Direct calculation of parton distribution functions with tensor network states

Manuel Schneider
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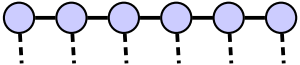

NATIONAL YANG MING CHIAO TUNG UNIVERSITY

理論與計算物理中心
Center for Theoretical and Computational Physics

[arXiv:2504.07508]
[arXiv:2409.16996]

Workshop on parton distribution functions in the EIC era
Academia Sinica, Taipei
16 June 2025

Collaborators

Mari Carmen Bañuls

Krzysztof Cichy

C.-J. David Lin

Outline

Outline

Parton Model and Deep Inelastic Scattering

- PDF: probability of constituent with momentum fraction ξ

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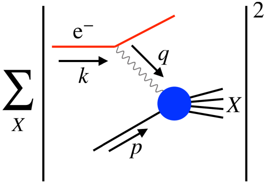
- PDF: probability of constituent with momentum fraction ξ
- test: deep inelastic scattering (DIS)
- large momentum transfer $Q^2 = -q^2$
- Bjorken parameter $\xi = \frac{Q^2}{2P\cdot q}$

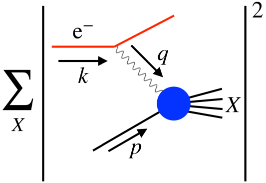


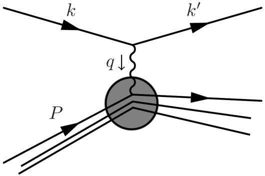
Parton Model and Deep Inelastic Scattering

- PDF: probability of constituent with momentum fraction ξ
- test: deep inelastic scattering (DIS)
- large momentum transfer $Q^2 = -q^2$
- Bjorken parameter $\xi = \frac{Q^2}{2P\cdot q}$
- QCD asymptotically free: scattering on free parton
- scattering amplitude factorizes:

$$\sigma(\xi, Q^2) = \hat{\sigma}(\xi, Q^2)\ f_\Psi(\xi)$$

experiment (σ) — perturbative ($\hat{\sigma}$) — non-perturbative (f_Ψ)



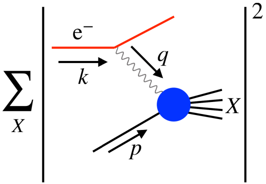


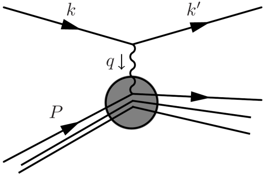
[Schwartz 2014]

Parton Model and Deep Inelastic Scattering

Parton Distribution Function (PDF):

- universal
- PDF: probability of parton with momentum fraction ξ





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Parton Model and Deep Inelastic Scattering

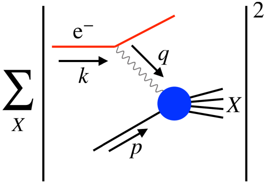
Parton Distribution Function (PDF):

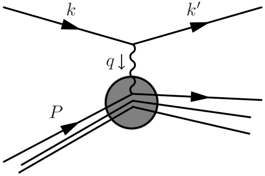
- universal
- PDF: probability of parton with momentum fraction ξ

$$f(\xi) = \sum\!\!\!\!\!\!\!\int_X \int d^4p\,\delta(P^\mu - p^\mu - p_X^\mu)\,\delta(\xi n\cdot P - n\cdot p)\left|\langle P|\psi^\dagger(0)|X\rangle\right|^2$$

$n\cdot p = \xi n\cdot P$, n light cone vector

$$\rightarrow \boxed{f(\xi) = \int dz^- e^{-i\xi P^+ z^-}\left\langle P\middle|\bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0)\middle|P\right\rangle}$$





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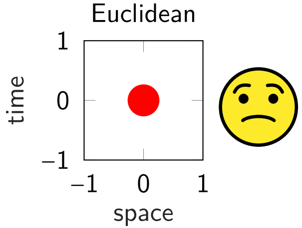
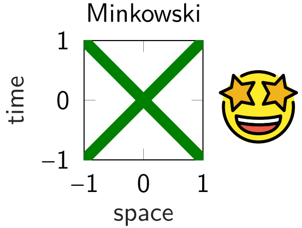
- integration along lightcone direction z^+
- lattice QCD in euclidean space: lightcone $\rightarrow$ point
- Hamiltonian formalism: lightcone in Minkowski space
- $\rightarrow$ use tensor network states/quantum devices





Outline

Tensor Networks

- generic state scales exponentially

$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \Psi^{s_1 s_2 \ldots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \ldots \otimes |s_N\rangle$$

Tensor Networks

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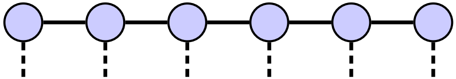
- tensor network state as ansatz

$$\Psi^{s_1 s_2 \ldots s_N} = \sum_{\{i_x\}} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \ldots A^{N,s_N}_{i_{N-1}}$$

- 1d: matrix product state (MPS)

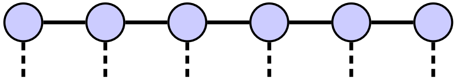
$=$ 

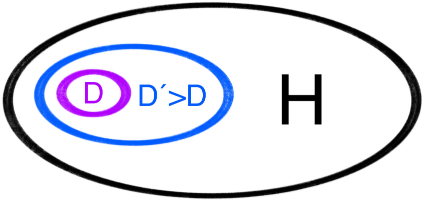
Tensor Networks

- generic state scales exponentially
- tensor network state as ansatz
- 1d: matrix product state (MPS)
- truncation to bond dimenion D
- polynomial resource scaling

$$|\psi\rangle = \sum_{s_1, s_2, \dots, s_N} \Psi^{s_1 s_2 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle$$

$$\Psi^{s_1 s_2 \dots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \dots A^{N,s_N}_{i_{N-1}}$$

$=$ 



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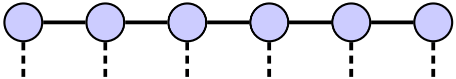
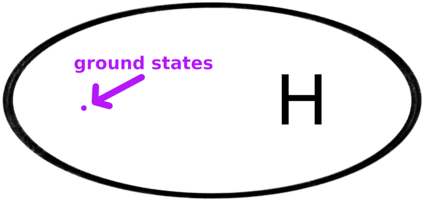
$=$ 

- truncation to bond dimenion D
- polynomial resource scaling
- good approximation for ground states and low excited states
- area laws of entanglement entropy [Hastings 2007]



Singular Value Decomposition (SVD)

$$M_{a,b} = \sum_{i=1}^{\dim(a)} \sum_{j=1}^{\dim(b)} U_{a,i} \cdot S_{i,j} \cdot V_{j,b}$$

unitary: $U^\dagger U = \mathbb{1}$

diagonal: $S_{i,j} = s_i \cdot \mathbb{1}_{i,j}$ with $s_i > 0$

unitary: $VV^\dagger = \mathbb{1}$

truncated Singular Value Decomposition (SVD)

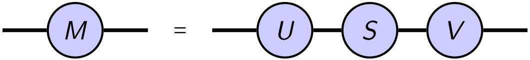
$$M_{a,b} = \sum_{i=1}^{D} \sum_{j=1}^{D} U_{a,i} \cdot S_{i,j} \cdot V_{j,b}$$

column-unitary: $U^\dagger U = \mathbb{1}$

diagonal: $S_{i,j} = s_i \cdot \mathbb{1}_{i,j}$ with $s_i > 0$

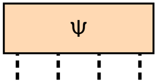
row-unitary: $VV^\dagger = \mathbb{1}$

- exact for $D = \text{rank}(M)$
- approximation for $D < \text{rank}(M)$



MPS through SVD

$$|\psi\rangle = \sum_{s_1, s_2, s_3, s_4} \Psi^{s_1 s_2 s_3 s_4} |s_1\rangle \otimes |s_2\rangle \otimes |s_3\rangle \otimes |s_4\rangle$$



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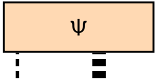

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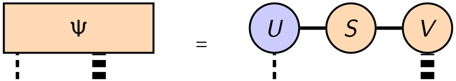

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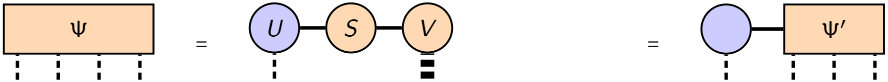

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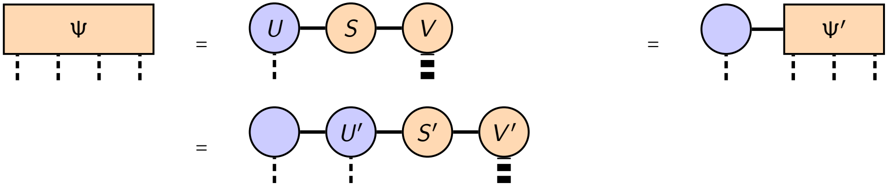

