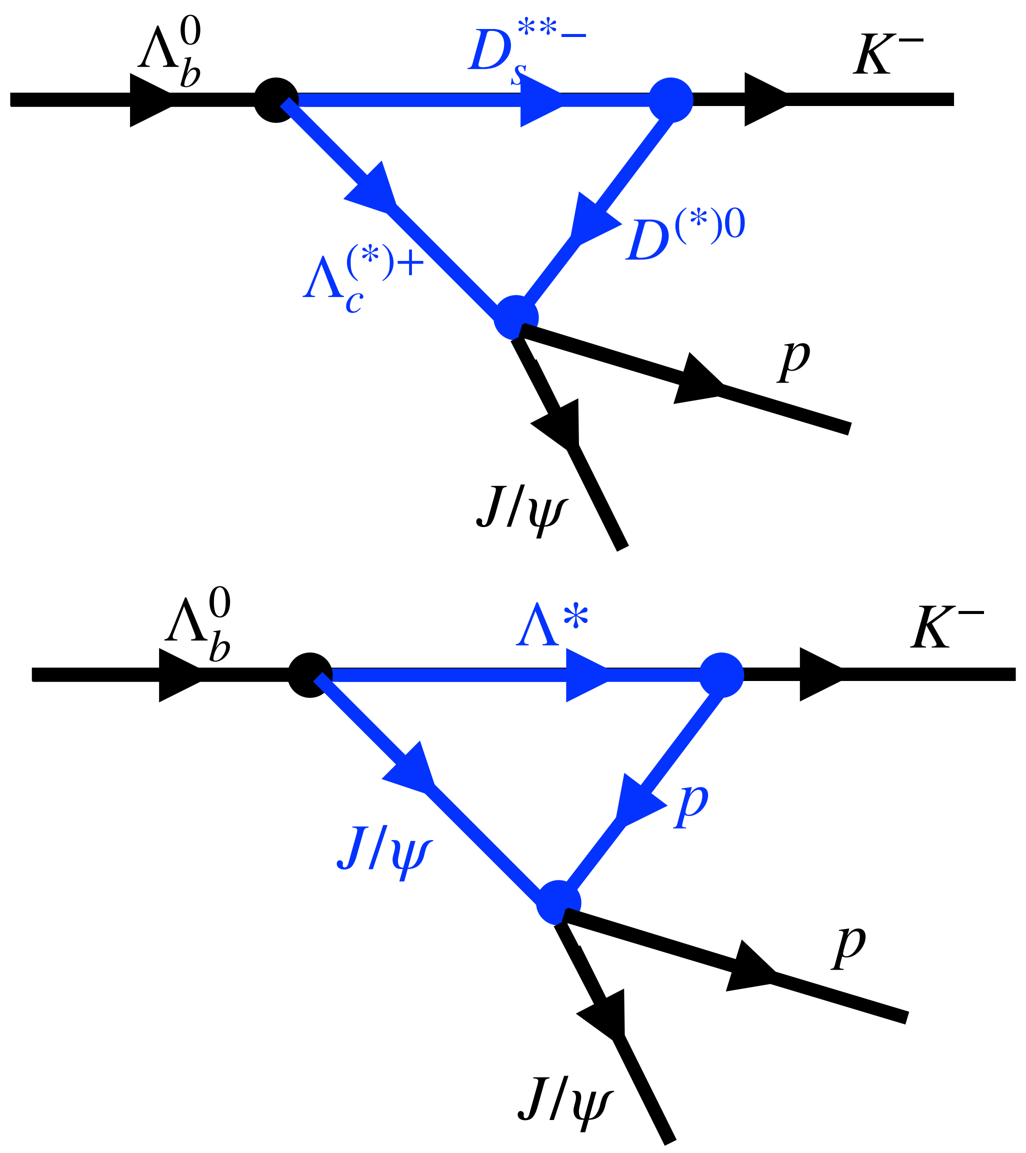


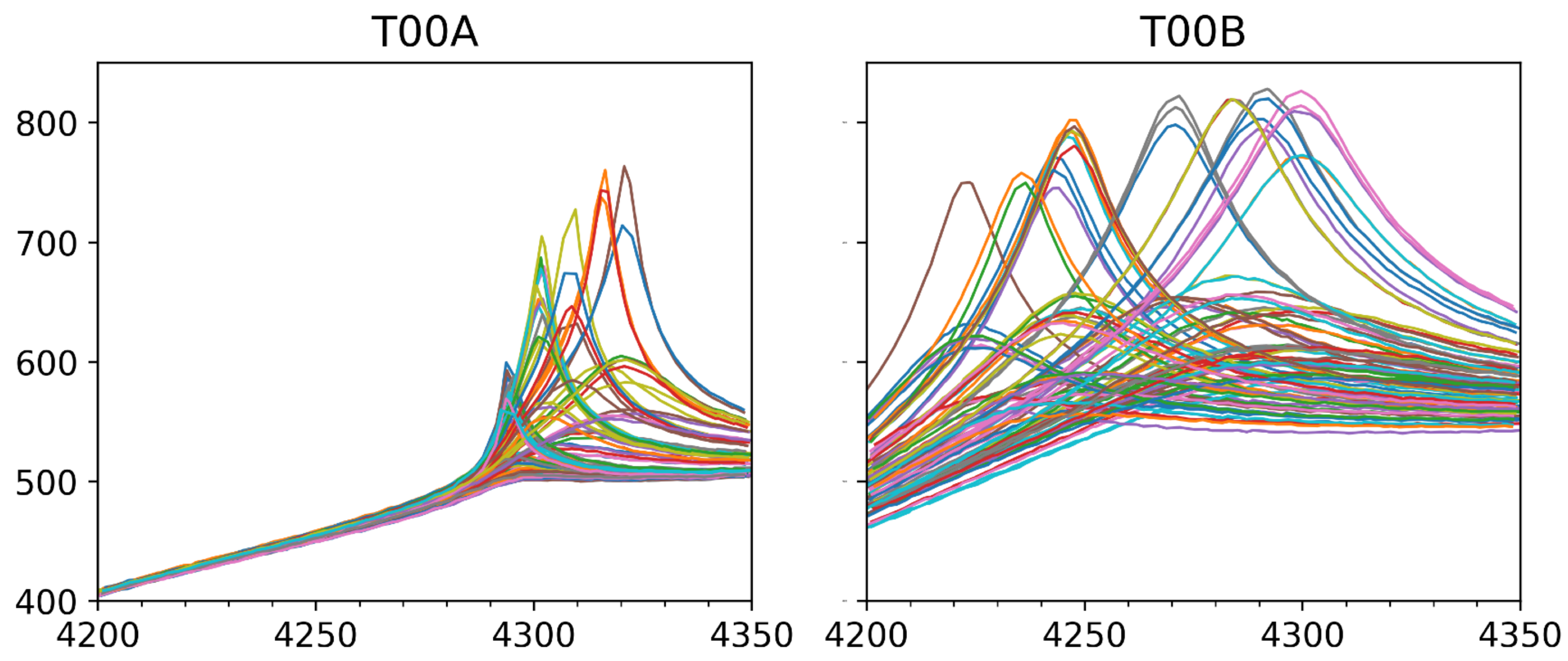
# Sample training datasets

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034

## Triangle singularity

| Parameter | Range of values [MeV] |
|---|---|
| $m_{\Lambda_b^0}$ | $5619.60 \pm 0.17$ |
| $m_{K^-}$ | $493.677 \pm 0.016$ |
| $m_{\Lambda_c^+}$ | $2286.46 \pm 0.14$ |
| $m_{\bar{D}^{*0}}$ | $2006.85 \pm 0.05$ |
| $m_{J/\psi}$ | $3096.9 \pm 0.006$ |
| $m_{D_s^{**}}$ | $[3209.80,\ 3315.00]$ |
| $m_{\Lambda^*}$ | $[2490.00,\ 2522.70]$ |
| $\Lambda$ | $[2000.0,\ 2500.0]$ |
| $\epsilon$ | $[1.0,\ 10.0]$ |




## Pole-based enhancements

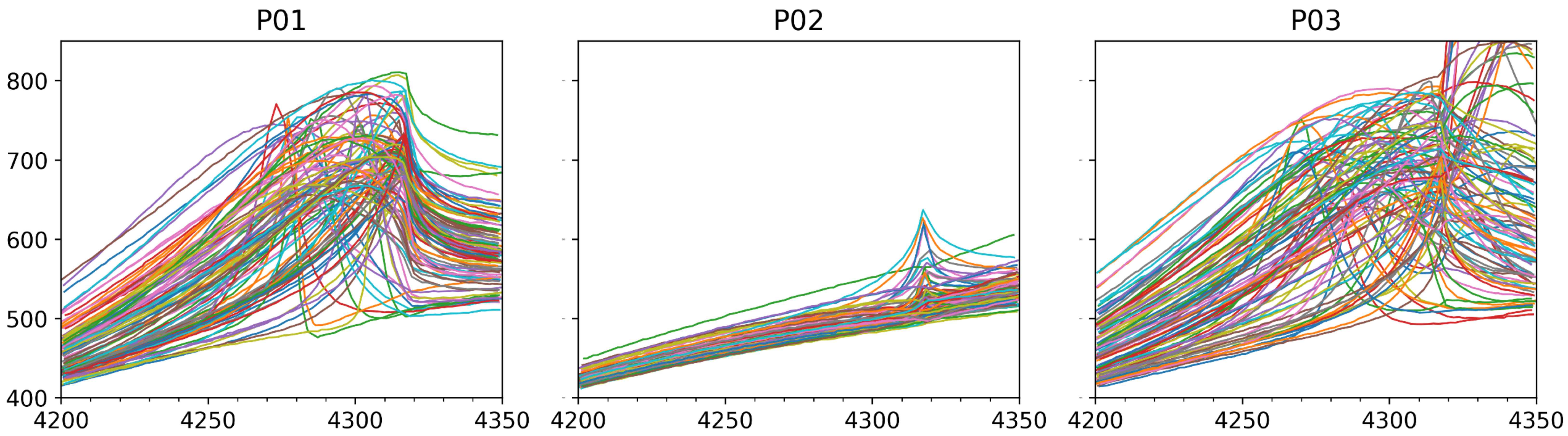
$$\begin{cases} T_2 - 50 \leq \mathrm{Re}\, E_{\mathrm{pole}} \leq 4350 & \text{all RS} \\ -100 \leq \mathrm{Im}\, E_{\mathrm{pole}} < 0 & [bt]\ \&\ [bb] \\ 0 < \mathrm{Im}\, E_{\mathrm{pole}} \leq 100 & [tb] \end{cases}$$

$4 \times 10{,}000$

Training dataset generated

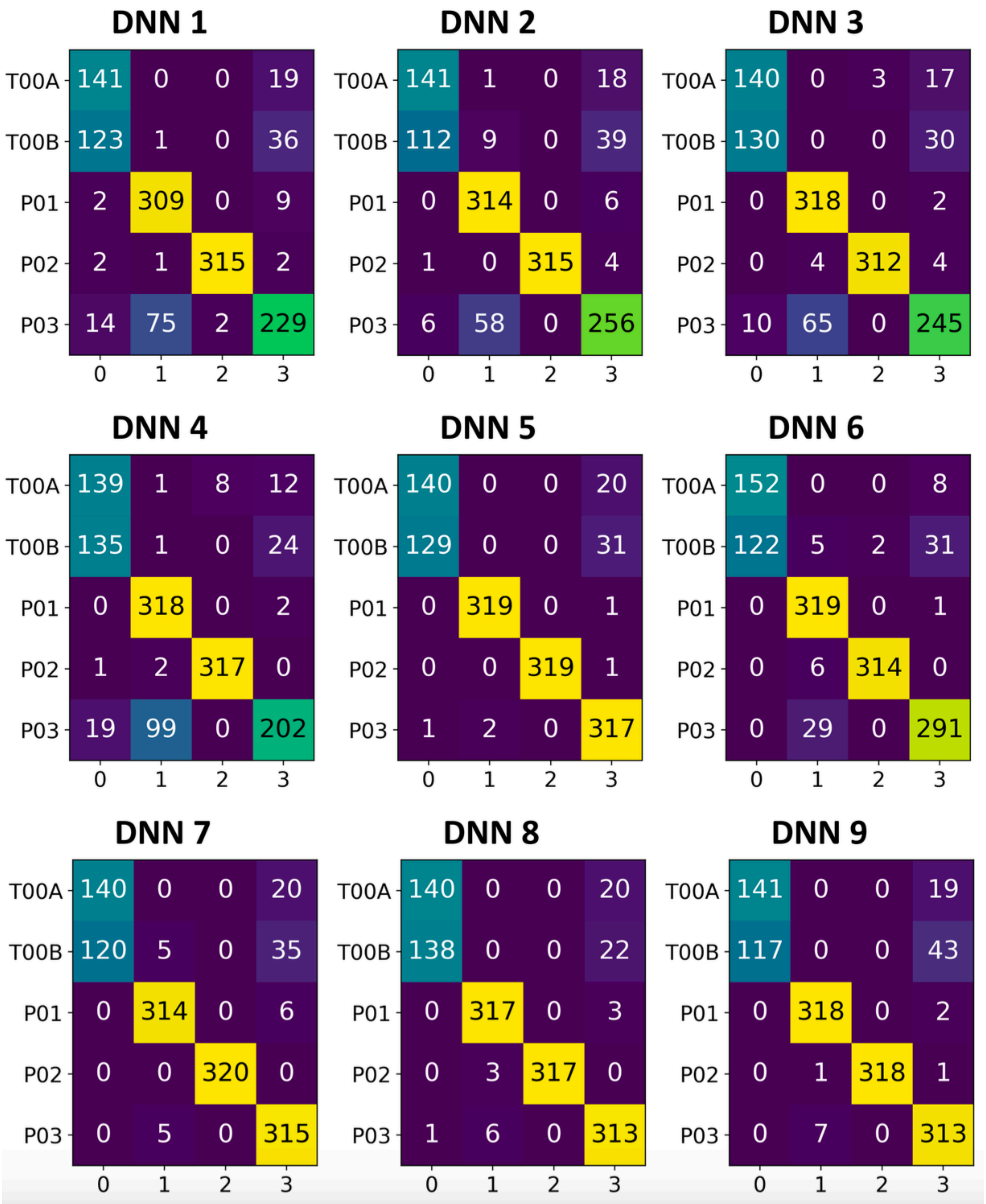
$4 \times 320$

Testing dataset generated

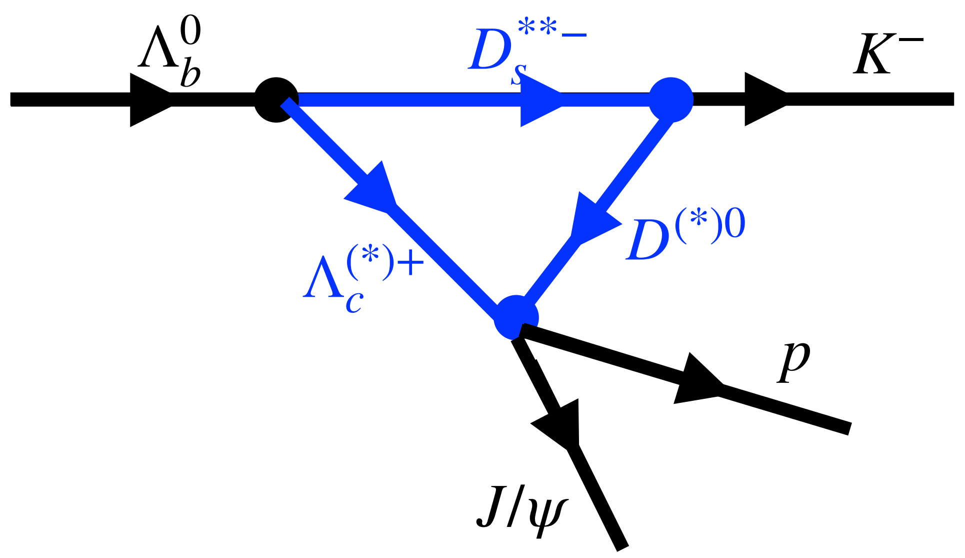


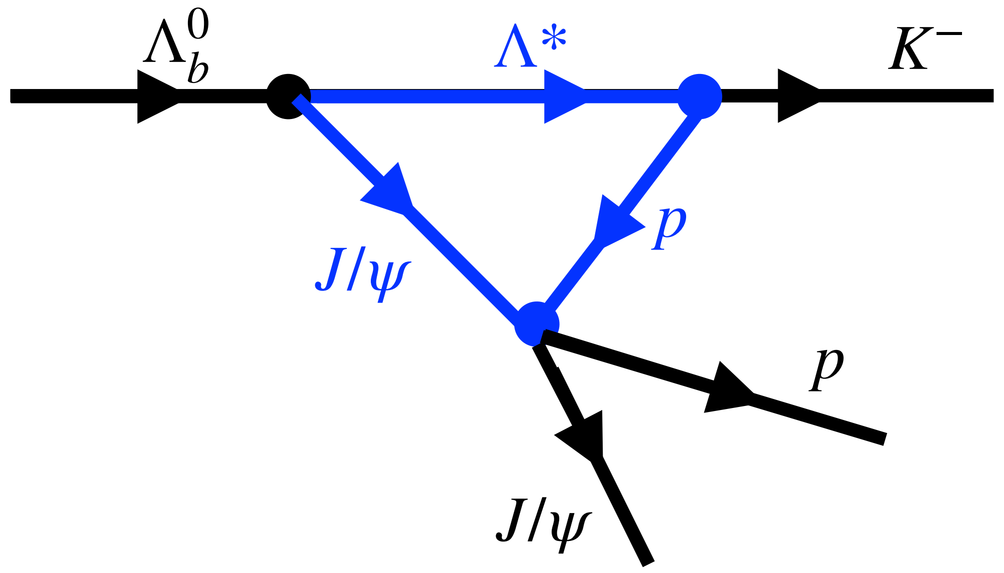
# Confusion Matrix

DAO Co, VAA Chavez, DLBS,
PRD 110 (2024) 114034




Generate a different set of 320 validation dataset per classification.