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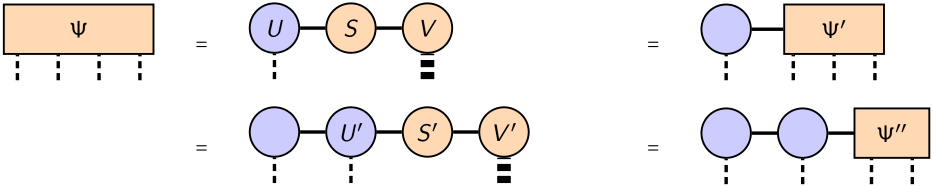

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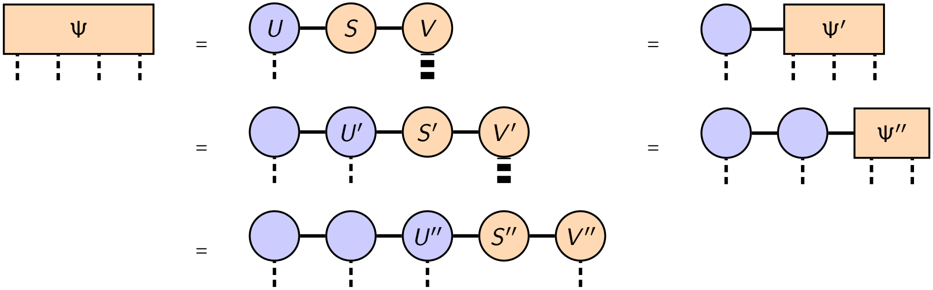

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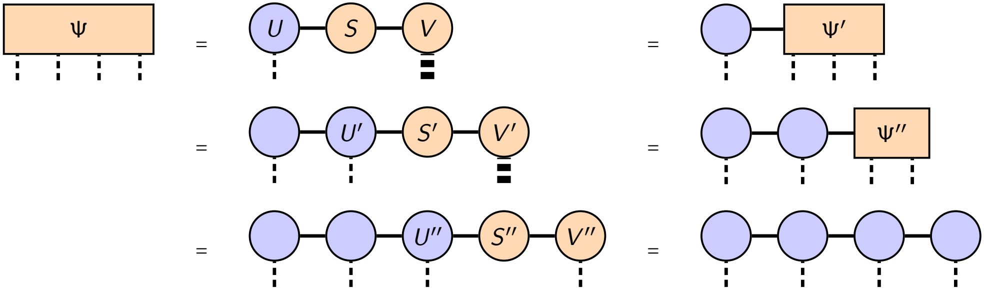
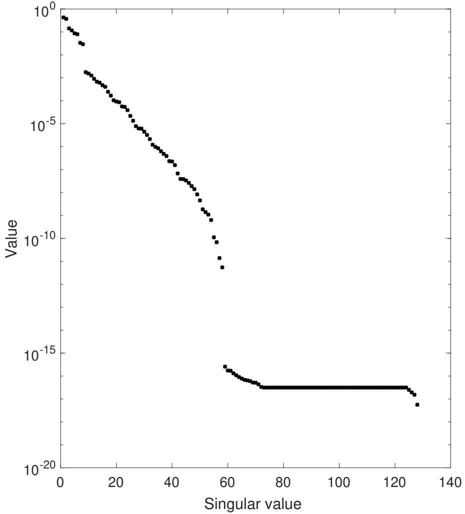
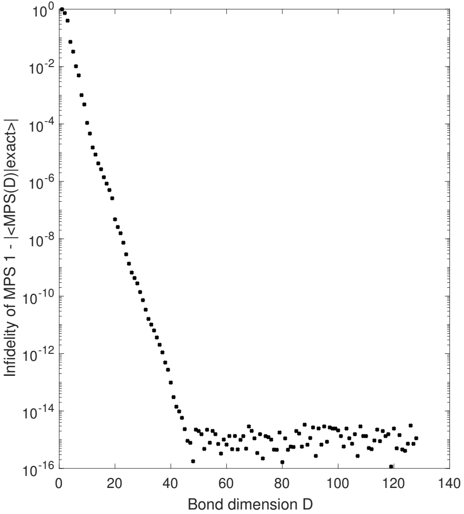


Singular values and cutoff

Schwinger model, $L = 14$, $\mu = 0.125$, $x = 10$, 2nd excitation



Singular values for cut in the middle

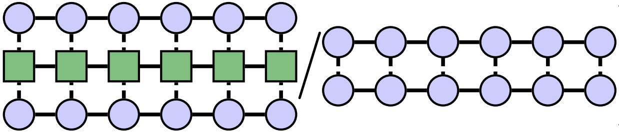


Infidelity on MPS with exact state
$1 - |\langle\Psi(D)|\Psi_{\text{exact}}\rangle|$

Efficient Tensor Network operations

- Find groundstate and excited states

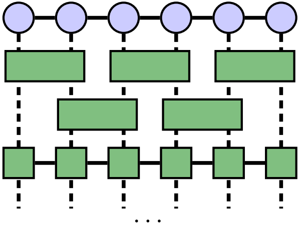
$$\min\left(E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \right.$$



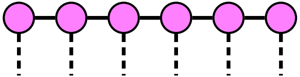
$$\left.\right)$$

- Apply operators / time evolution

$$\hat{O} | \Psi \rangle =$$

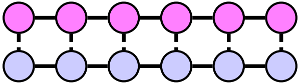


$$\longrightarrow | \Phi \rangle =$$



- Calculate overlap

$$\langle \Psi | \Phi \rangle =$$



Outline

The Schwinger Model [Hamer et al. 1997]

- quantum electrodynamics in 1+1 dimensions, $U(1)$ symmetry
- fermion couples to gauge boson → partons
- bound states → hadrons [Bañuls et al. 2013]
- ⇒ can calculate PDF [Dai et al. 1995]
- Lagrange density:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - g\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- for TN/QC: transform action into spin-model Hamiltonian

Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - gA\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$

Legendre transformation
$(E = F_{01})$

Spin formulation of the Schwinger Model

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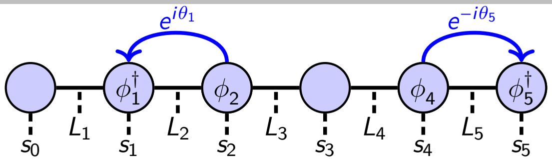

$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger e^{i\theta_n}\phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

staggered fermions
$(\theta = agA_1, gL = E)$

$$\phi_n \sim \begin{cases}\Psi_{\text{upper}}(x) & \text{if } n \text{ even}\\ \Psi_{\text{lower}}(x) & \text{if } n \text{ odd,}\end{cases}$$

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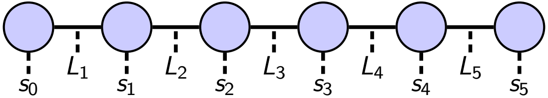

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decoupling

$$\phi_n \to \prod_{k<n}\left(e^{-i\theta_k}\right)\phi_n$$

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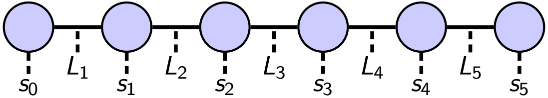

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$$H = \frac{1}{2a}\sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_{n+1}^- \sigma_n^+) + \frac{m}{2}\sum_n [1 + (-1)^n \sigma_n^z] + \frac{ag^2}{2}\sum_n L_n^2$$

Jordan-Wigner transformation

$$\hat{\phi}_n = \prod_{k<n} (i\sigma_k^z)\,\sigma_n^-$$

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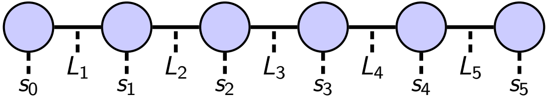
$$H = \frac{1}{2a}\sum_n(\sigma_n^+\sigma_{n+1}^- + \sigma_{n+1}^-\sigma_n^+) + \frac{m}{2}\sum_n[1 + (-1)^n\sigma_n^z] + \frac{ag^2}{2}\sum_n L_n^2$$



Gauss's law:
$(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2})$

$$L_n - L_{n-1} = \frac{1}{2}[(-1)^n + \sigma_n^z] + q_n$$

Spin formulation of the Schwinger Model

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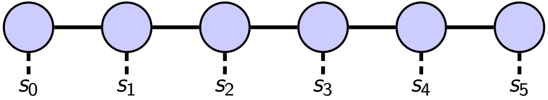
$$\mathcal{H} = -i\bar{\Psi}\gamma^1(\partial_1 - igA_1)\Psi + m\bar{\Psi}\Psi + \frac{1}{2}E^2$$



$$H = -\frac{i}{2a}\sum_n\left(\phi_n^\dagger \cancel{e^{i\theta_n}}\phi_{n+1} - \phi_{n+1}^\dagger \cancel{e^{-i\theta_n}}\phi_n\right) + m\sum_n(-1)^n\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_n L_n^2$$

$$H = \frac{1}{2a}\sum_n(\sigma_n^+\sigma_{n+1}^- + \sigma_{n+1}^-\sigma_n^+) + \frac{m}{2}\sum_n[1 + (-1)^n\sigma_n^z] + \frac{ag^2}{2}\sum_n L_n^2$$

$$H = x\sum_{n=0}^{N-2}[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+] + \frac{\mu}{2}\sum_{n=0}^{N-1}[1 + (-1)^n\sigma_n^z] + \sum_{n=0}^{N-2}\left[\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z + 2q_k\right)\right]^2$$

Gauss's law:

$$L_n - L_{n-1} = \frac{1}{2}[(-1)^n + \sigma_n^z] + q_n$$

$\left(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2}\right)$

Lightfront matrix elements

$$\left\langle P \middle| \bar{\Psi}(z^+)\gamma^+ W(z^+ \leftarrow 0)\Psi(0) \middle| P \right\rangle$$
$$\rightarrow \mathcal{M}_{(\mathrm{e,e})} + \mathcal{M}_{(\mathrm{o,o})} - \mathcal{M}_{(\mathrm{o,e})} - \mathcal{M}_{(\mathrm{e,o})}$$
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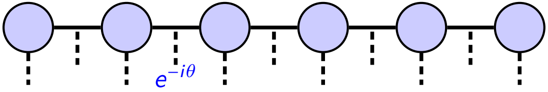
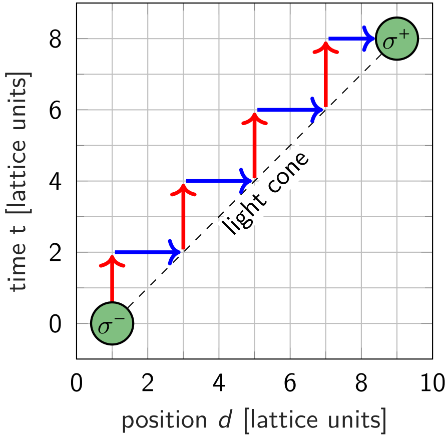
- lightcone
  → small time- and space-like steps
- time evolution:
  $e^{-i\tau H} \approx \left(e^{-i\delta\tau H_{eo}} e^{-i\delta\tau H_{oe}} e^{-i\delta\tau H_L}\right)^{N_\tau}$
- spatial evolution:
  change electric field along the path
  →move static charges





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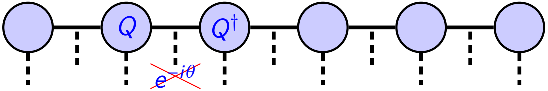

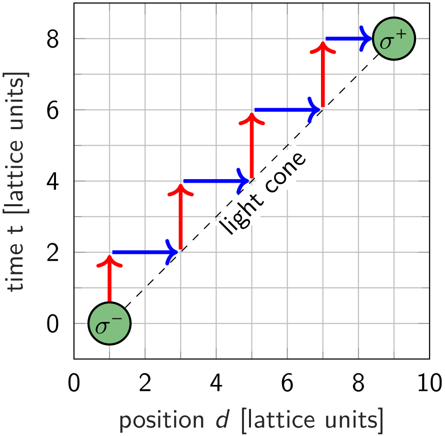

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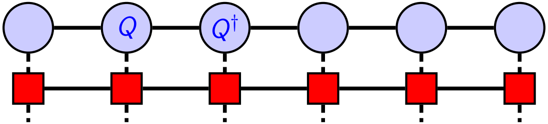

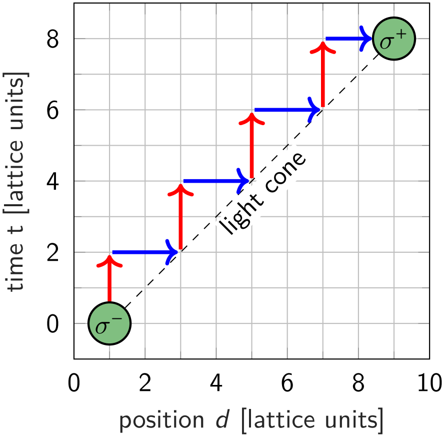

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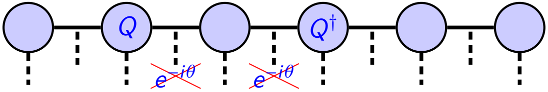

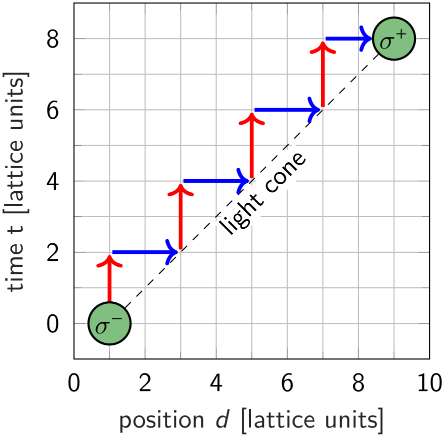

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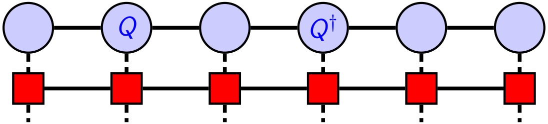

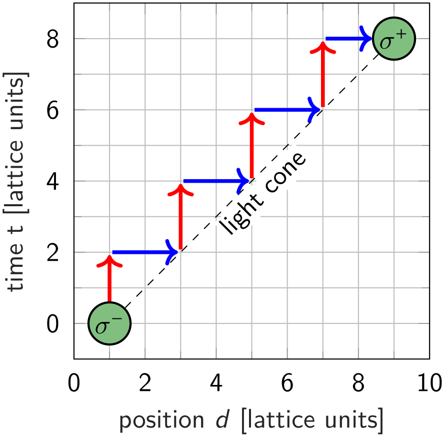

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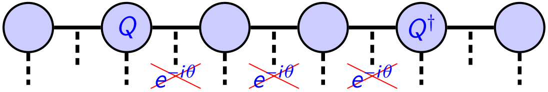

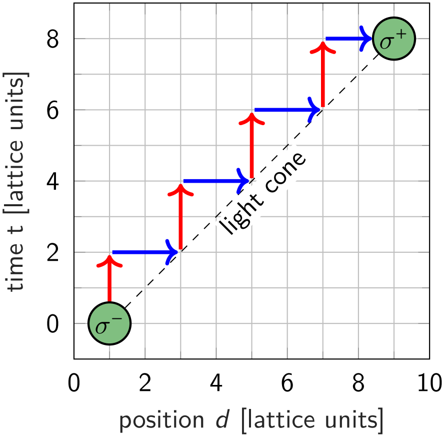

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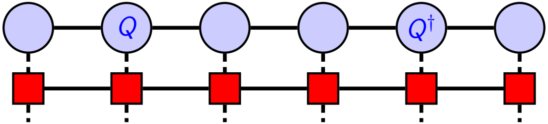
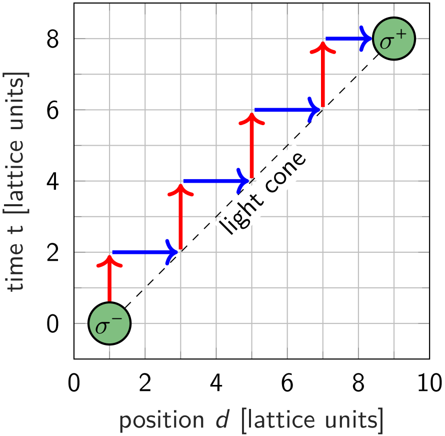
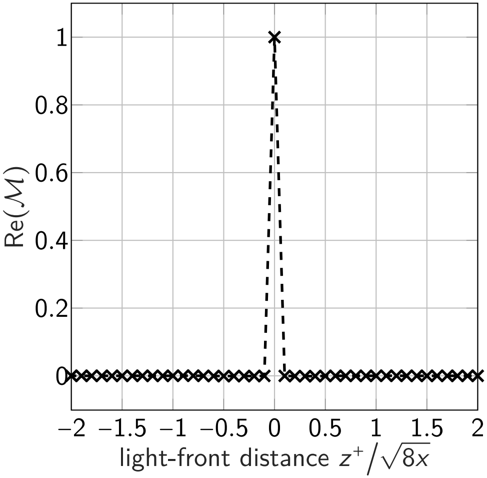