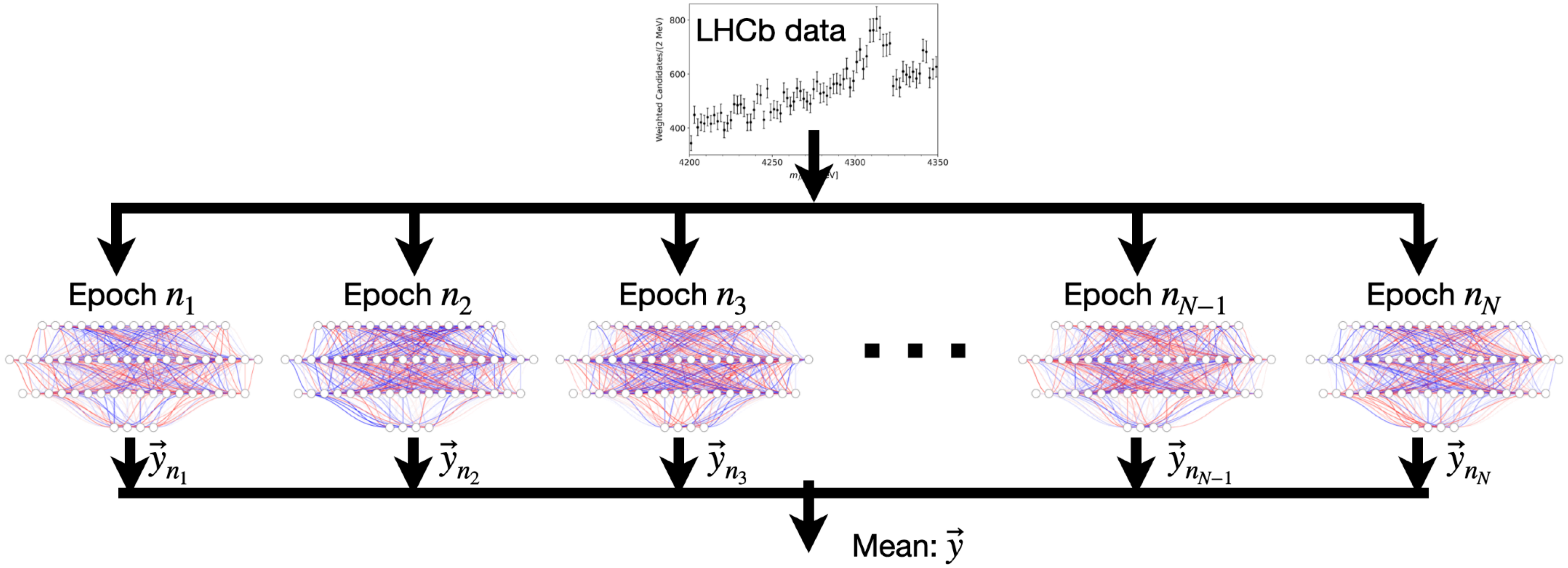

| Label | Description |
|---|---|
| 0 | Triangle mechanism |
| 1 | 1 pole in 2nd RS |
| 2 | 1 pole in 4th RS |
| 3 | 1 pole in 2nd and 1 pole in 3rd RSs |

Slight confusion between TS and BW-like line shapes.

If the experimental data will favor either the TS or class 3, further analysis must be done.

# Inference stage (snapshot ensemble)

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034




- Bootstrapped 3000 line shapes from the experimental data. (Uniform distribution)
- Feed to the DNN state at epoch n
- Get the mean count

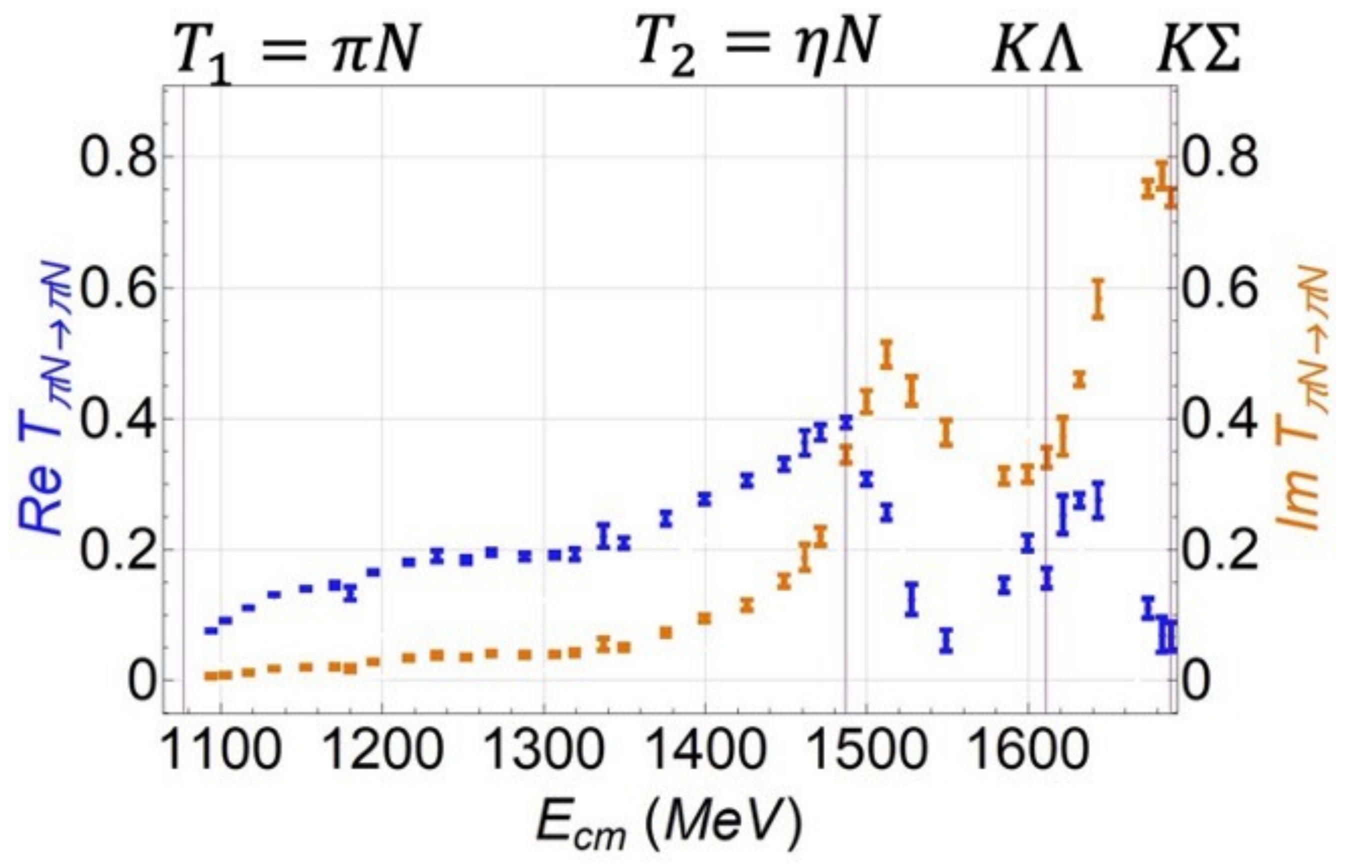
| Model | TS | 1 pole in RS2 | 1 pole in RS4 | 1 pole each in RS2 & RS4 |
|---|---|---|---|---|
| DNN 1 | 261 | 1930 | 0.248 | 804 |
| DNN 2 | 18.4 | 1998 | 969 | 14.5 |
| DNN 3 | 0.653 | 2960 | 0 | 36 |
| DNN 4 | 0 | 2990 | 0 | 14.2 |
| DNN 5 | 0 | 2660 | 0 | 337 |
| DNN 6 | 0.436 | 2999 | 0 | 0.891 |
| DNN 7 | 1 | 2590 | 0 | 414 |
| DNN 8 | 0 | 2999 | 0 | 1.06 |
| DNN 9 | 0 | 2970 | 0 | 127 |

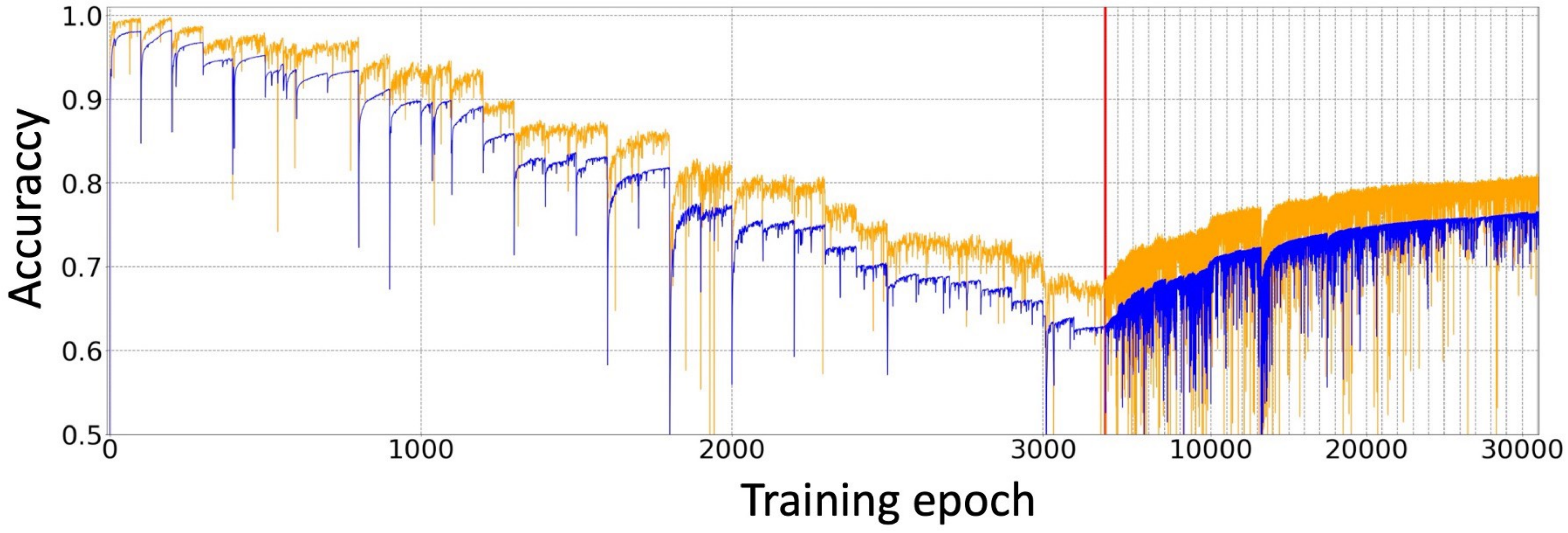
- TS is ruled out by pure line shape analysis!
- It is possible to distinguish kinematical cusp vs pole despite the presence of experimental uncertainty.

The data favored the pole-based interpretation: 1 pole in 2nd RS

# Probe deeper: Pole structure of $P_{c\bar{c}}(4312)^+$

DNN designed to probe the pole structure is difficult to train.




| Label | S-matrix pole configuration |
|---|---|
| 0 | no nearby pole |
| 1 | 1 pole in $[bt]$ |
| 2 | 2 poles in $[bt]$ |
| ⋮ | ⋮ ⋮ ⋮ ⋮ |
| 32 | 1 pole in $[bt]$, 2 poles in $[bb]$ and 1 pole in $[tb]$ |
| 33 | 1 pole in $[bt]$, 1 pole in $[bb]$ and 2 poles in $[tb]$ |
| 34 | 1 pole in $[bt]$, 1 pole in $[bb]$ and 1 pole in $[tb]$ |

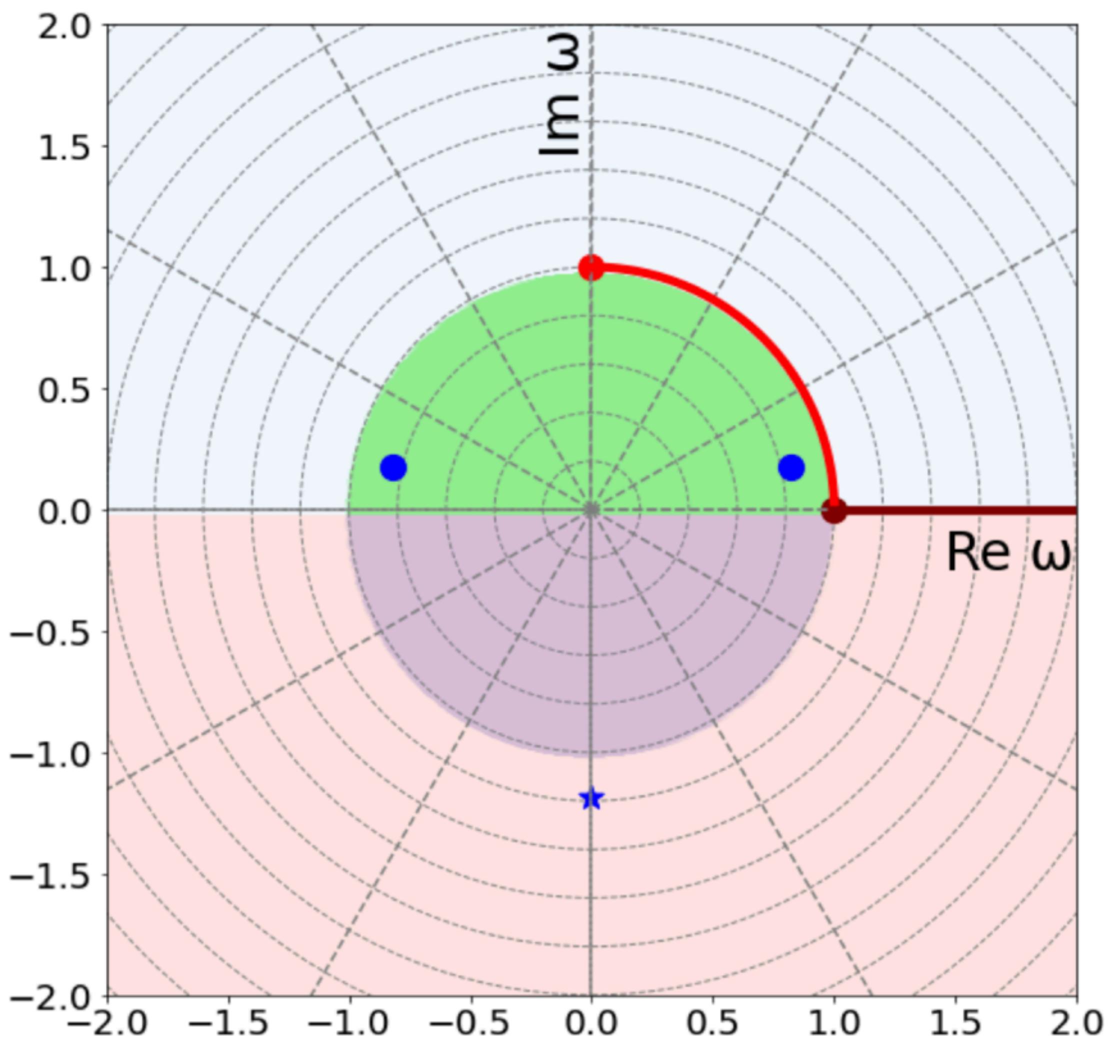
DLBS, YI, TS, AH PRD 104 3 036001 (2021)

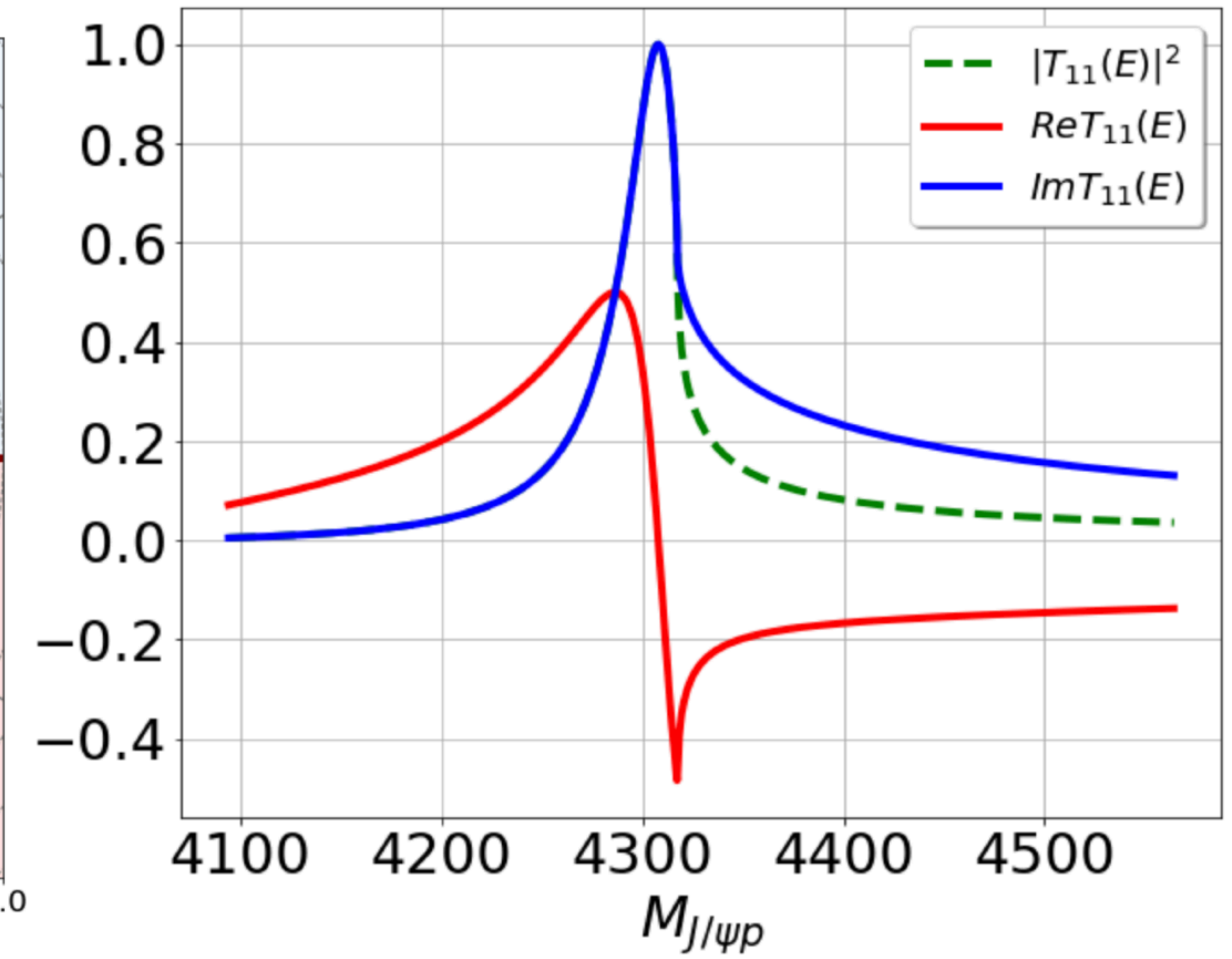
Line shape - pole structure map, intrinsically ambiguous.

Complications introduced by pole-shadow pair.