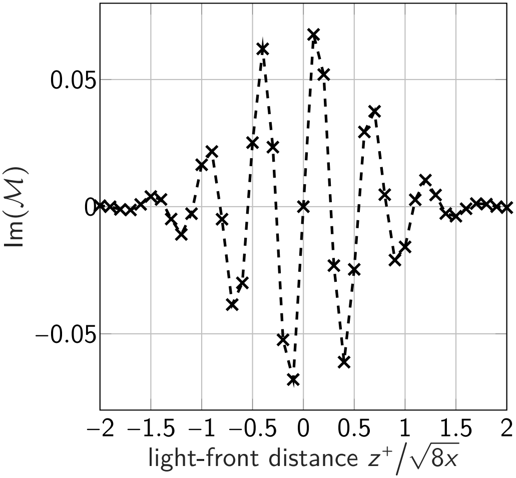


Outline

Results: Matrix elements

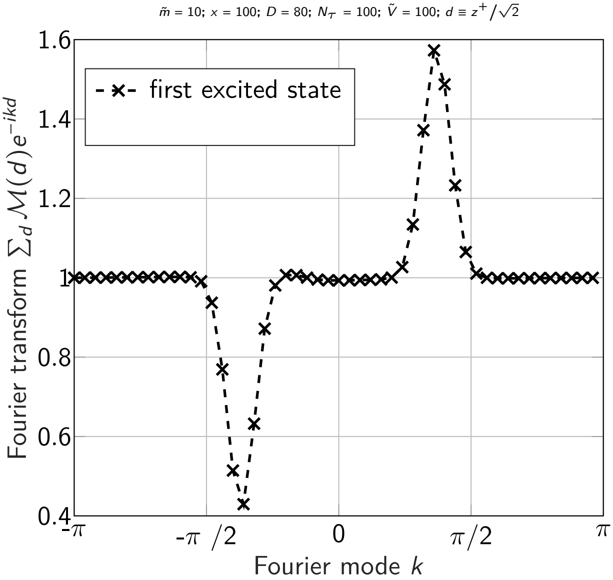
$\tilde{m} = \frac{m}{g}\sqrt{\pi} = 10;\ x = 100;\ D = 80;\ N_\mathcal{T} = \frac{\mathcal{T}}{\delta\mathcal{T}} = 100;\ \tilde{V} = \tilde{m}\frac{N}{\sqrt{x}} = 100$





$$\mathcal{M} = \mathcal{M}_{ee} + \mathcal{M}_{oo} - \mathcal{M}_{eo} - \mathcal{M}_{oe}$$

Results: Fourier transform of matrix elements



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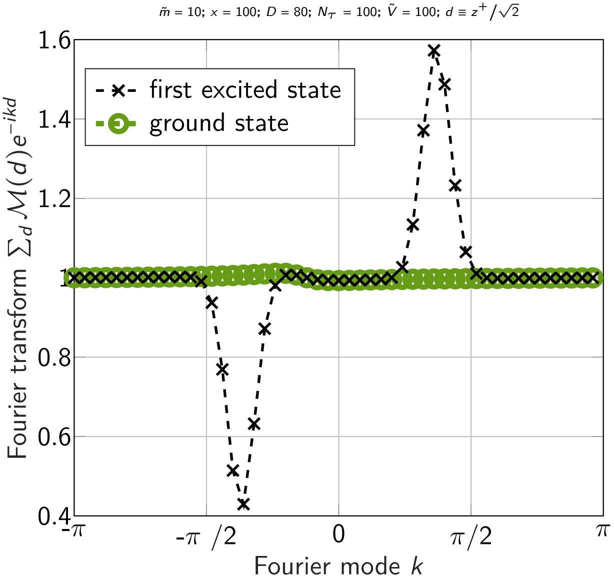

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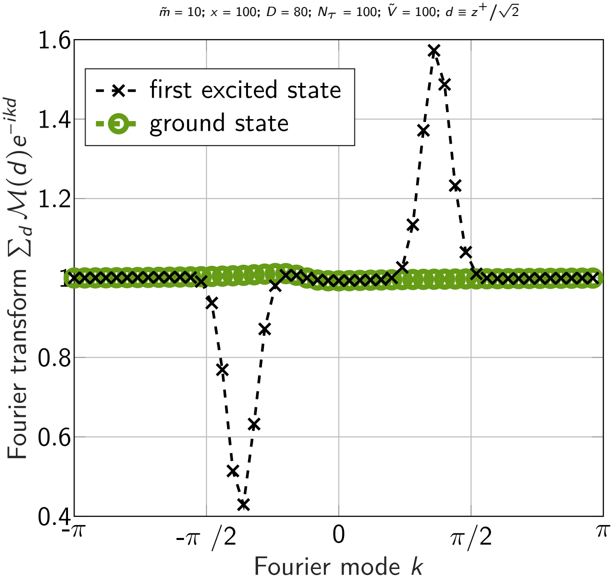
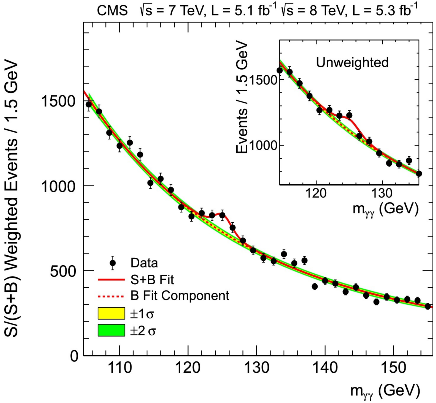


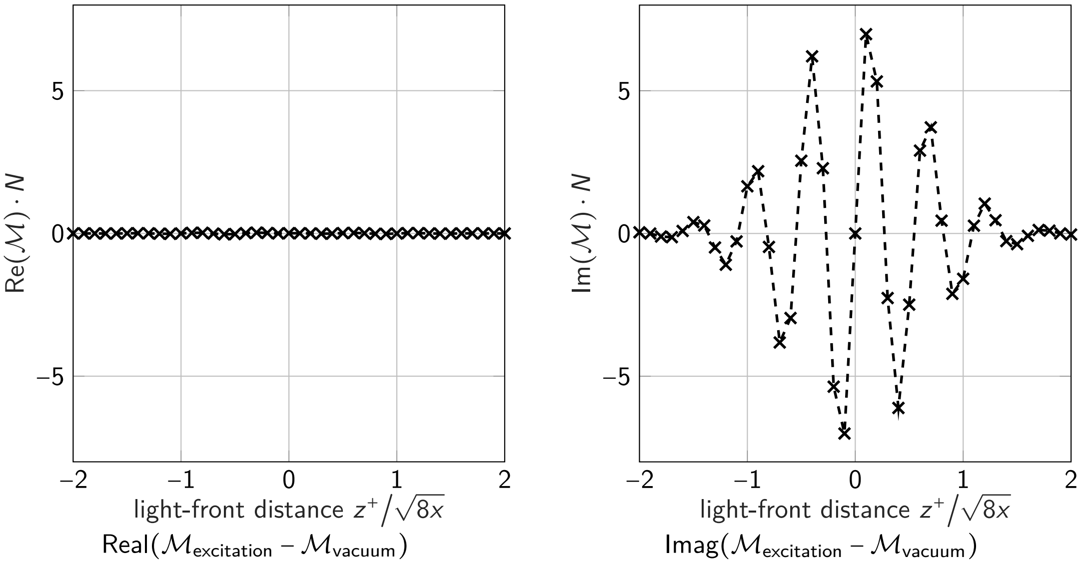


[CMS collaboration 2012]

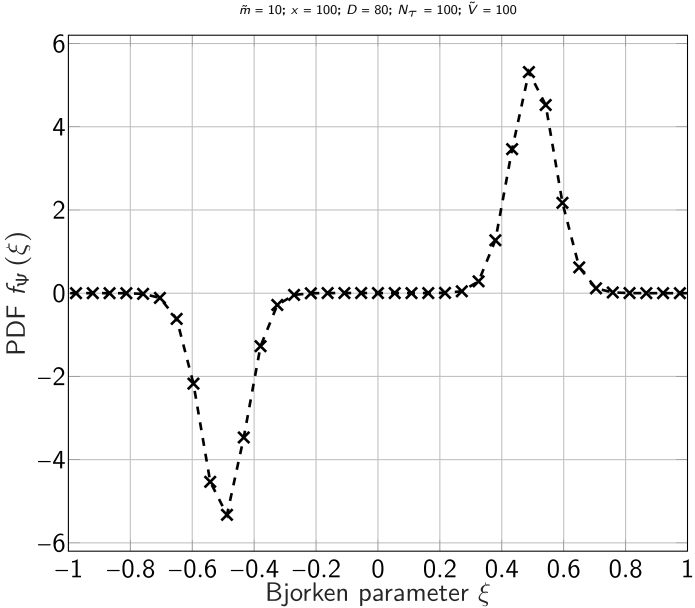
Do we observe a particle here?