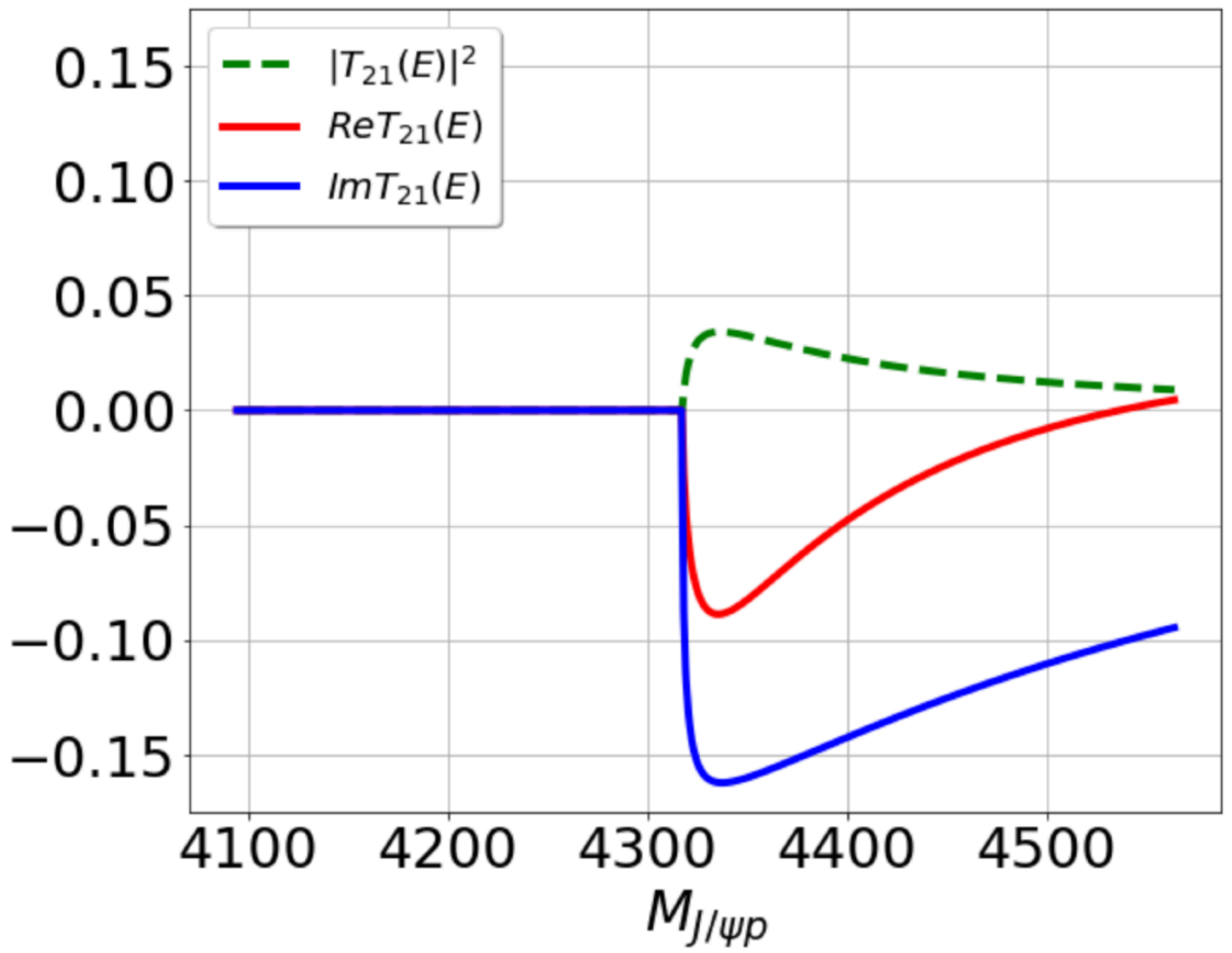
# Probe deeper: Pole structure of $P_{c\bar{c}}(4312)^+$

1 pole in the 2nd RS

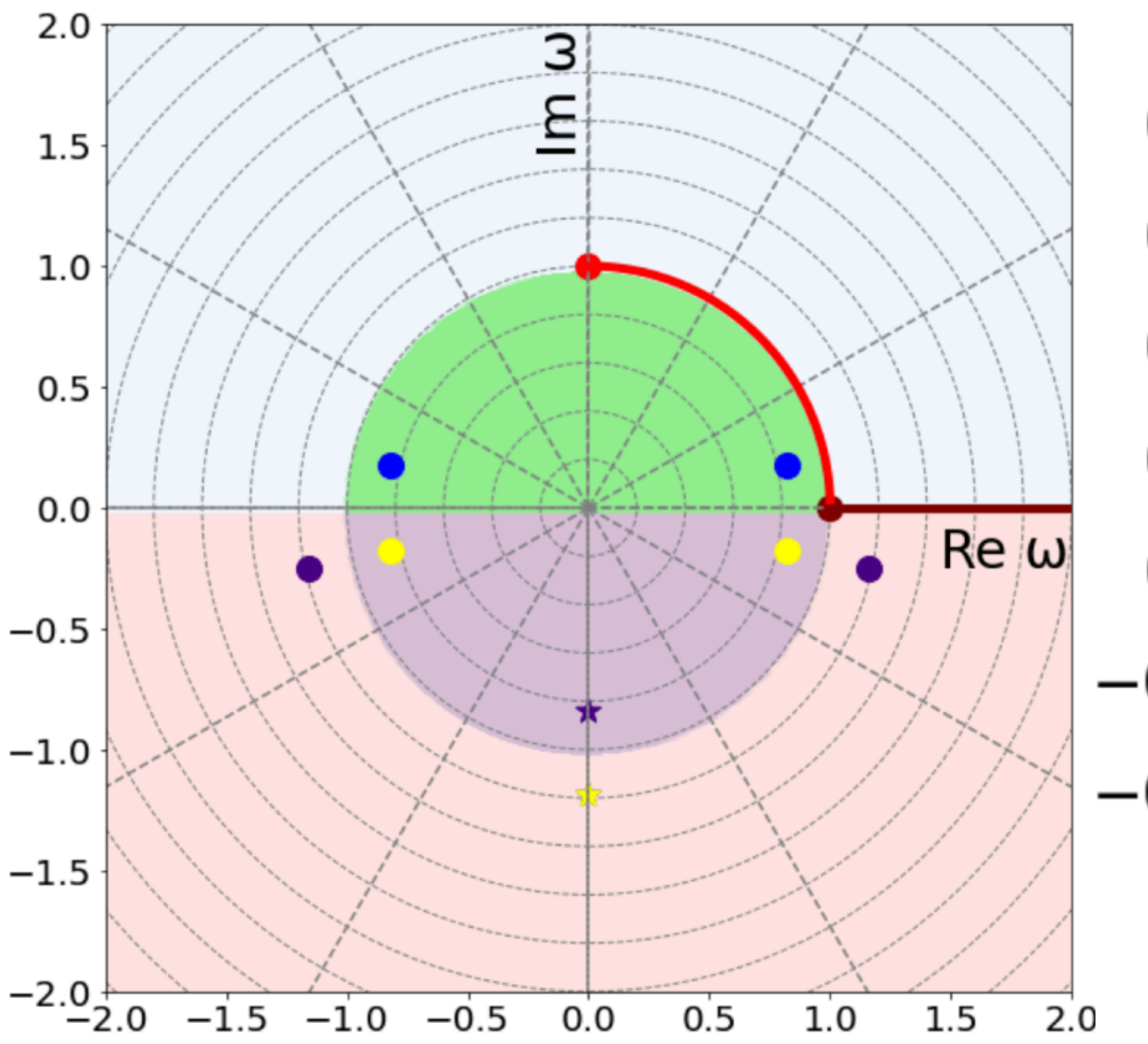


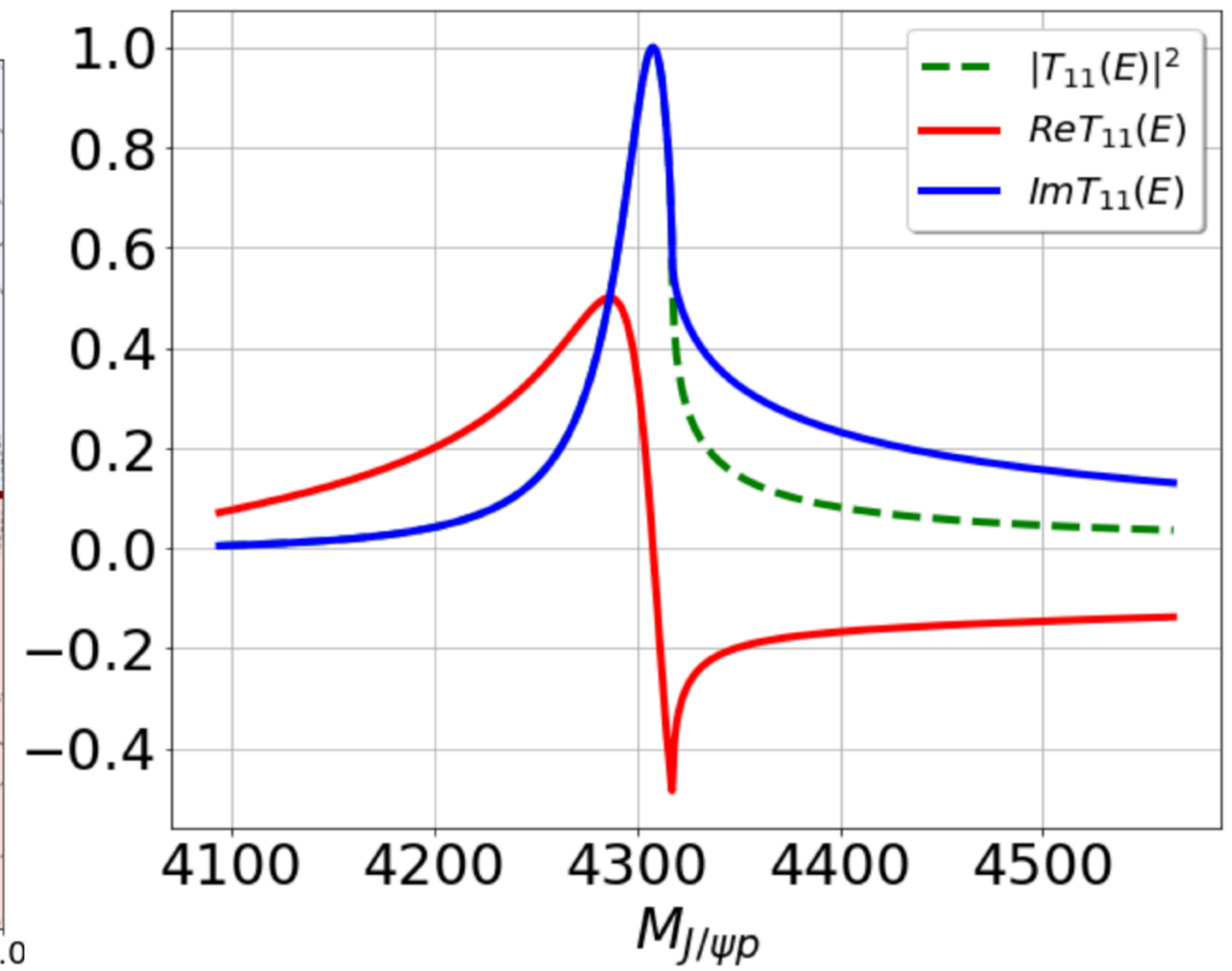



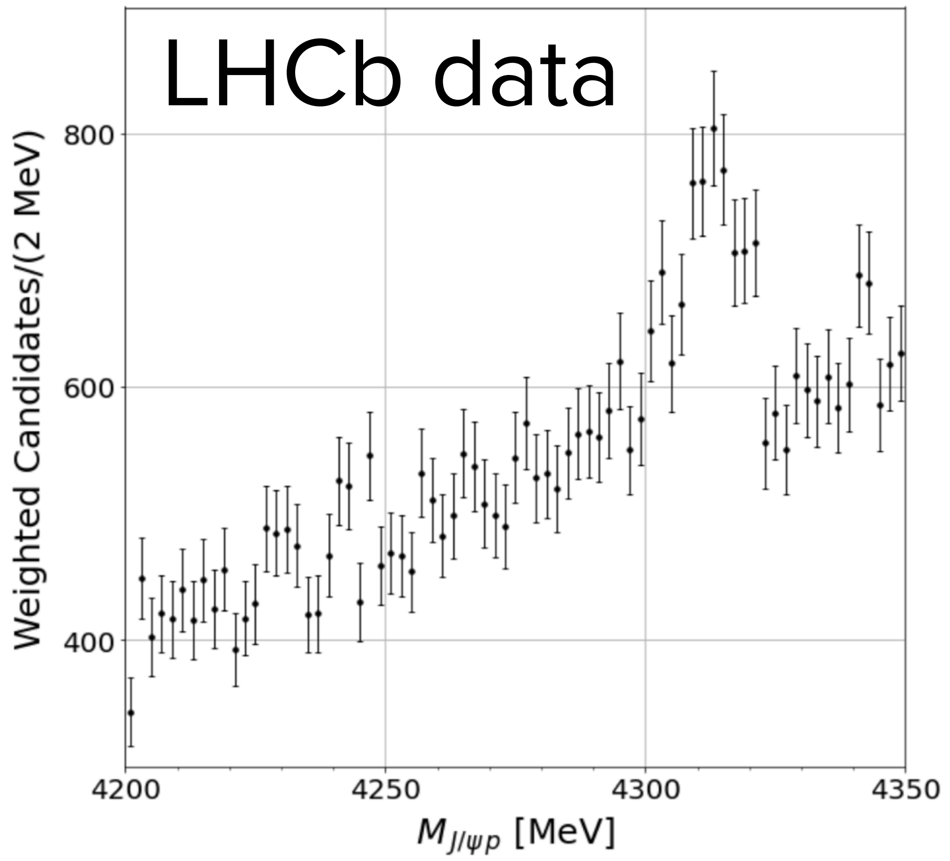

It is not enough to use one element of the T-matrix.
We need to include the off-diagonal element.



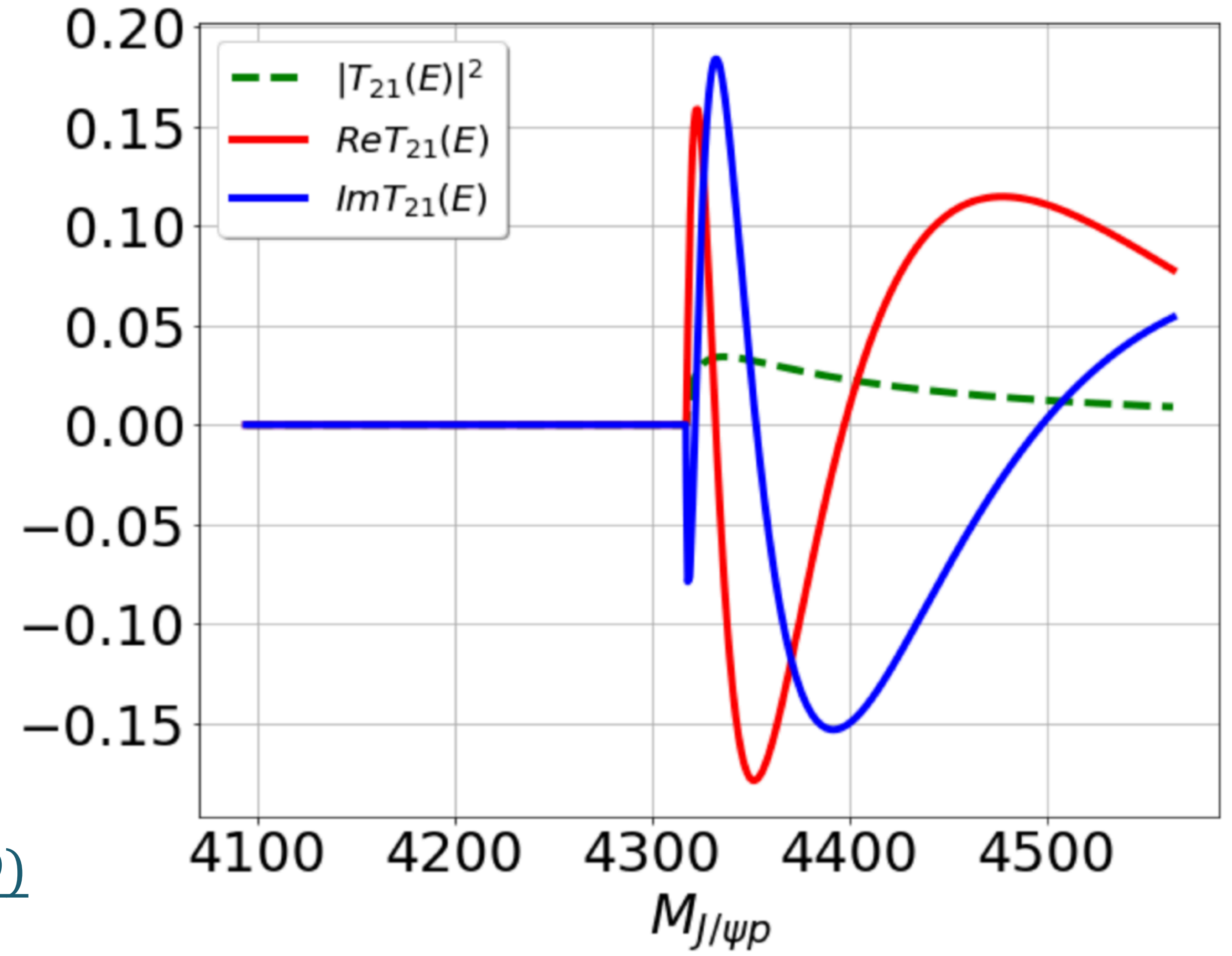

1 pole in each RS



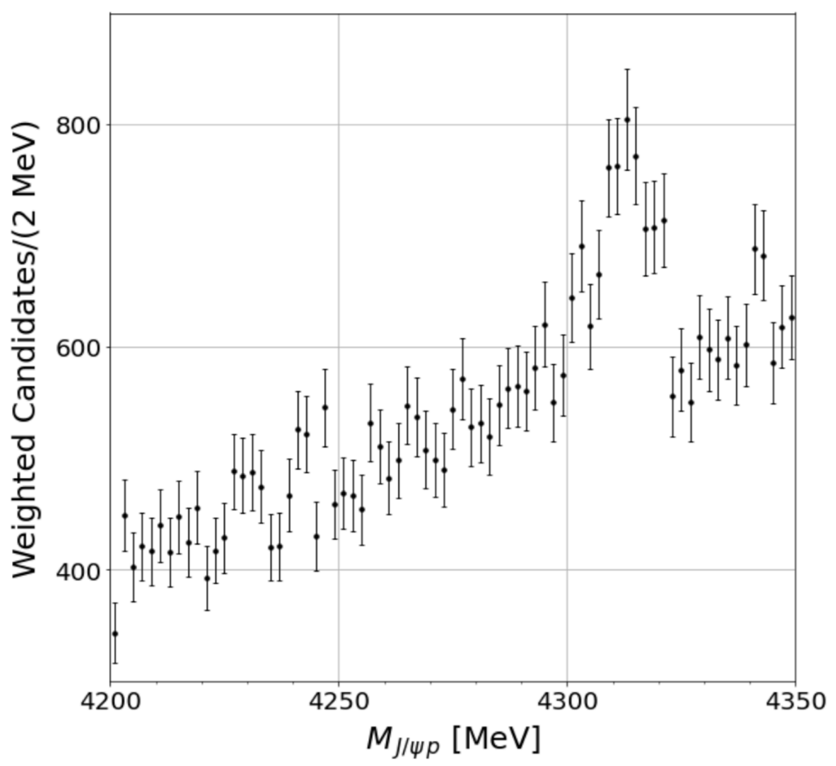



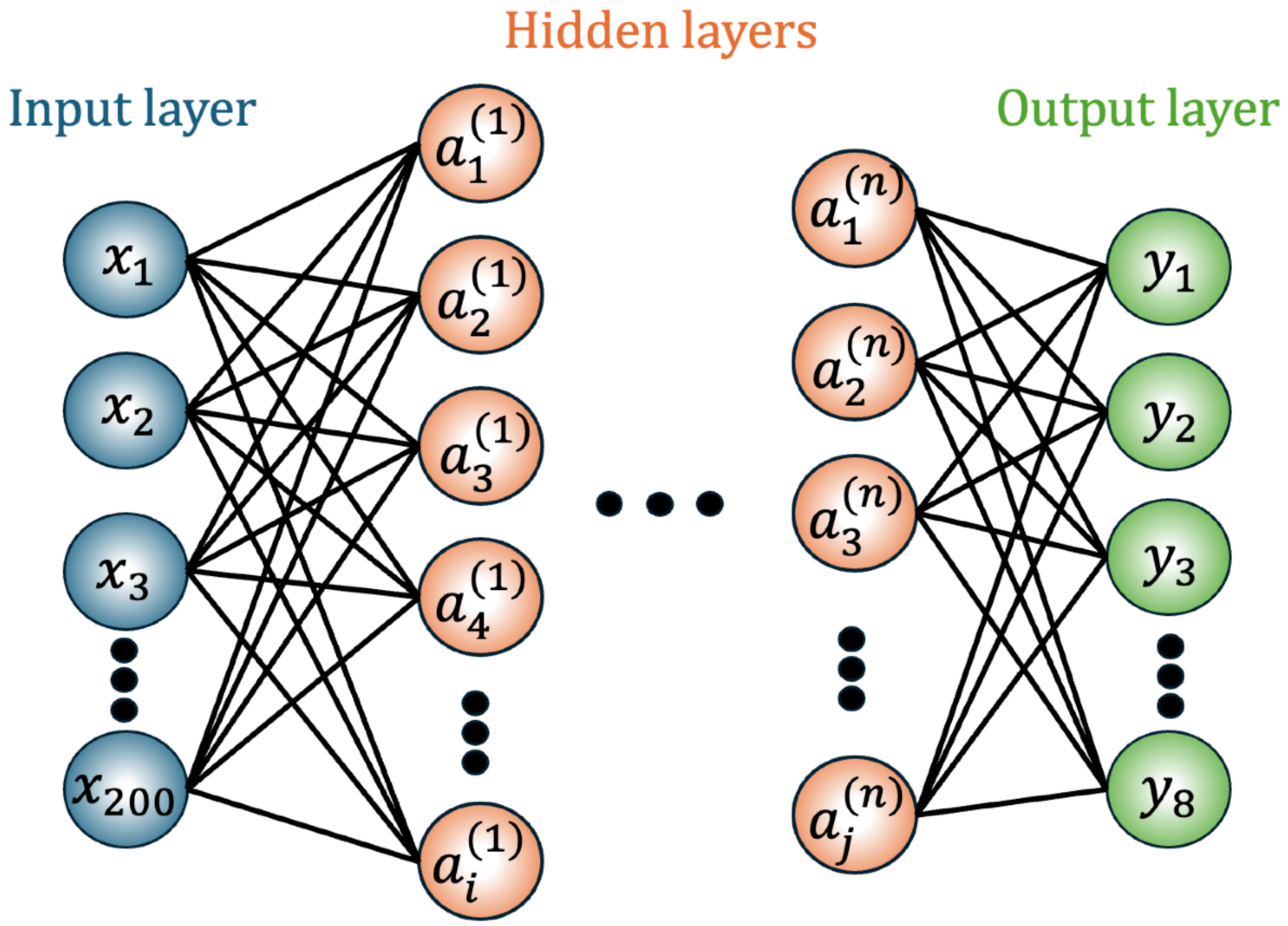

LHCb, PRL 122 222001 (2019)




LMS, DLBS PRC 108 045204 (2023)

# Probe deeper: Pole structure of $P_{c\bar{c}}(4312)^+$




| Class label | $S$-matrix pole configuration |
|---|---|
| 0 | 1 pole on $[bt]$ |
| 1 | 1 pole on $[bb]$ |
| 2 | 1 pole on $[tb]$ |
| 3 | 1 pole on $[bt]$ and 1 pole on $[bb]$ |
| 4 | 1 pole on $[bb]$ and 1 pole on $[tb]$ |
| 5 | 1 pole on $[bb]$, 1 pole on $[tb]$, 1 pole on $[bt]$ |
| 6 | 2 poles on $[bb]$, 1 pole on $[tb]$ |
| 7 | 1 pole on $[bb]$, 2 poles on $[tb]$ |

$$\frac{dN}{d\sqrt{s}} = \rho(s)\Big[\,|F(s)|^2 + B(s)\Big];$$

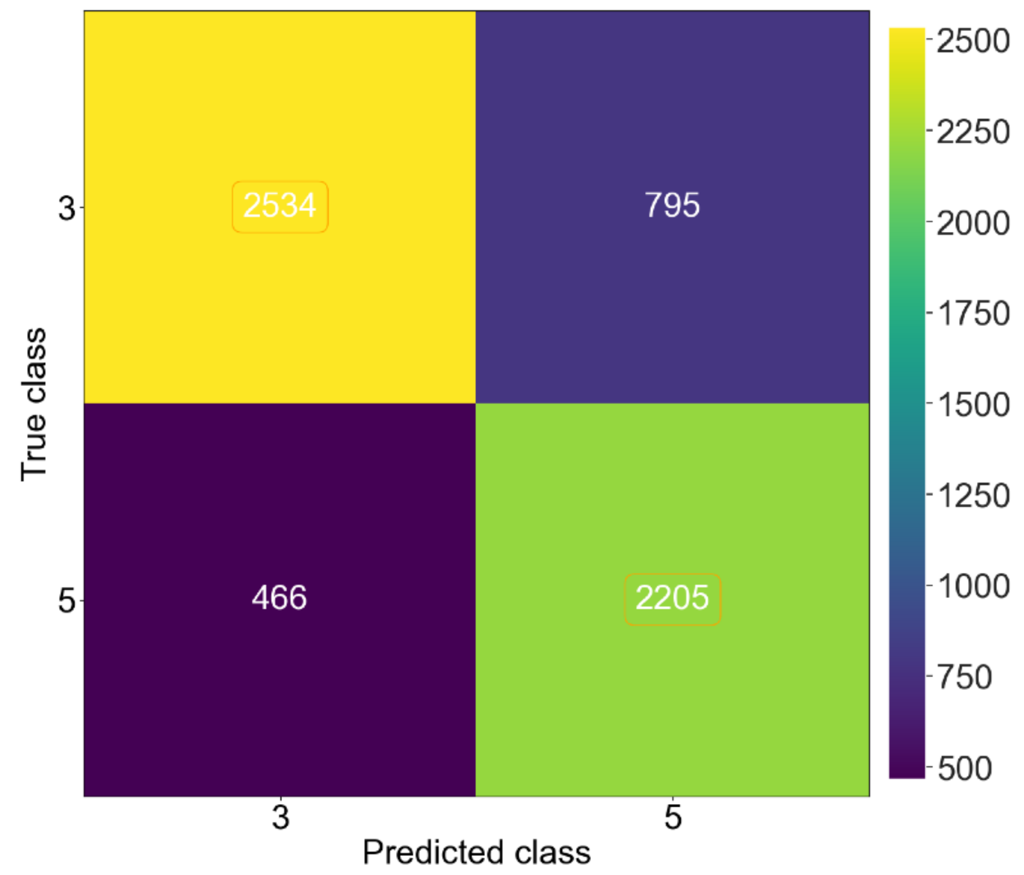
$$F(s) = \alpha_1 T_{11}(s) + \alpha_2 T_{21}(s)$$

$$S_{11}(\omega) = \prod_m \frac{D_m(-1/\omega)}{D_m(\omega)} \qquad S_{22}(\omega) = \prod_m \frac{D_m(1/\omega)}{D_m(\omega)}$$

$$S_{11}S_{22} - S_{12}S_{21} = \prod_m \frac{D_m(-\omega)}{D_m(\omega)} \qquad \hat{S} = \hat{1} + 2i\hat{T}$$

Class 3:
1 pole in 2nd RS and
1 pole in the 3rd

Class 5:
1 pole in each
unphysical RS

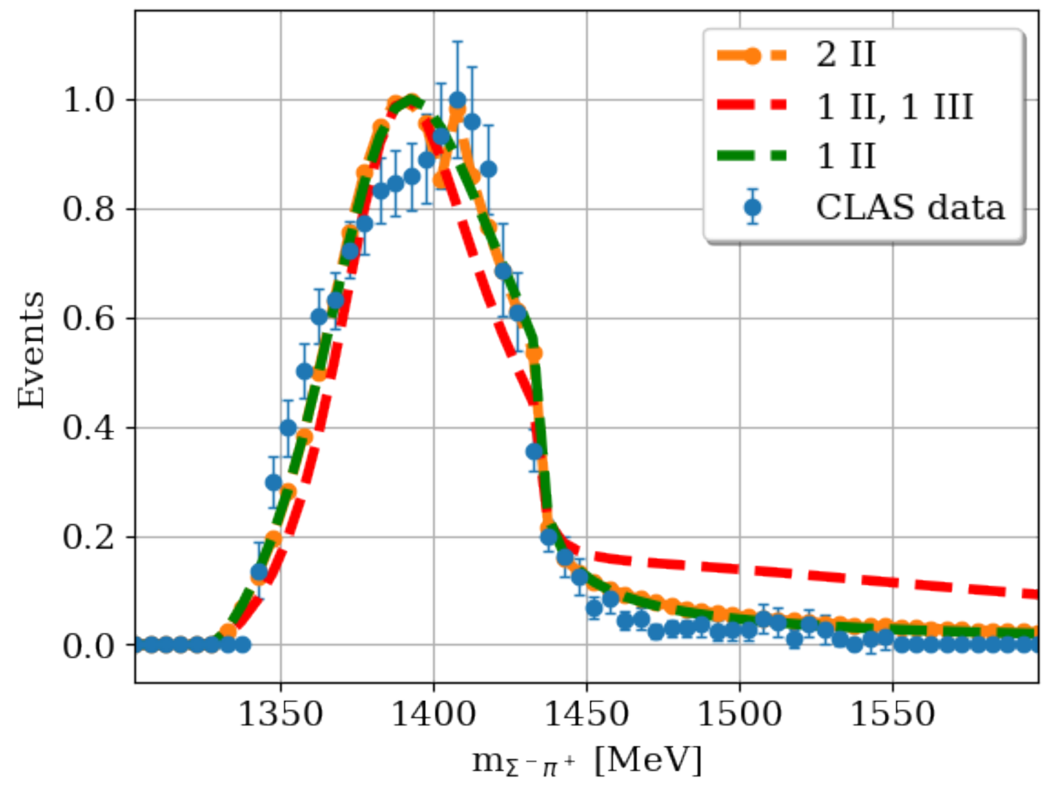



Pole in 2nd and 3rd RS
-resemble pole-shadow pair
-possible true resonance
decaying into $J/\psi p$

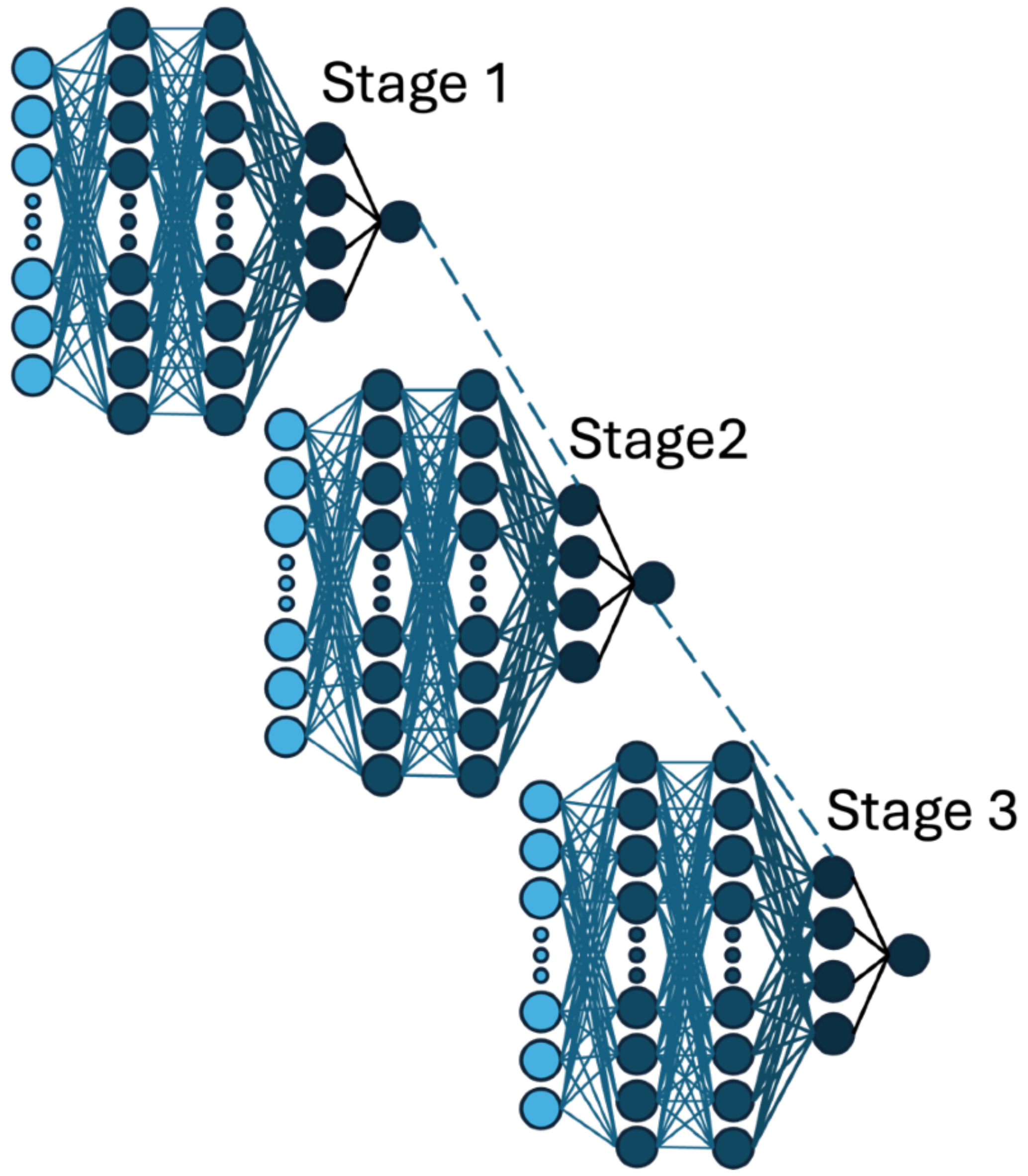
Pole in 4th RS
-Virtual state of $\Sigma_c\bar{D}$
coupled to $J/\psi p$