Results: Subtracted matrix elements [Collins 2011]

$\tilde{m} = 10;\ x = 100;\ D = 80;\ N_T = 100;\ \tilde{V} = 100$



Results: PDF



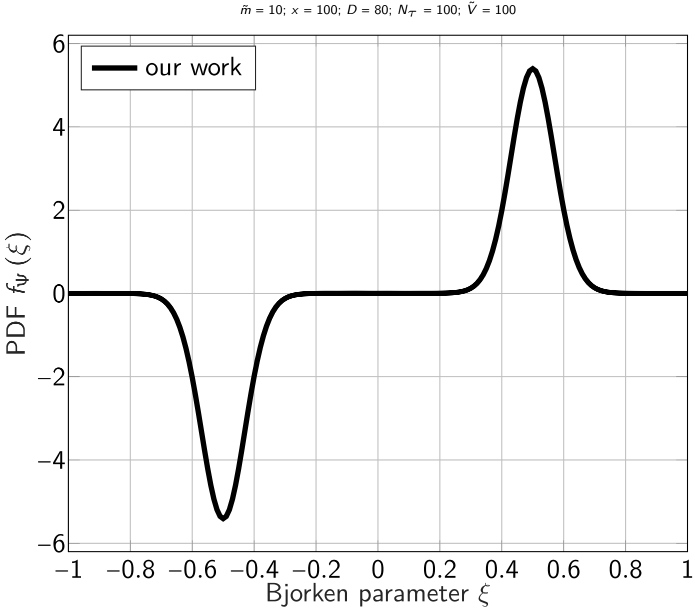
observations:

- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- antifermion PDF from negative ξ:

$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

- observed symmetry
  $\rightarrow f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓

Results: PDF



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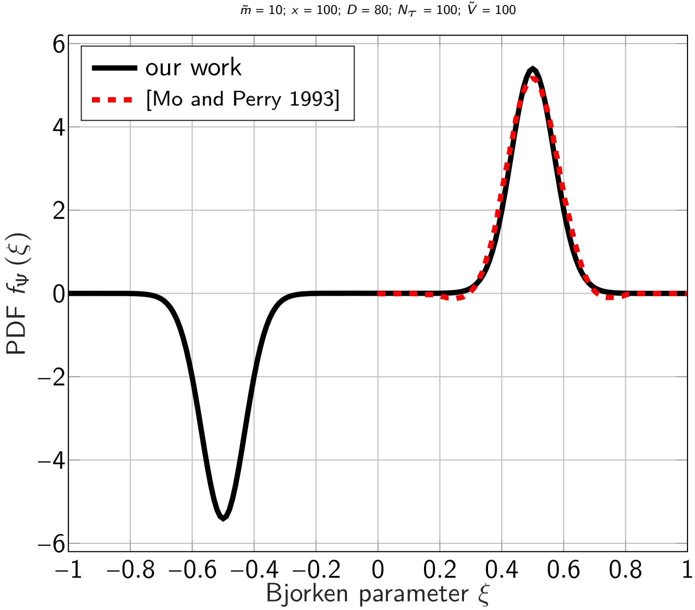

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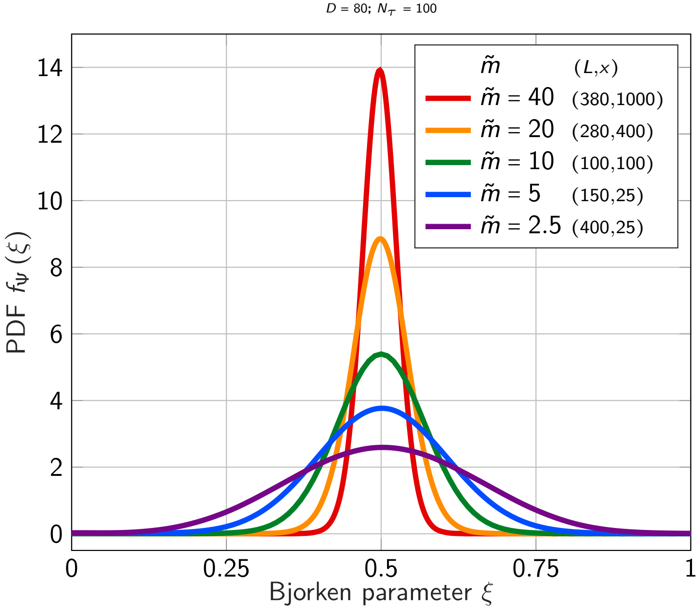

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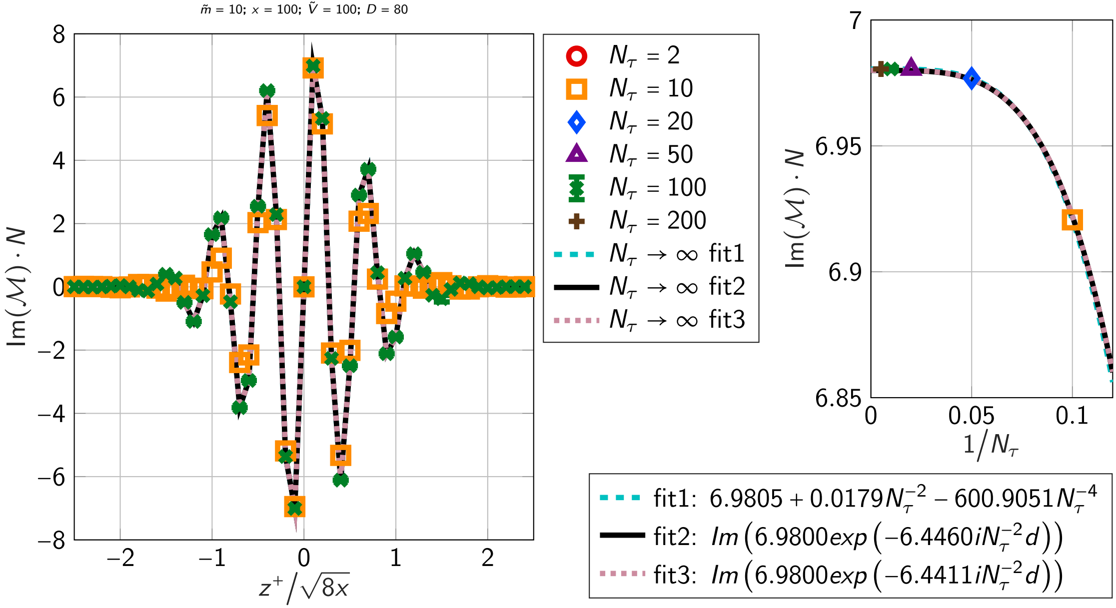
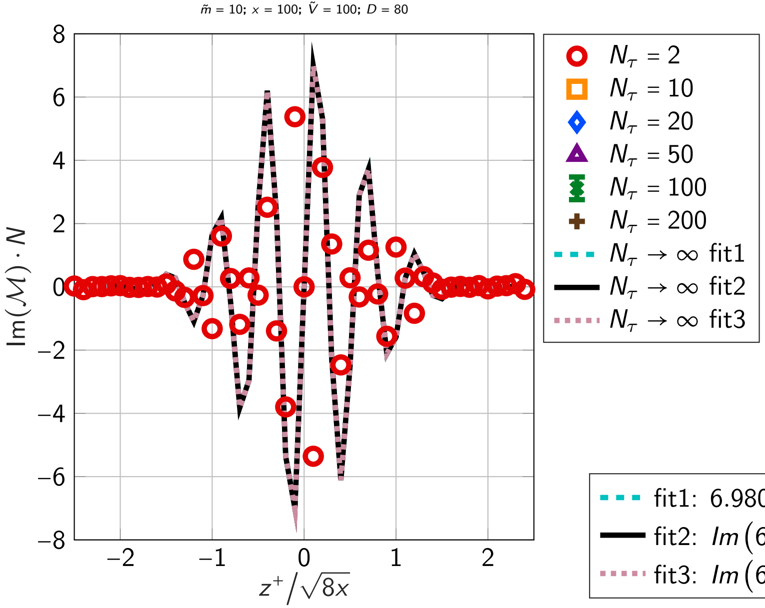
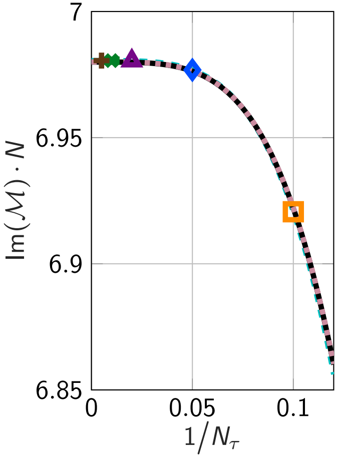
  $\rightarrow f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓
- peak broadens with decreasing fermion mass ✓