LM Santos, VAA Chavez, DLBS, JPG (2025) 52 015104

# Pole structure of $\Lambda(1405)$: Preliminary Results

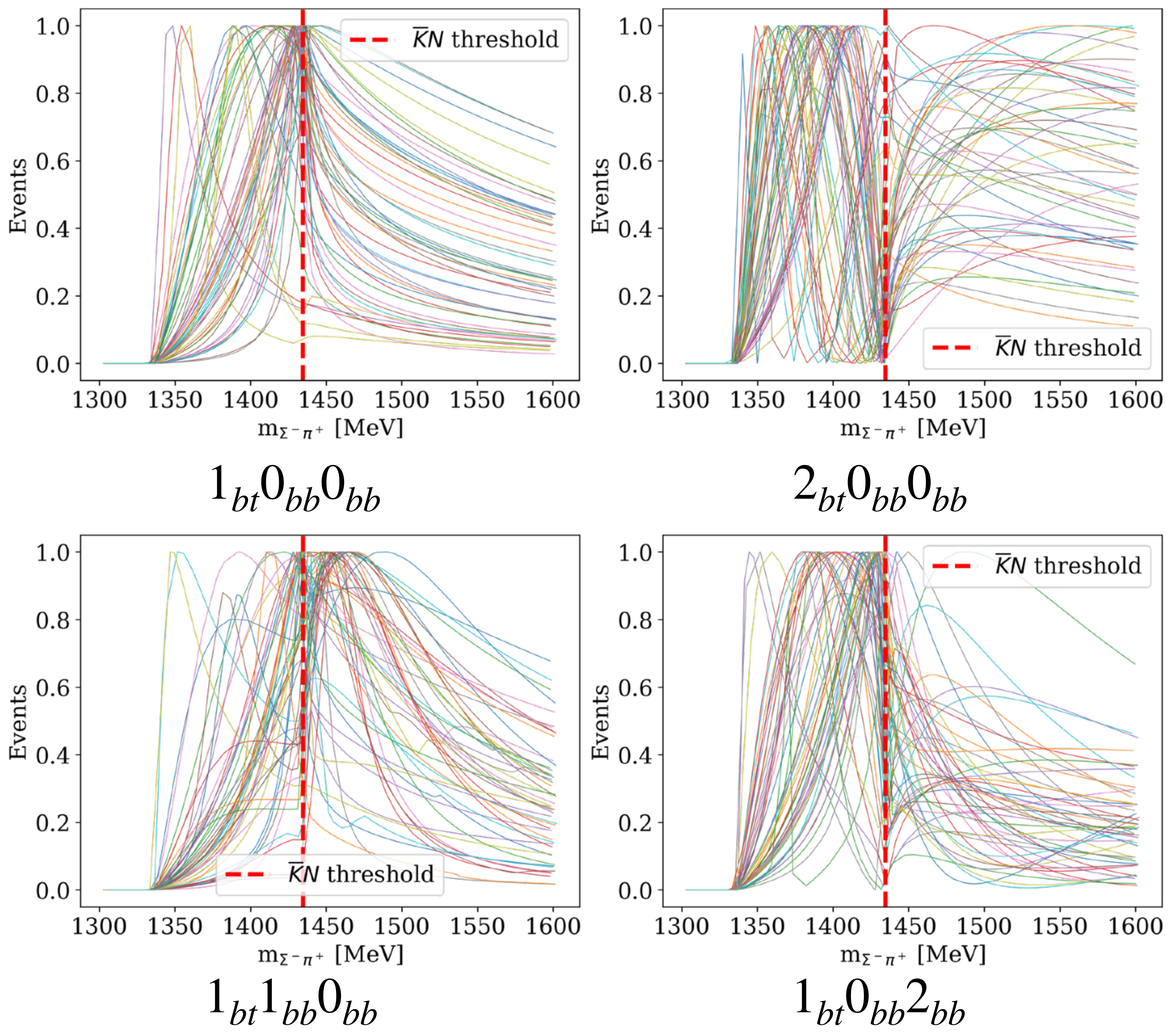


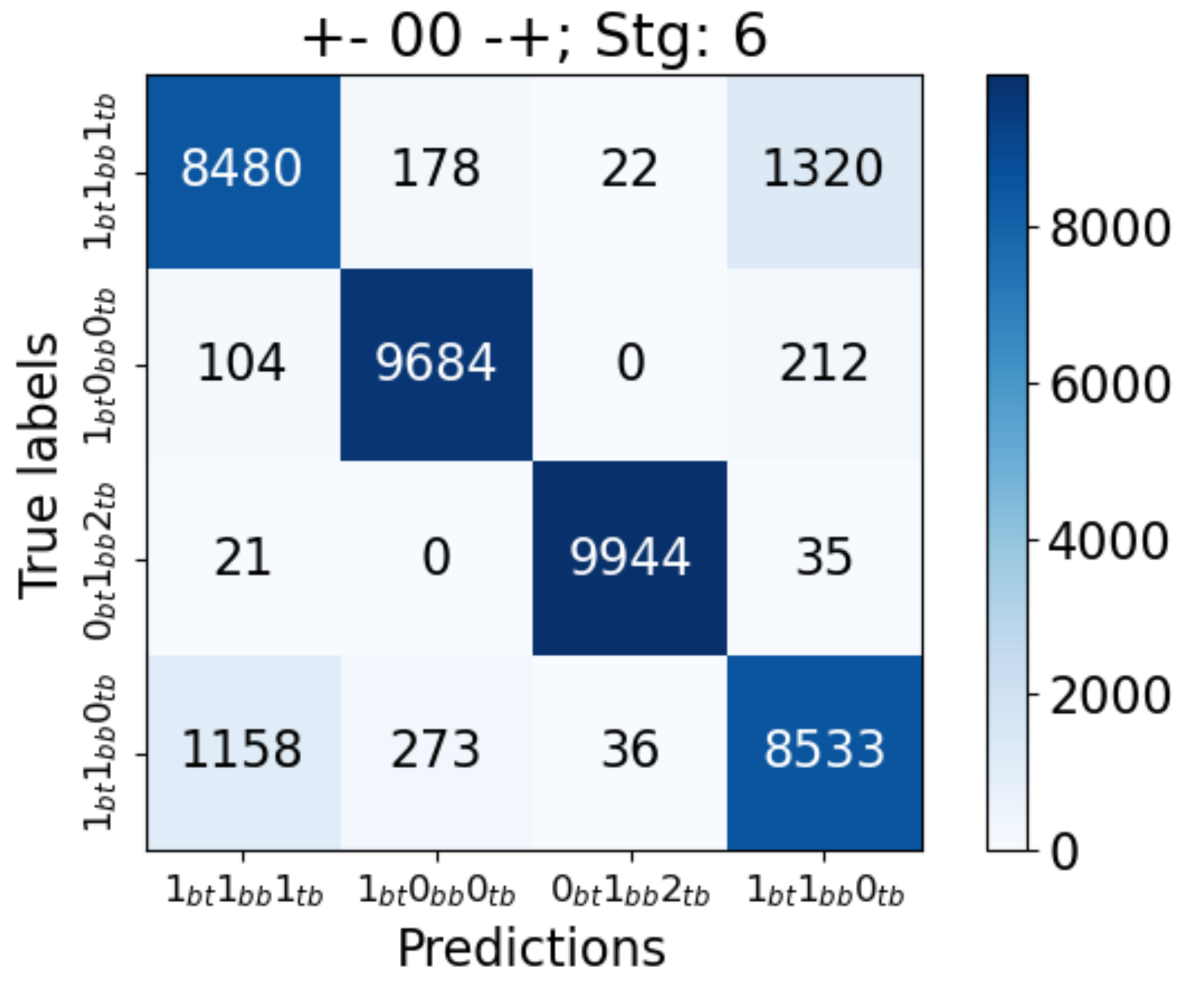

Ambiguous Pole Structures



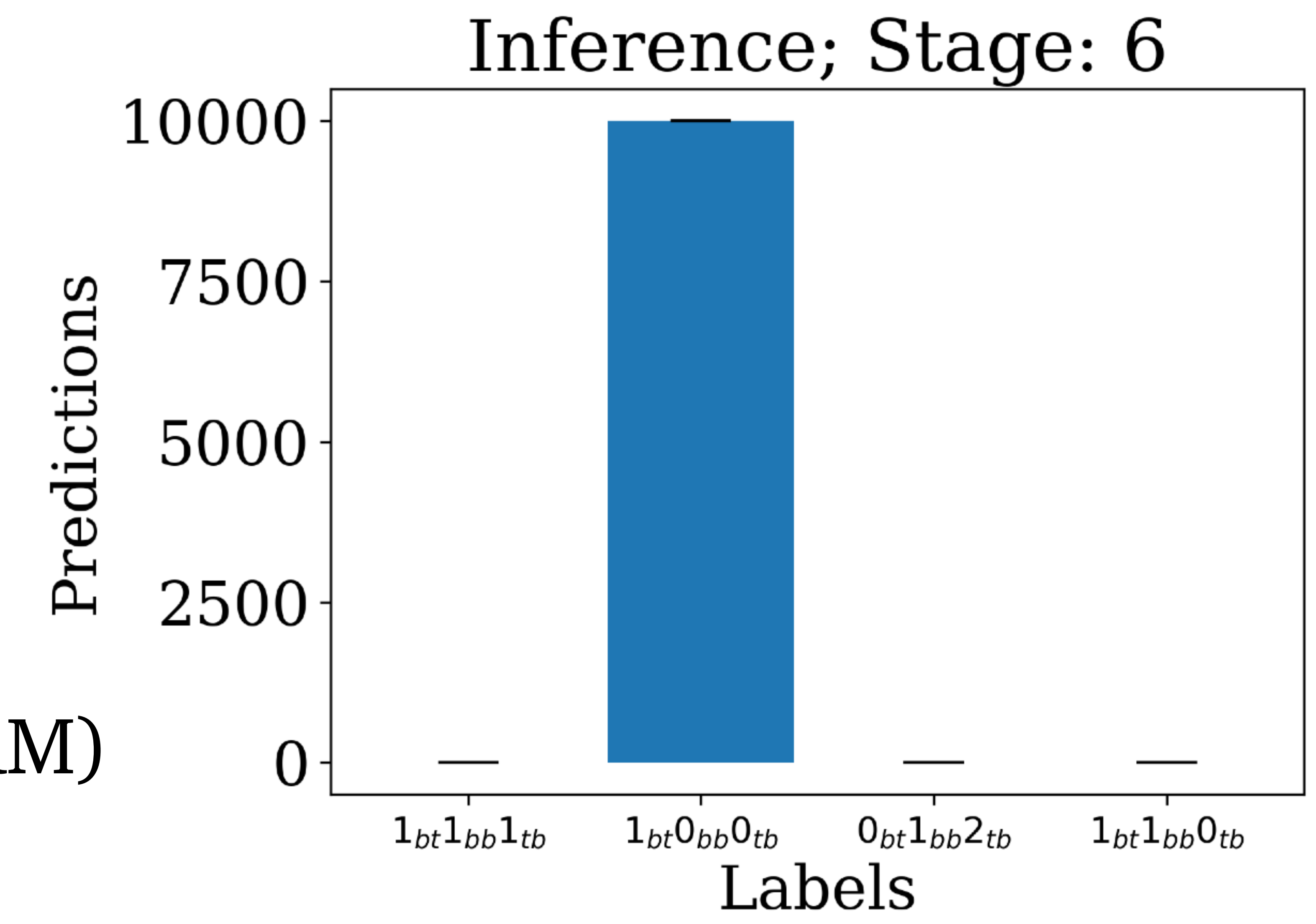

Stage Training

Dataset Generation



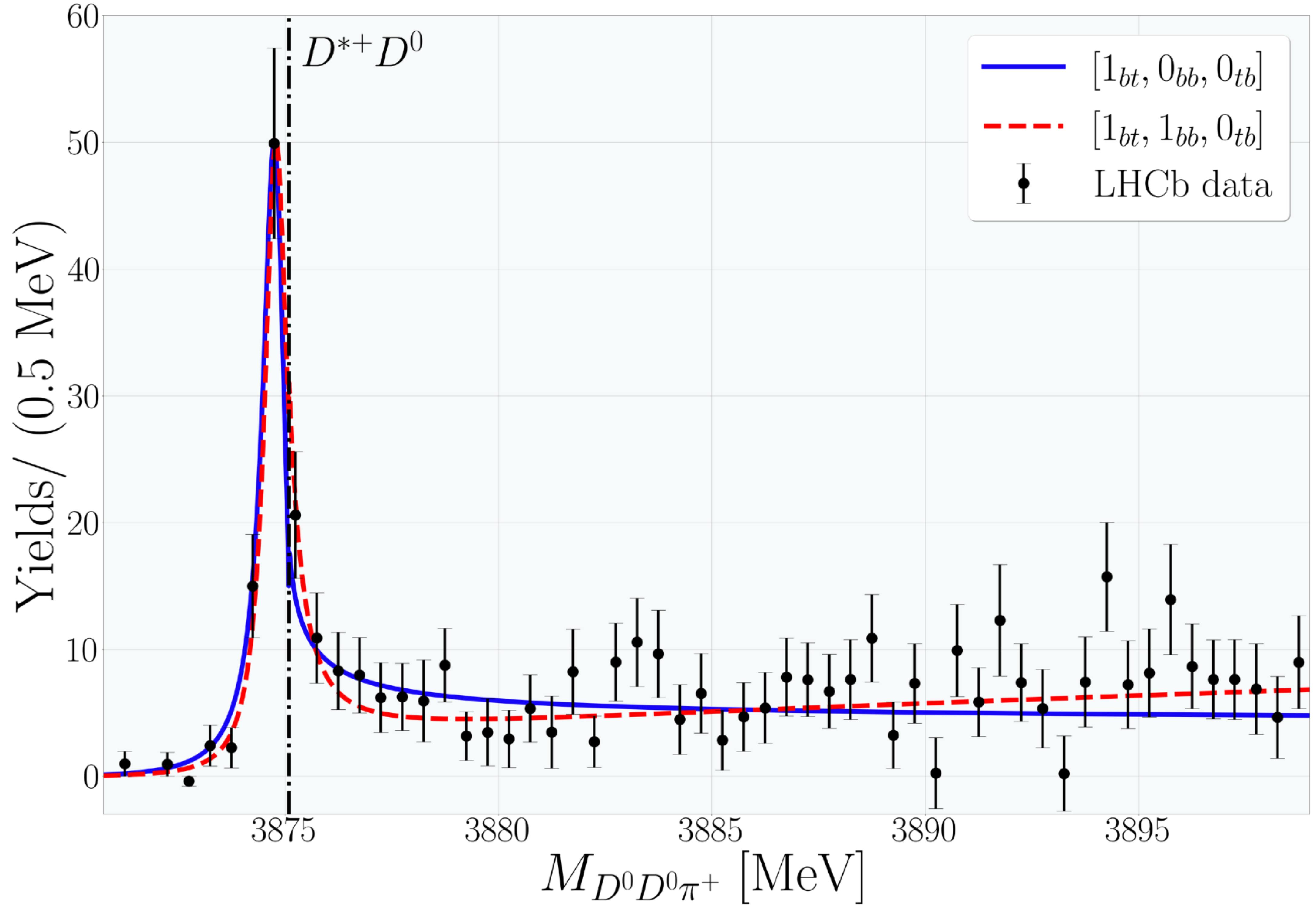

Confusion Matrix

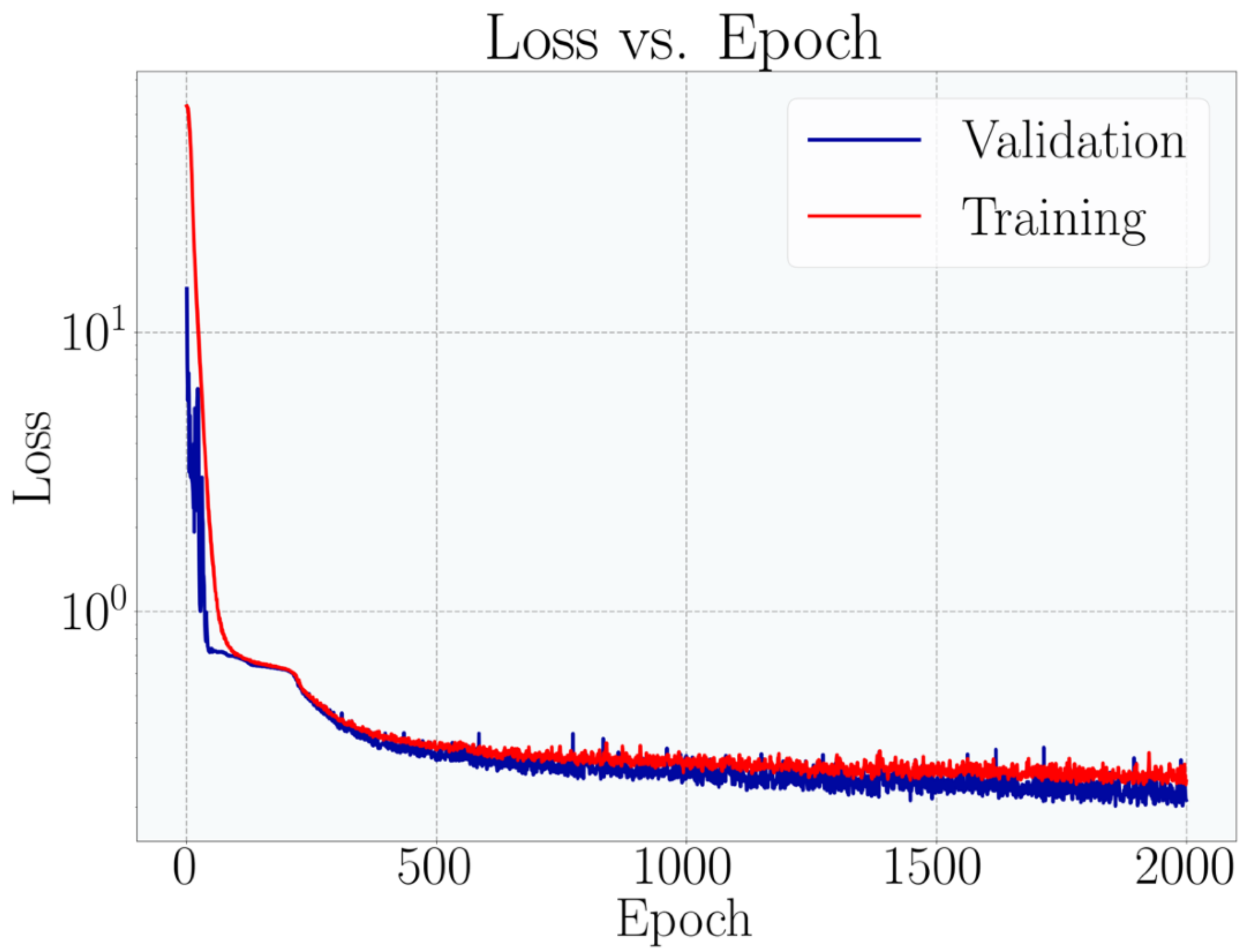


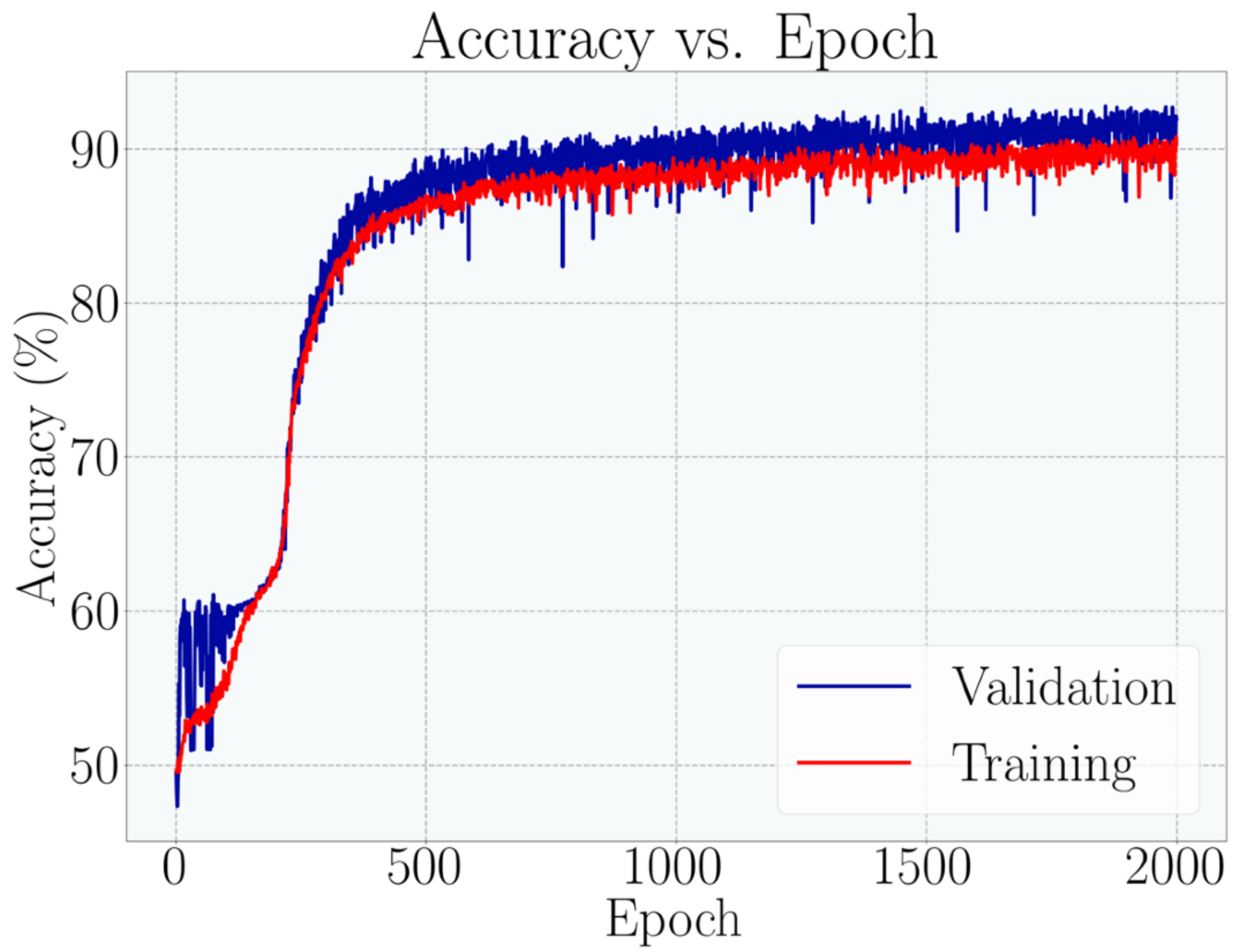

Two-pole (on the same RM) cannot be confirmed.

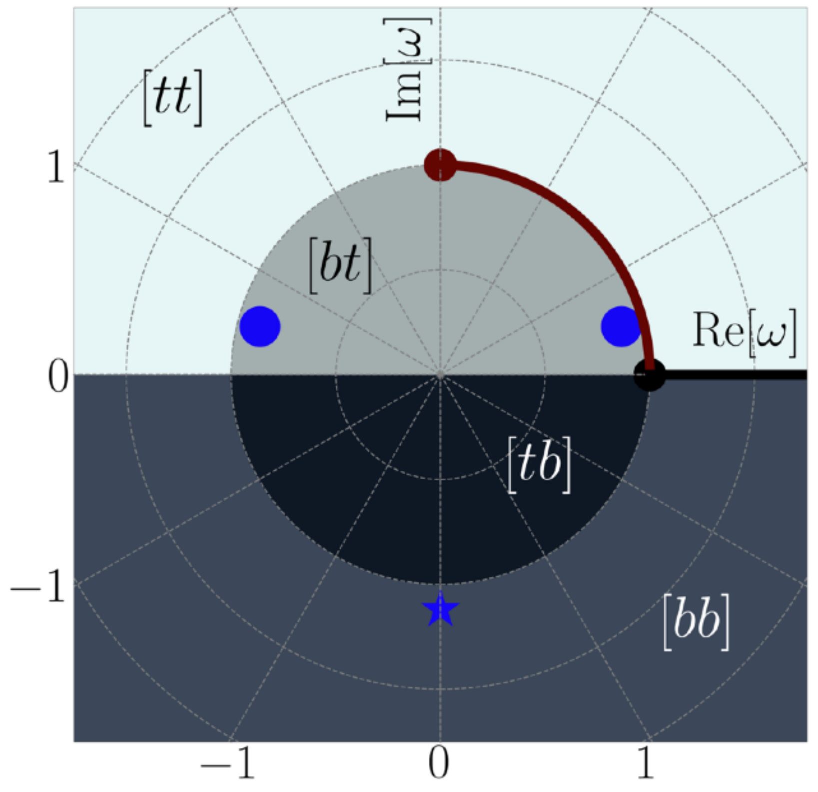
VAA Chavez, DLBS, In preparation

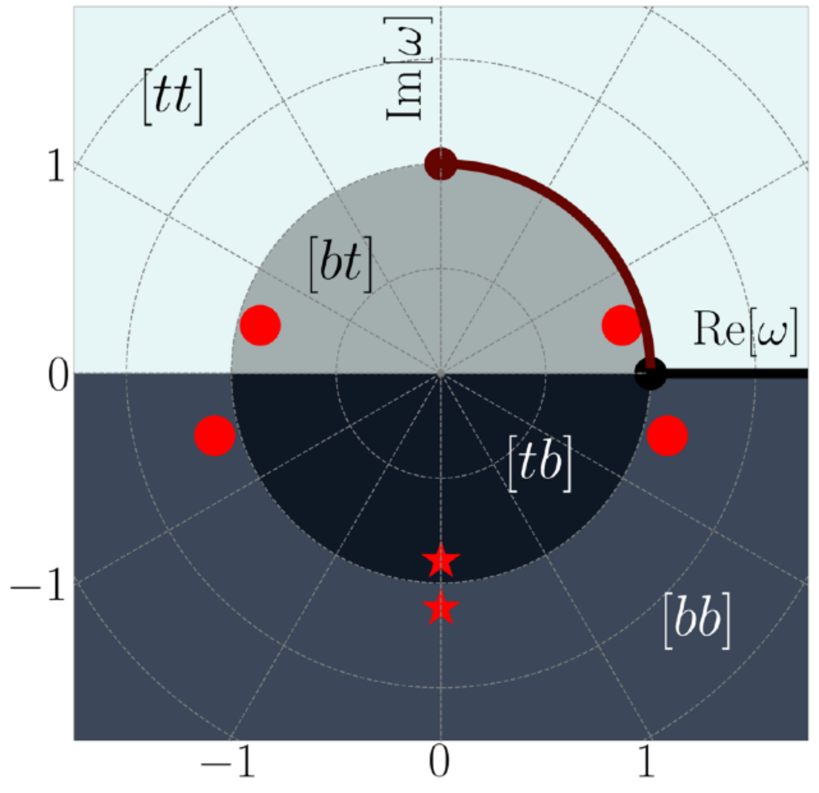
# Pole structure of $T_{cc}(3875)^+$: Preliminary Results

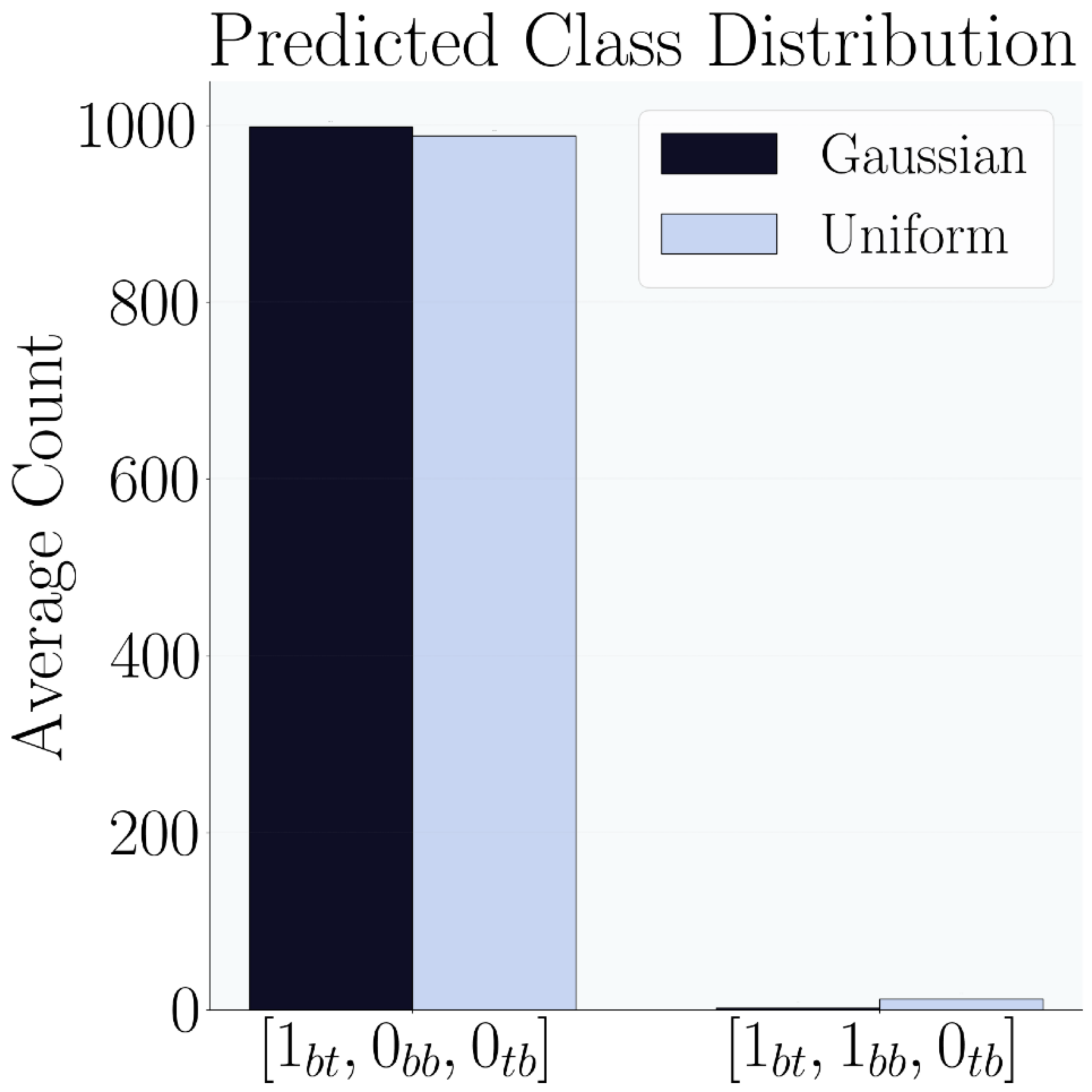


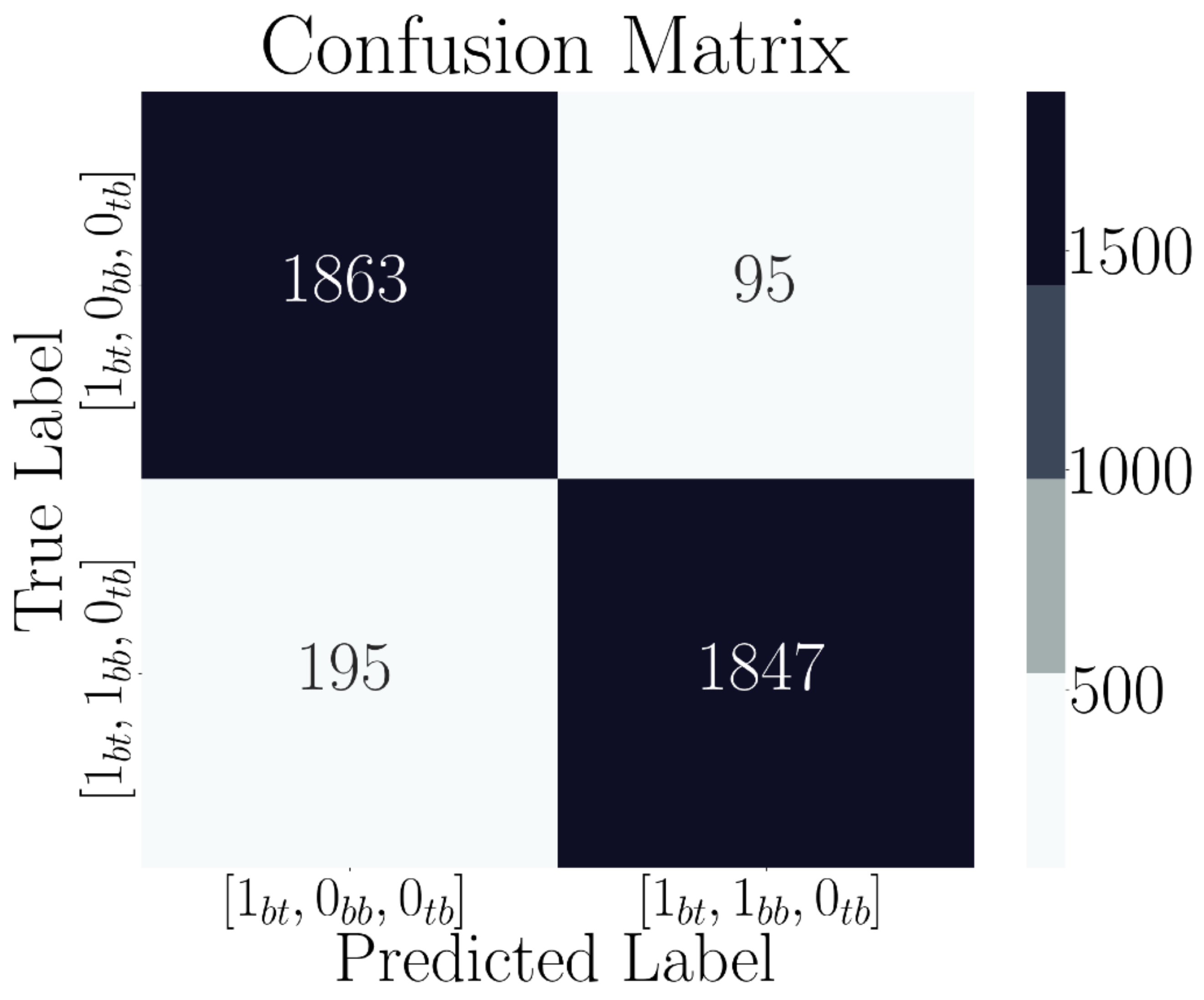




JB Pagayon, DLBS, In preparation

# Conclusion and Outlook

- It is possible to distinguish kinematical enhancements with dynamical pole-based enhancements despite the presence of experimental uncertainty.

- To fully utilized the power of ML, use it a as a model-selection framework.
    - Multi-parameter of a DNN can be used to cover a wider model space.

- Using the ML approach, we have shown that
    - $P_{c\bar{c}}(4312)^+$ is NOT due to (single) triangle singularity
    - $P_{c\bar{c}}(4312)^+$ is a possible true resonance that is contaminated by the coupled-channel interaction of $\Sigma_c\bar{D}$ (having a virtual state) with $J/\psi p$.

Outlook

- Apply the method to other near-threshold phenomena.
- Apply the method to correlation function.

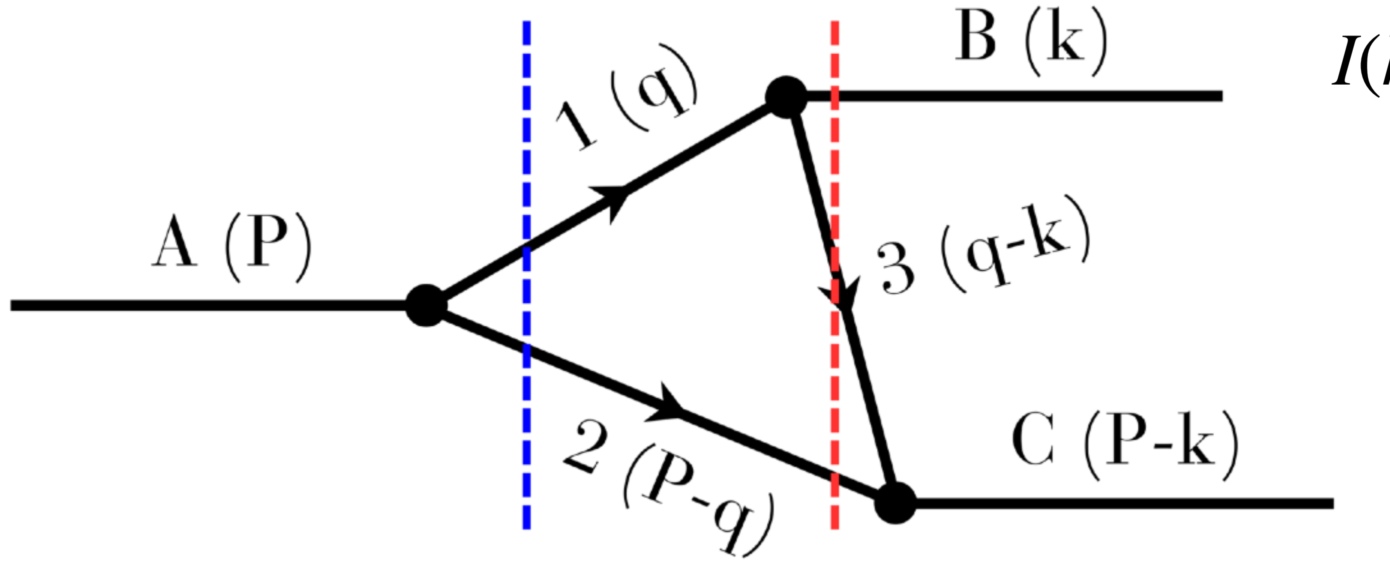
**Thank you for your attention.**

# Back up: TS Ruled out

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034

M Bayar, F Aceti, FK Guo, E Oset, PRD 126 074039 (2016)

FK Guo, XH Liu, S Sakai, PPNP 122 103757 (2020)




$$I(k) \propto \int_0^\infty \frac{q^2 f(q)\,dq}{P^0 - \sqrt{m_1^2+q^2} - \sqrt{m_2^2+q^2} + i\epsilon}$$

$$f(q) = \int_{-1}^{1} \frac{dz}{E_C - \omega(q) - \sqrt{m_3^2+q^2+k^2-2qkz} + i\epsilon}$$

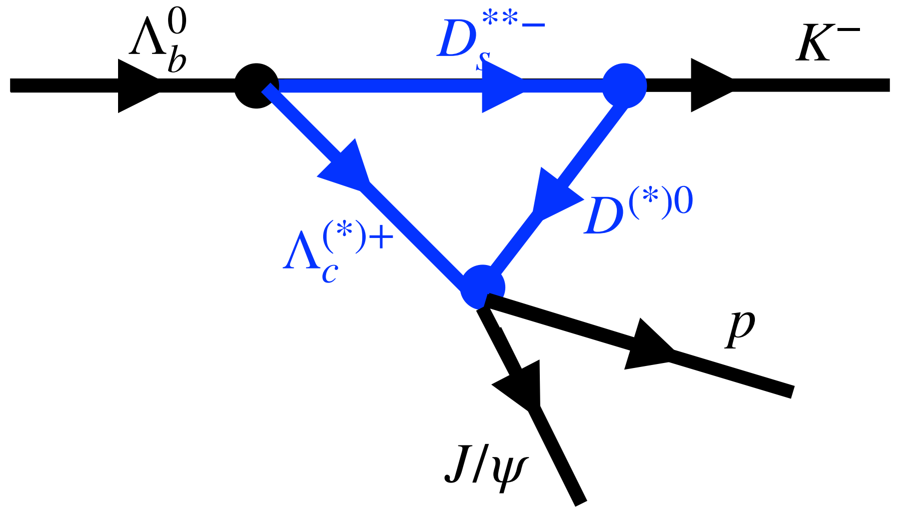
$$m_1^2 \in \left[\frac{M_A^2 m_3 + M_B^2 m_2}{m_2+m_3} - m_2 m_3,\ (M_A - m_2)^2\right]$$

$$m_C^2 \in \left[(m_2+m_3)^2,\ \frac{M_A m_3^2 - M_B^2 m_2}{M_A - m_2} + M_A m_2\right]$$

The amplitude has no singularity (no pole)

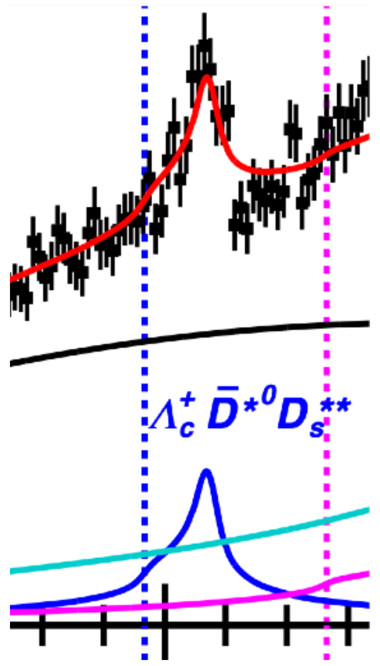
No need to introduce new hadronic state to explain the enhancement.