Results: N_τ-dependence



Results: N_τ-dependence – too small



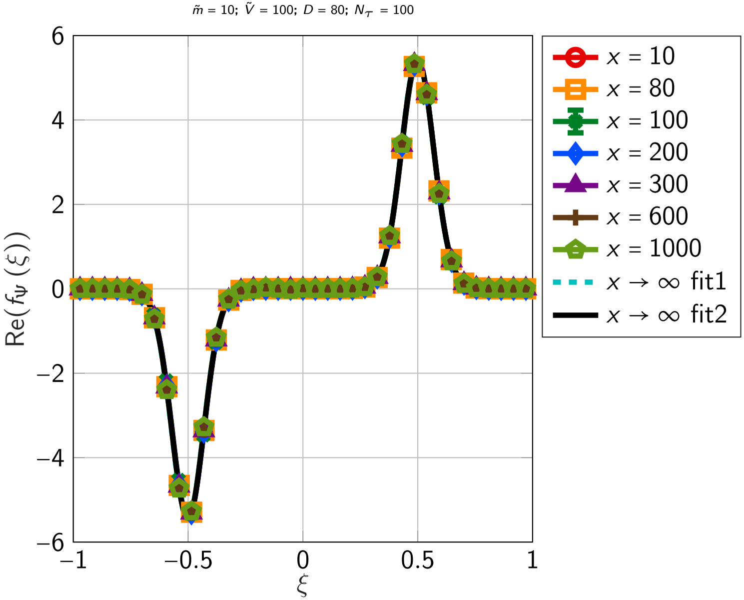


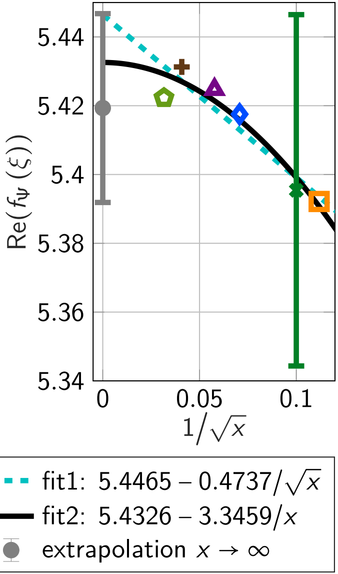
fit1: $6.9805 + 0.0179 N_\tau^{-2} - 600.9051 N_\tau^{-4}$

fit2: $Im\left(6.9800 exp\left(-6.4460 i N_\tau^{-2} d\right)\right)$

fit3: $Im\left(6.9800 exp\left(-6.4411 i N_\tau^{-2} d\right)\right)$

Results: x-dependence





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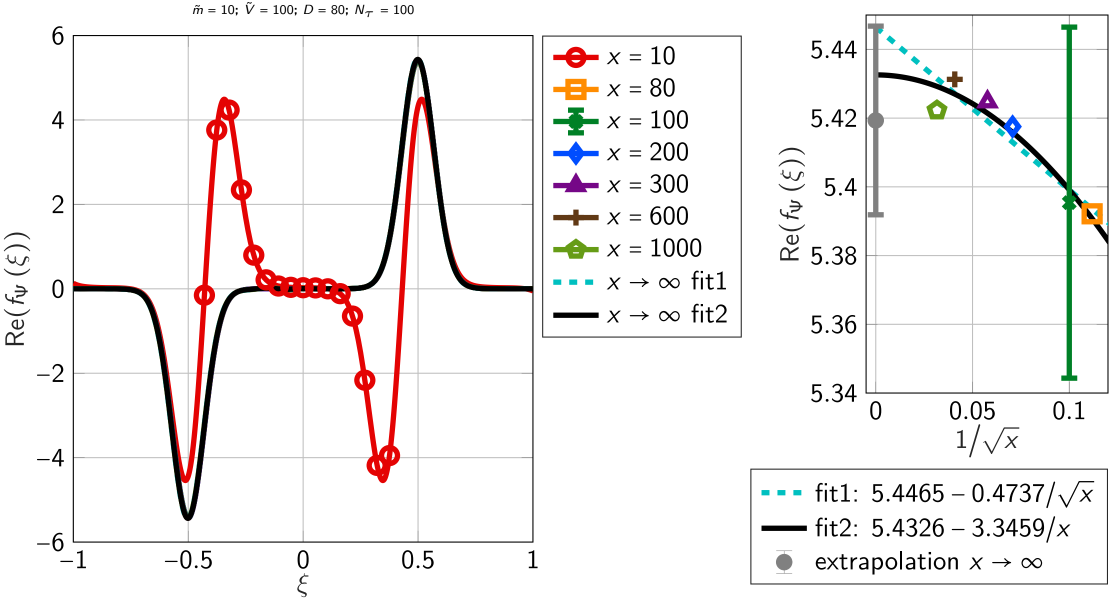
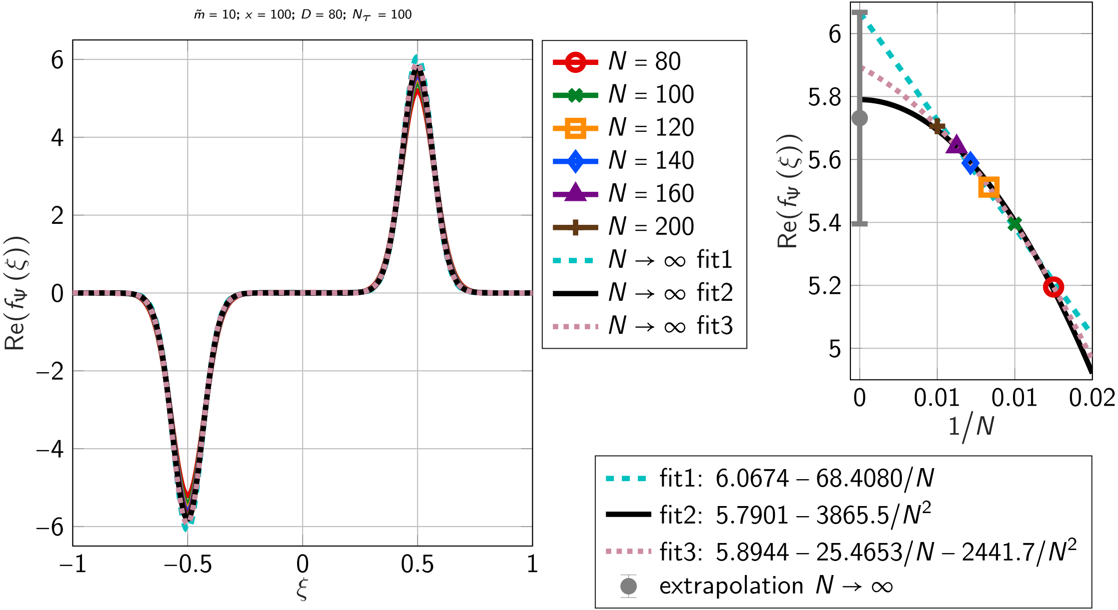
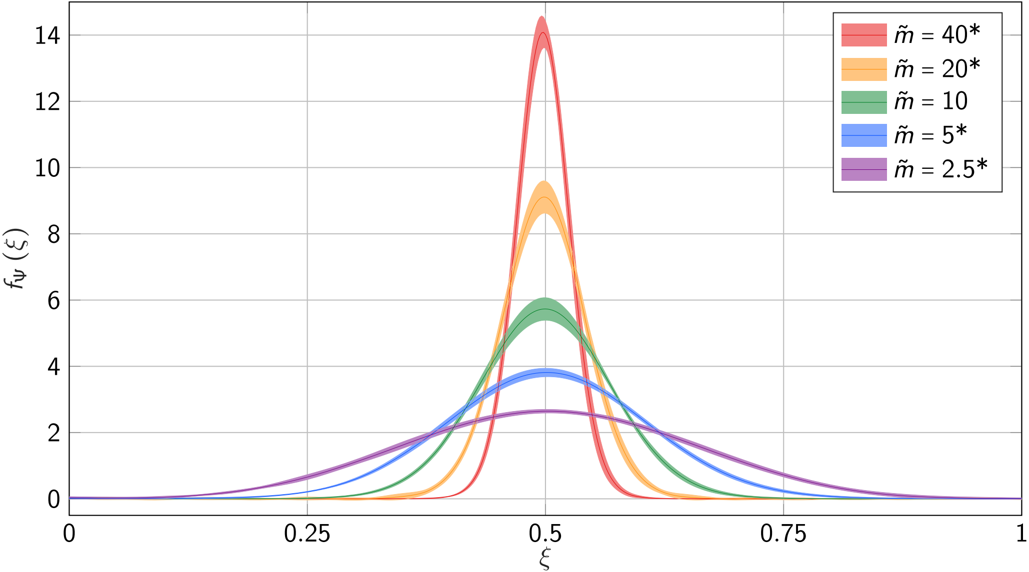