Mass condition - crucial in generating mock dataset

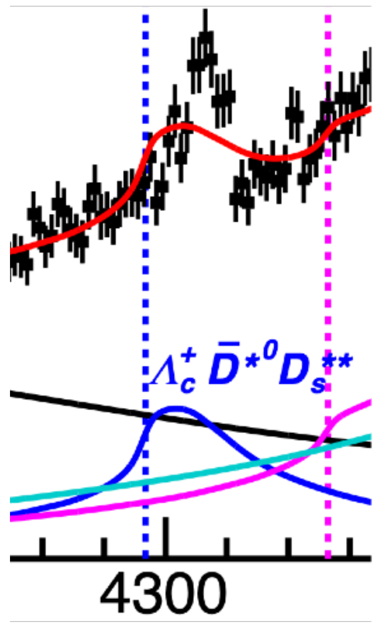



Good fit but with unrealistic $\Gamma \sim 1$MeV for $D_s^{**}$

$D_s^{**-}(3288)$ Not in PDG

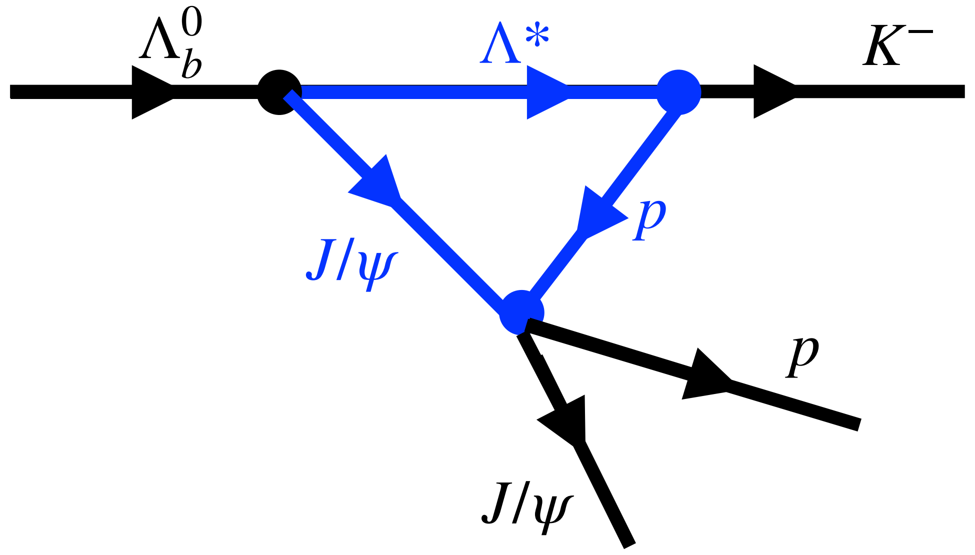



Bad fit with realistic $\Gamma \sim 50$MeV for $D_s^{**}$



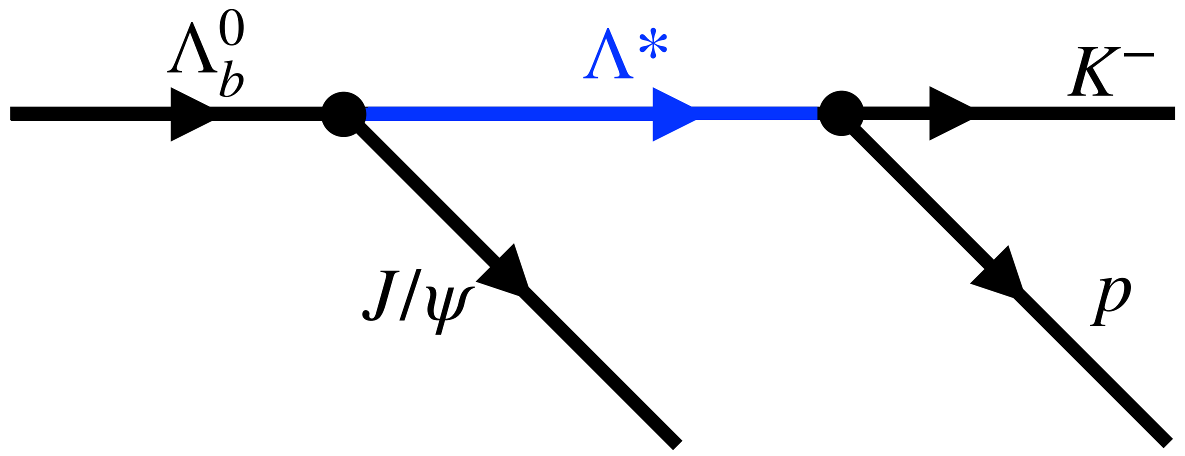

TS interpretation for the $P_{c\bar{c}}(4312)^+$ - ruled out

LHCb, PRL 122 222001 (2019)




TS enhancement is drowned by the tree-level decay. (Schmid theorem)

VR Debastiani, S Sakai, E Oset, EPJC, 79, 69 (2019)



# General S-matrix parametrization

DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034

$$S_{11}(p_1,p_2) = \prod_m \frac{D_m(-p_1,p_2)}{D_m(p_1,p_2)}$$

KJ Le Couteur, Proc. Roy. Soc (Lo
RG Newton J. Math. Phys. 2, 188

$$p_k \to q_k; \quad s = q_k^2 + \epsilon_k^2; \quad \omega = \frac{q_1+q_2}{\sqrt{\epsilon_2^2-\epsilon_1^2}}$$

One pole per $D_m(q_1,q_2)$ : $\omega_m$

$D_m(q_1,q_2) = D_m(\omega)$ M Kato, Ann. Phys., 31 1 (1965)

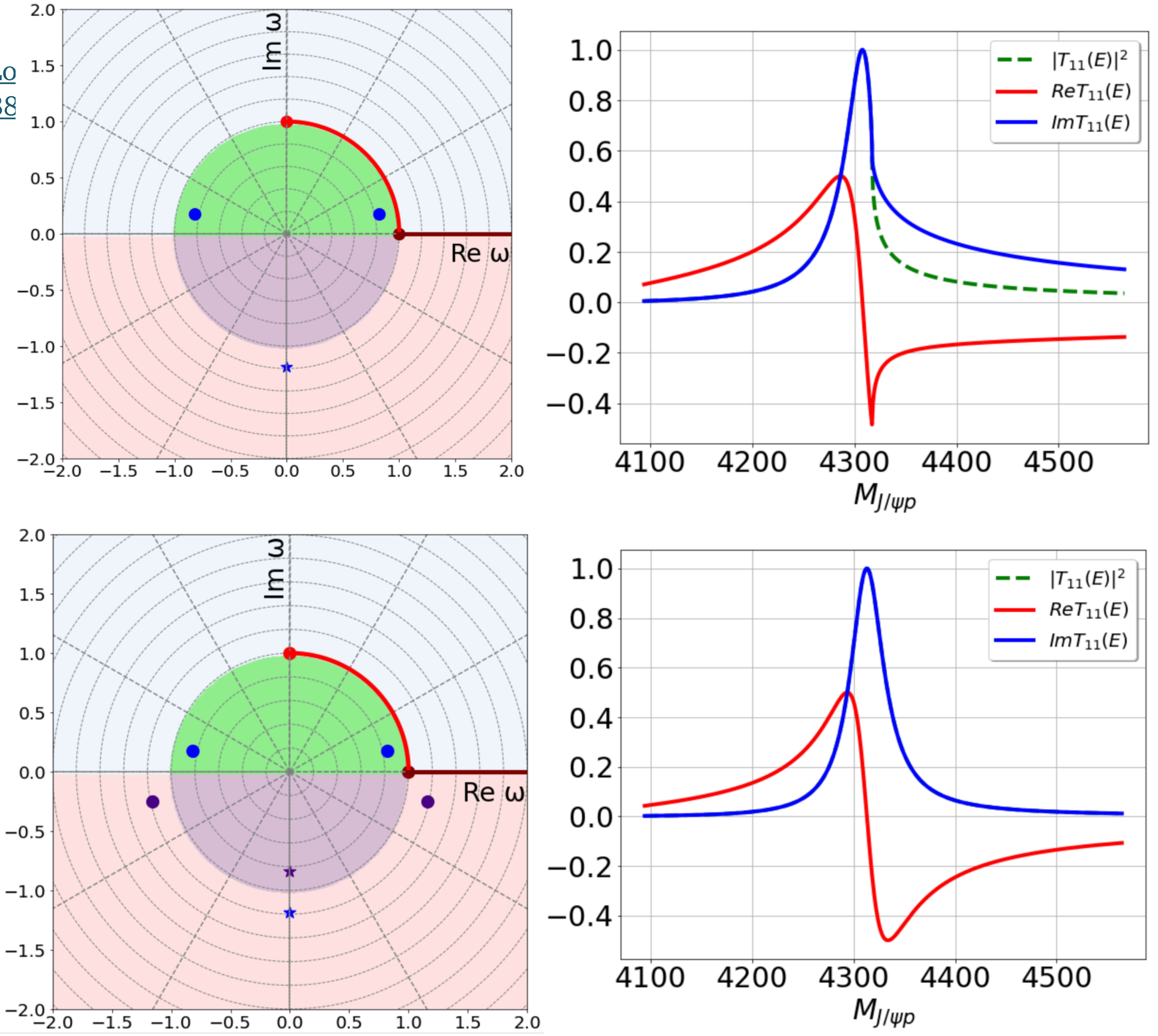
$$= \frac{1}{\omega^2}\left(\omega-\omega_m\right)\left(\omega+\omega_m^*\right)\left(\omega-\omega_{\bar{m}}\right)\left(\omega+\omega_{\bar{m}}^*\right)$$

$\omega_{\bar{m}}$ needed only to ensure $\lim_{\omega\to\infty} S_{11} \to 1;$

$$|\omega_m\omega_{\bar{m}}| = 1$$

- Poles are introduced independently.
- Cusp at the threshold can be controlled via pole placement.

LMS, DLBS PRC 108 045204 (2023)

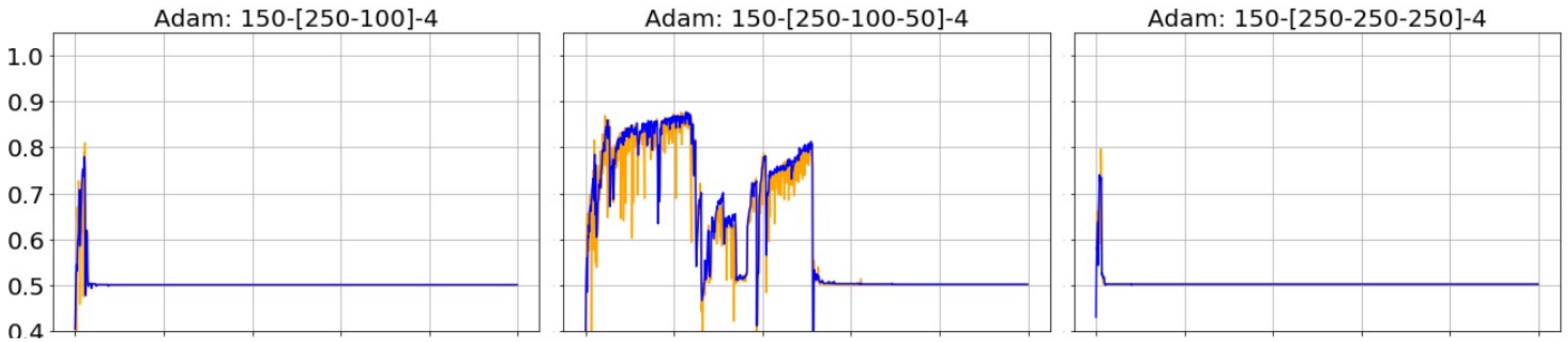


# Back-up:Training performance

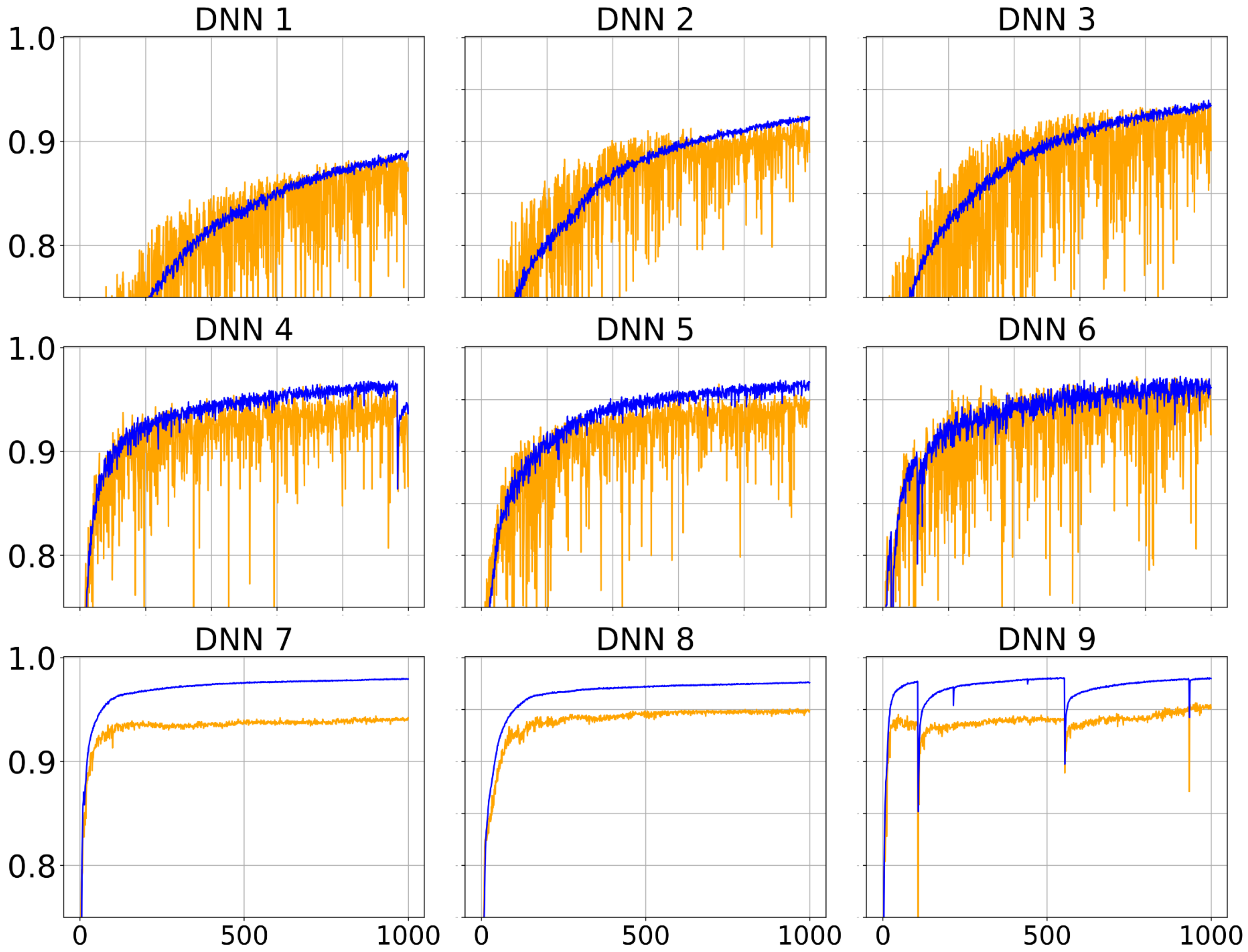
DAO Co, VAA Chavez, DLBS, PRD 110 (2024) 114034

| Model | Optimizer and Architecture |
|---|---|
| DNN 1 | AdaGrad: 150-[250-100]-4 |
| DNN 2 | AdaGrad: 150-[250-100-50]-4 |
| DNN 3 | AdaGrad: 150-[250-250-250]-4 |
| DNN 4 | AMSGrad: 150-[250-100]-4 |
| DNN 5 | AMSGrad: 150-[250-100-50]-4 |
| DNN 6 | AMSGrad: 150-[250-250-250]-4 |
| DNN 7 | SMORMS3: 150-[250-100]-4 |
| DNN 8 | SMORMS3: 150-[250-100-50]-4 |
| DNN 9 | SMORMS3: 150-[250-250-250]-4 |

Other optimizers cannot even learn the classification task.

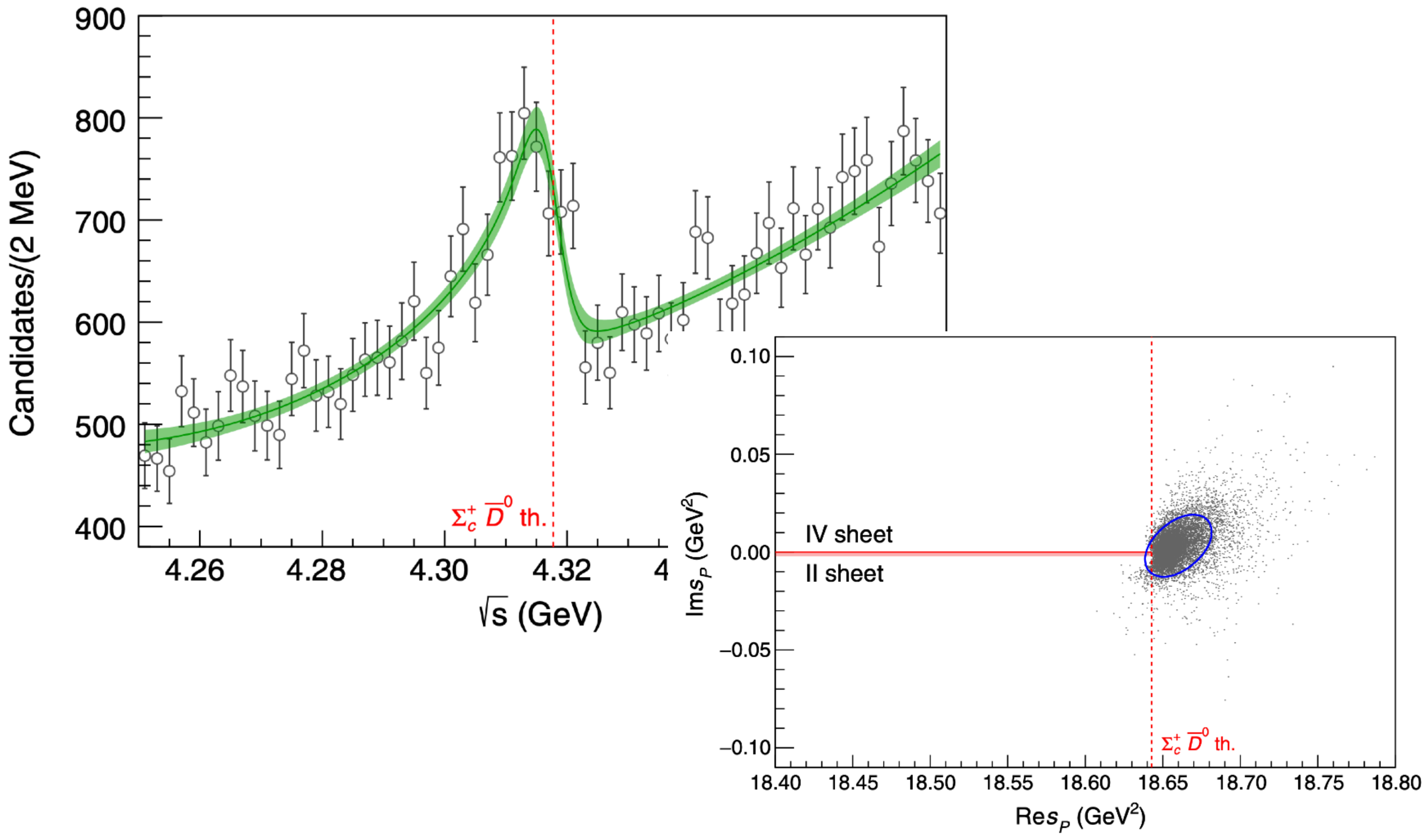



The classification task is non-trivial.