Results: N-dependence

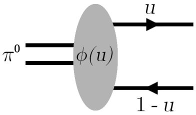


Results: PDF [*preliminary]



Outlook: lightcone distribution amplitude (LCDA) [very preliminary]

LCDA: decay or hadronization in inclusive processes



$$\pi^0 \to q(u)\bar{q}(1-u)$$
$$\to \gamma\gamma *$$

$$if\phi(u) = \int dz^- e^{iuP^+z^-} \langle 0 | \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) | P \rangle$$

Outlook: lightcone distribution amplitude (LCDA) [very preliminary]

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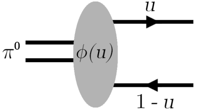

$\pi^0 \rightarrow q(u)\bar{q}(1-u)$
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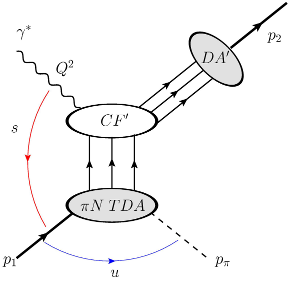

[EIC collaboration 2022]
$N_1\gamma \rightarrow N_2\pi_0$

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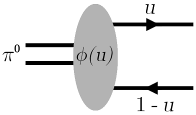

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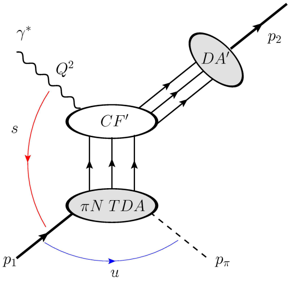

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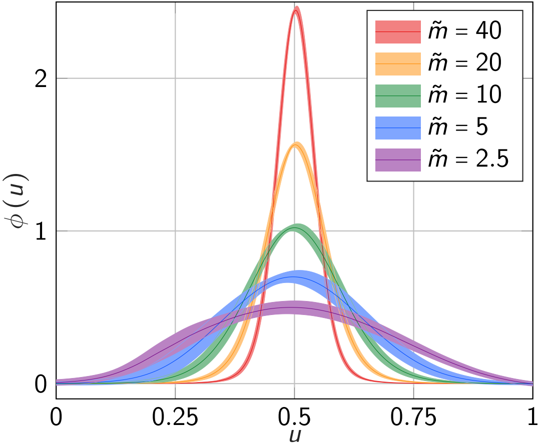

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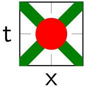
Outline

Summary

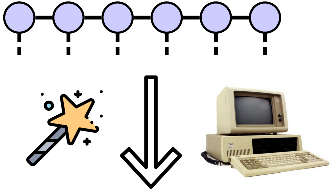
Summary:

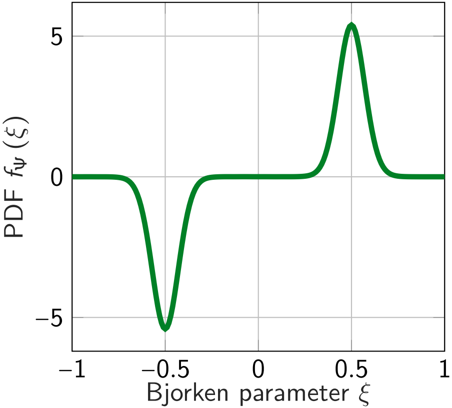
- PDF → universal structure of hadrons

- Euclidean space: lightcone → point



- ⇒ use tensor network states / quantum devices
- Schwinger model:
  fermion- and anti-fermion-PDF for the vector meson





Outlook:

- further lightcone observables
- same analysis for QCD

[arXiv:2504.07508]
[arXiv:2409.16996]

[1] M. C. Bañuls, K. Cichy, C. J. D. Lin, and M. Schneider, "Parton distribution functions in the schwinger model from tensor network states," arXiv e-prints, arXiv:2504.07508 (2025) doi:10.48550/arXiv.2504.07508.

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[3] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, (Cambridge University Press, Mar. 2014), ISBN: 978-1-107-03473-0, 978-1-107-03473-0, doi:10.1017/9781139540940.

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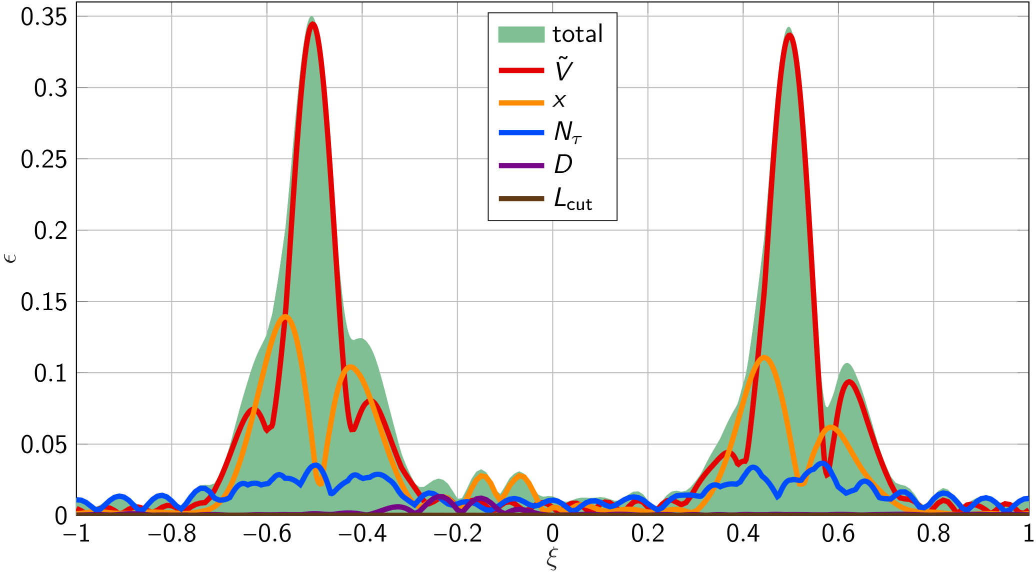
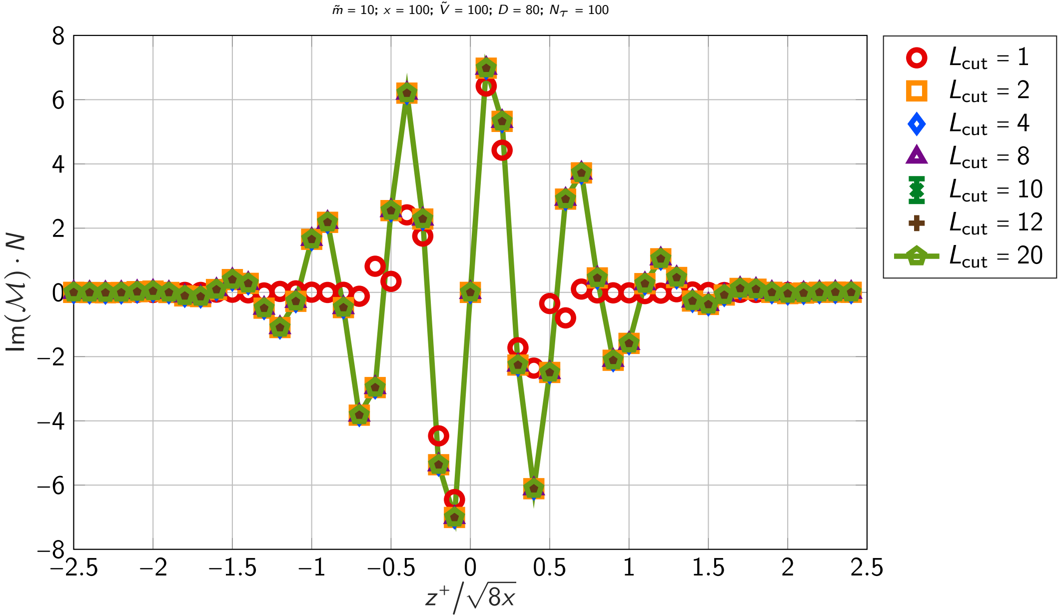
Outline