Training epoch vs accuracy

# Conventional fitting vs Deep learning




JPAC, PRL 123 092001 (2019)

Fitting:

Constrain the parameter space of ERE:

$$T(s) = \frac{m_{22} - ik_2}{(m_{11} - ik_1)(m_{22} - ik_2) - m_{12}^2}$$

Result:
$P_{c\bar{c}}(4312)^+$ is a virtual state of $\Sigma_c^+ \bar{D}^0$

TABLE I. Softmax output probabilities [26] for the three experimental datasets by LHCb [8].

| | $b\|2$ | $b\|4$ | $v\|2$ | $v\|4$ |
|---|---|---|---|---|
| $\cos\theta_{P_c}$-weighted | 0.6% | <0.01% | 1.1% | 98.3% |
| $m_{Kp} > 1.9$ GeV | 1.4% | <0.1% | 1.6% | 97.0% |
| $m_{Kp}$ all | 5.4% | <0.1% | 21.0% | 73.6% |

JPAC, PRD 105 L091501 (2022)

Train DNN using ERE: $T(s) = \frac{m_{22} - ik_2}{(m_{11} - ik_1)(m_{22} - ik_2) - m_{12}^2}$

Result/ interpretation:
$P_{c\bar{c}}(4312)^+$ is a virtual state of $\Sigma_c^+ \bar{D}^0$

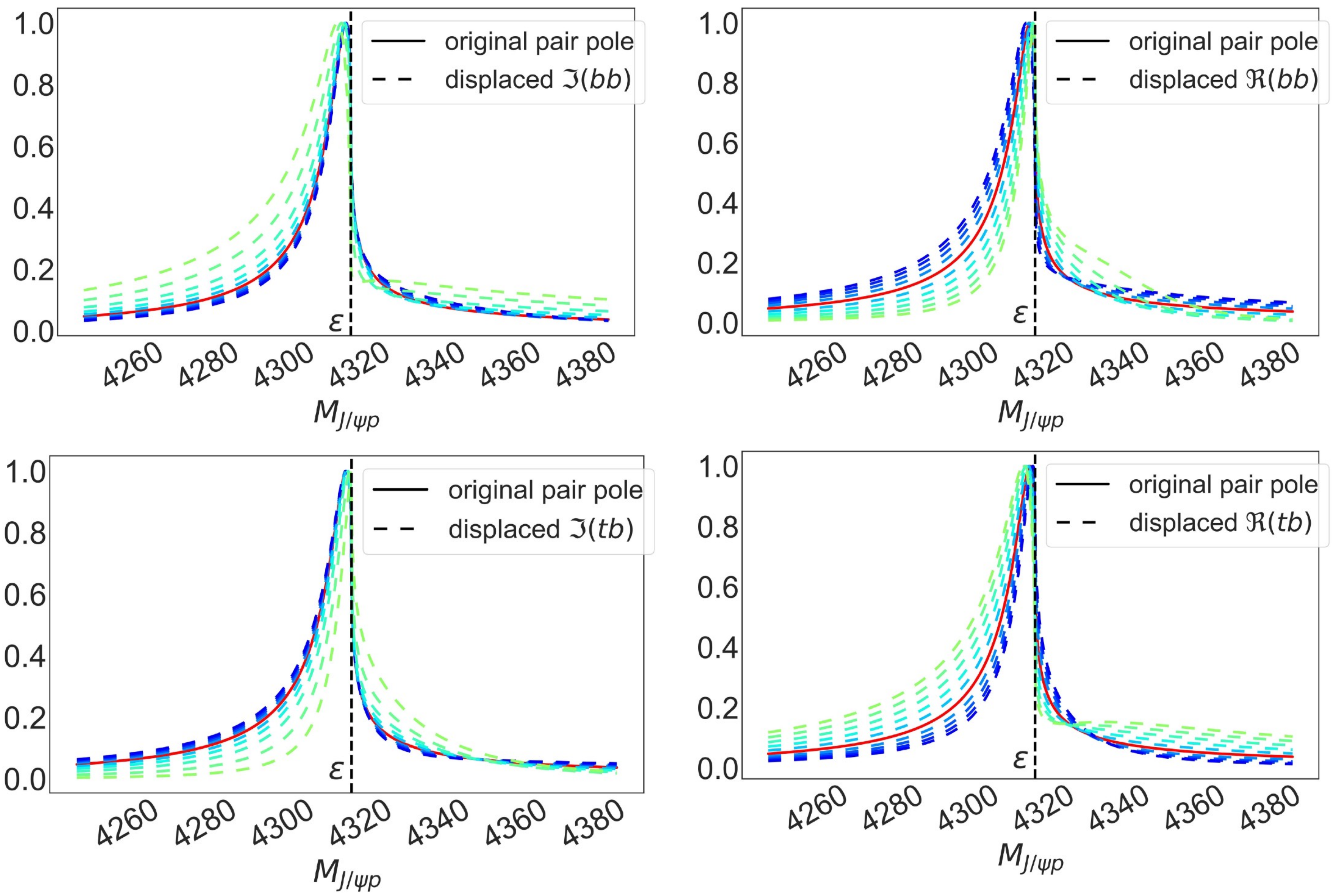
DL result is consistent with standard fitting method.

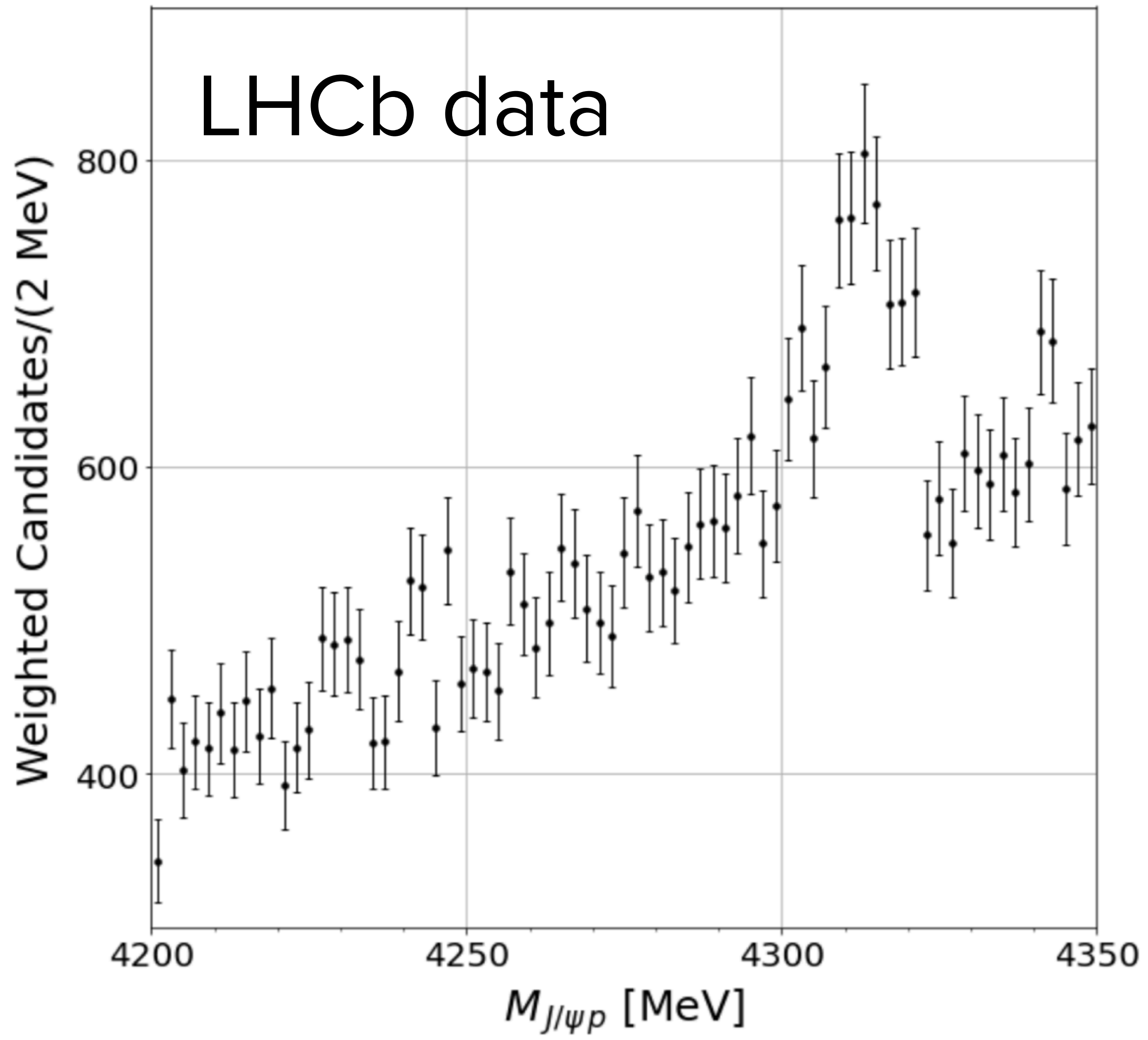
Is DL just a “glorified” alternative to fitting?
Maybe we can do more.

Proposal:
- Use DL to constrain the model space.
- Use fitting when constraining the parameters of a model.

# Back Up: $P_{c\bar{c}}(4312)^+$

Maybe unlikely to happen -
to produce ambiguous pole structure, the poles must have the same position.




LHCb, PRL 122 222001 (2019)

Slight variation in the line shape

Experimental uncertainty might hide them.

LMS, DLBS PRC 108 045204 (2023)

# Back Up: General parametrization

$$S_{11}(p_1, p_2) = \prod_m \frac{D_m(-p_1, p_2)}{D_m(p_1, p_2)}$$

KJ Le Couteur, Proc. Roy. Soc (London) A256 (1960)
RG Newton J. Math. Phys. 2, 188 (1961)

ERE - Scattering length approx.

$$D(p_1, p_2) = \left(M_{11} - ip_1\right)\left(M_{22} - ip_2\right) - M_{12}^2$$

$D(p_1, p_2)$ can only vanish when the $\bar{p}_1$ and $\bar{p}_2$ have opposite imaginary parts. W Frazer and A Hendry, Phys. Rev., 134, B1307 (1964)

Flatté-like parametrization

$$D(p_1, p_2) = E - M + ip_1 + i\gamma_2 p_2$$

Shadow poles may appear on the physical sheet.

RJ Eden, JR Taylor, Phys. Rev. 133, B1575 (1974)

$$\left[\left(p_1 - i\beta_1\right)^2 - \alpha_1^2\right] + \lambda\left[\left(p_2 - i\beta_2\right)^2 - \alpha_2^2\right] = 0$$

$$\left[\left(p_1 - i\beta_1\right)^2 - \alpha_1^2\right]\left[\left(p_1 - i\beta_1\frac{1-\lambda}{1+\lambda}\right)^2 - \left(\alpha_1^2 + \frac{4\lambda\beta_2^2}{(1+\lambda)^2}\right)\right] = 0$$

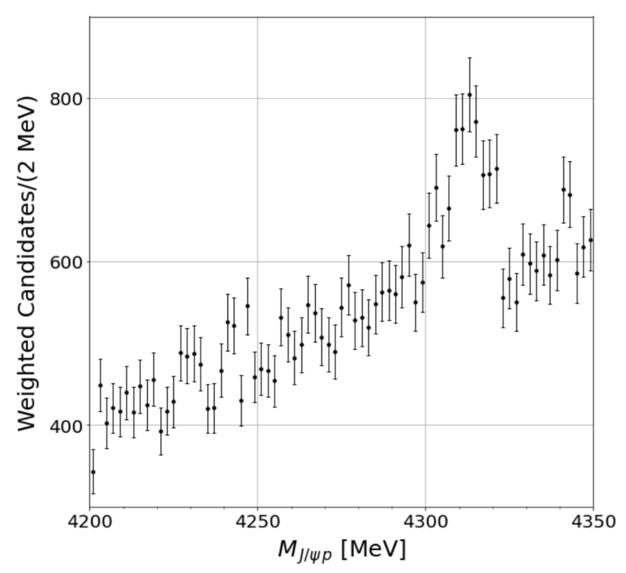
$$\left[\left(p_2 - i\beta_2\right)^2 - \alpha_2^2\right]\left[\left(p_2 + i\beta_2\frac{1-\lambda}{1+\lambda}\right)^2 - \left(\alpha_2^2 + \frac{4\lambda\beta_1^2}{(1+\lambda)^2}\right)\right] = 0$$

Main pole | Shadow poles

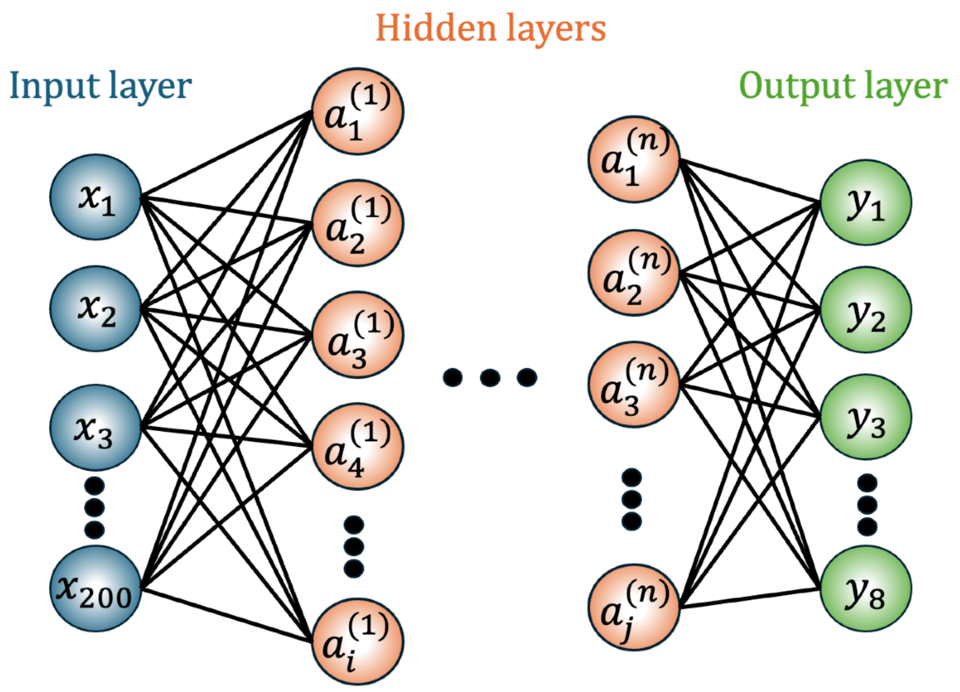
We need a general parametrization:

- Pole position can be controlled and RS can be assigned.
- Poles are independent of each other.

# Back Up: $P_{c\bar{c}}(4312)^+$



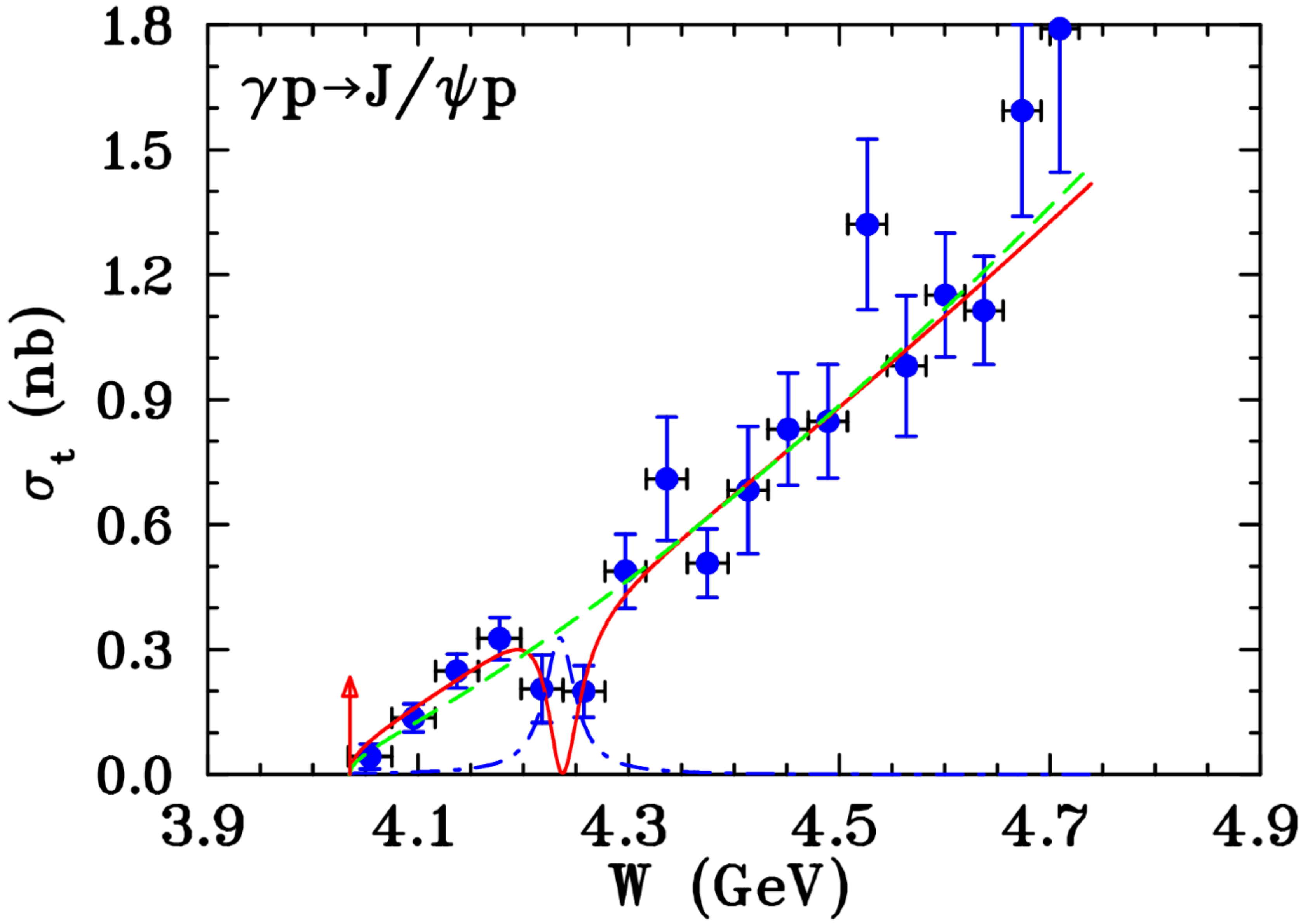






*Pole in 2nd and 3rd RS*
*-resemble pole-shadow pair*
*-possible true resonance decaying into $J/\psi p$*

*Pole in 4th RS*
*-Virtual state of $\Sigma_c\bar{D}$ coupled to $J/\psi p$*





*In contrast with the analysis of JPAC in 2019*

[JPAC, PRL 123 092001 (2019)]

*GlueX: $J/\psi$ photo-production 2023 result - structure found in the $J/\psi$-$p$ cross-section*

[GlueX, PRC 108 025201 (2023)]

*The dip structure can be interpreted as a resonance that interfere with the non-resonant background.*

[I Strakovsky, et al, PRC 108 015202 (2023)]

*No contribution from the $\Sigma_c\bar{D}$ - cannot be interpreted as molecular.*