Contributions to error



L_{cut}-dependence: truncation of electric field



Lattice forumlation of PDF

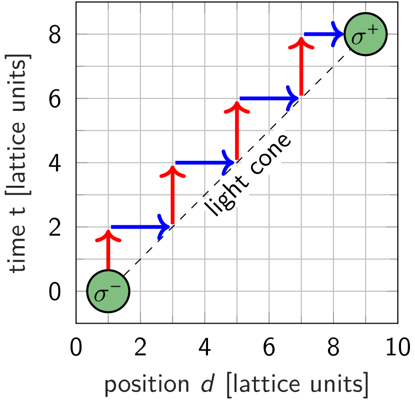
PDF for lattice spin model:

$$f_\Psi(\xi) = \frac{NM}{8\pi x} \sum_{d=0,2,4,\ldots,L} e^{-i\xi\frac{Md}{2x}} \mathcal{M}(d)$$

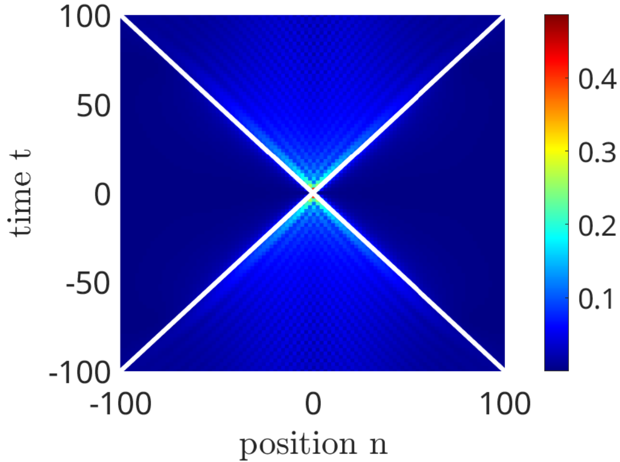
$$\mathcal{M}(d) = \mathcal{M}_{0,0}(d) - \mathcal{M}_{0,1}(d) - \mathcal{M}_{1,0}(d) + \mathcal{M}_{1,1}(d)$$

$$\mathcal{M}_{a,b}(d) = \langle h| e^{iHt_d} \prod_{k<d+a} (-i\sigma_k^z)\, \sigma_{d+a}^+ e^{-iH_{d-1}\delta t}$$

$$\ldots e^{-iH_3\delta t} e^{-iH_1\delta t} \prod_{k'<b} (i\sigma_{k'}^z)\, \sigma_b^- |h\rangle_c .$$



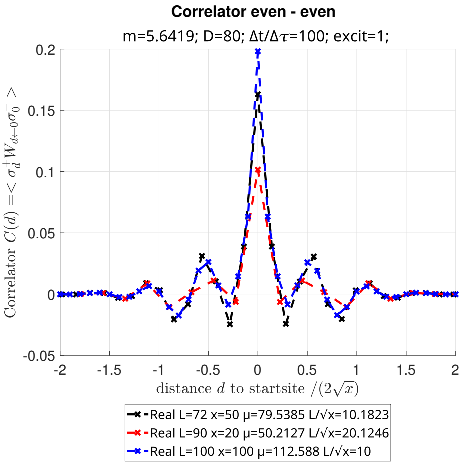
Light-cone structure



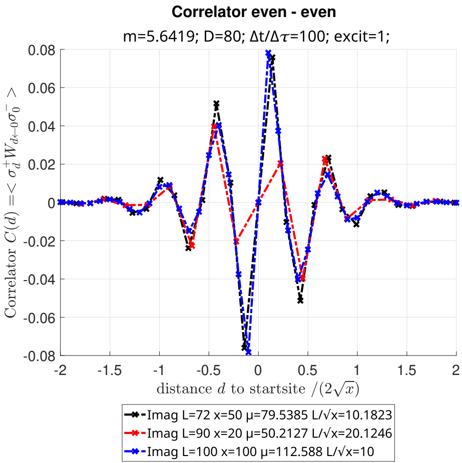
$$\langle P|\, e^{iHt} \prod_{k<n} (i\sigma_k^z)\, \sigma_n^+ e^{-iH_0 t} \prod_{k'<0} (-i\sigma_{k'}^z)\, \sigma_0^- \,|P\rangle$$

- even-to-even matrix element
- calculated to each site at each timeslice
- static charge fixed at orign

Results - correlators



$\mathrm{Re}(C_{ee})$



$\mathrm{Im}(C_{ee})$

Factorization

cross section:

$$\sigma \propto L^{\mu\nu}(k,q)\, W_{\mu\nu}(q,P)$$

hadronic Tensor:

$$W_{\mu\nu}(\xi,P) = \sum_i \int_x^1 \frac{dz}{z} f_i(z)\, \hat{W}_{\mu\nu}\left(\frac{\xi}{z}, Q\right)$$

leading order with $\hat{W} \propto \delta\left(1 - \frac{\xi}{z}\right)$:

$$W_{\mu\nu}(q,P) = 4\pi\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1 + \frac{8\pi x}{Q^2}\left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right) F_2$$

factorization (leading order):

$$F_1(\xi) = \frac{1}{2}\sum_i e_i^2 f_i(\xi)$$
$$F_2(\xi) = 2x F_1(\xi)$$

Light cone PDF details

$$f(\xi) = \sum\!\!\!\!\!\!\int_X \int d^4p\,\delta\left(P^\mu - p^\mu - p_X^\mu\right)\delta\left(\xi n\cdot P - n\cdot p\right)\left|\langle X|\psi(0)|P\rangle\right|^2 \quad (1)$$

$$= \sum\!\!\!\!\!\!\int_X \delta\left(\xi n\cdot P - n\cdot (P - p_X)\right)\left|\langle X|\psi(0)|P\rangle\right|^2 \quad (2)$$

$$= \sum\!\!\!\!\!\!\int_X \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-itn\cdot(\xi P - P + p_X)}\langle P|\psi^\dagger(0)|X\rangle\langle X|\psi(0)|P\rangle \quad (3)$$

$$= \sum\!\!\!\!\!\!\int_X \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-itn\cdot\xi P}\langle P|e^{itn\cdot\hat{P}}\psi^\dagger(0)\,e^{-itn\cdot\hat{P}}|X\rangle\langle X|\psi(0)|P\rangle \quad (4)$$

$$= \sum\!\!\!\!\!\!\int_X \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-it\xi n\cdot P}\langle P|\psi^\dagger(tn)|X\rangle\langle X|\psi(0)|P\rangle \quad (5)$$

$$= \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-it\xi n\cdot P}\langle P|\psi^\dagger(tn)\,\psi(0)|P\rangle \quad (6)$$

Wilson line (gauge invariance), γ-structure (projection of Dirac structure), $t \to z^+$, $n\cdot P = P^-$:

$$\boxed{f(\xi) = \int \frac{dz^+}{2\pi}e^{-iz^+\xi P^-}\left\langle P\left|\psi^\dagger(z^+)\gamma^+W(z^+\leftarrow 0)\psi(0)\right|P\right\rangle} \quad (7)$$

Error sources and extrapolations

- initial state: $D \to \infty$
- entanglement growth in time evolution: $\chi \to \infty$
- Trotter error: $N_\tau = \frac{\delta\tau}{\tau} \to 0$
- continuum Limit: $x \to \infty$ (which means $a \to 0$)
- volume: $\frac{L}{\sqrt{x}} \to \infty$
- electric field cutoff: $L_{\text{cut}} \to \infty$
- numerical optimization error: tol $\to 0$

Algorithm

$$\mathcal{M}_{(e,e)}(2n,2m) = \langle P|\, e^{iHt_nH} \prod_{k<2n} (i\sigma^z_k)\sigma^+_{2n}\, Q_{2n+1}\, e^{-iH\tau H_q} \dots Q^\dagger_{2m-3}\, Q_{2m-1}\, e^{-iH\tau H_q}\, Q^\dagger_{2m-1} \prod_{k'<2m} (-i\sigma^z_{k'})\sigma^-_{2m}\, |P\rangle$$

Input: $|P\rangle$
Output: $\mathcal{M}_{(e,e)}(2n,2m)$

1: apply $\prod_{k'<2m+1}(-i\sigma^z_{k'})\sigma^-_{2m+1}$ to $|P\rangle$
2: insert static charge $Q^\dagger_{2m}$ in Hamiltonian
3: **for** $k = \{2m-2, 2m-4 \dots 2n\}$ **do**
4: trotterized real time evolution $e^{-i\tau H_q}$
5: move static charge from k to $k+2$
6: **end for**
7: (remove static charge)
8: apply $\prod_{k<2n}(i\sigma^z_k)\sigma^+_{2n}$
9: calculate overlap with $\langle P|$
10: multiply phase $e^{it_n m}$