Lattice QCD Calculation of the First and Second Moment of the Kaon Distribution Amplitude

Speaker: Alex Chang (張聖彬)

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THE HOPE COLLABORATION

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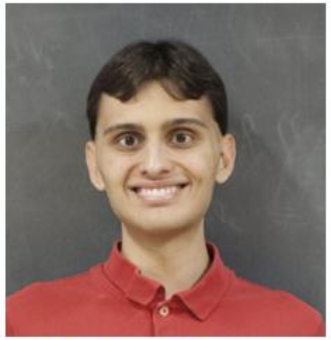

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THE HOPE COLLABORATION

HPCI High Performance Computing Infrastructure

RIKEN R-CCS RIKEN Center for Computational Science

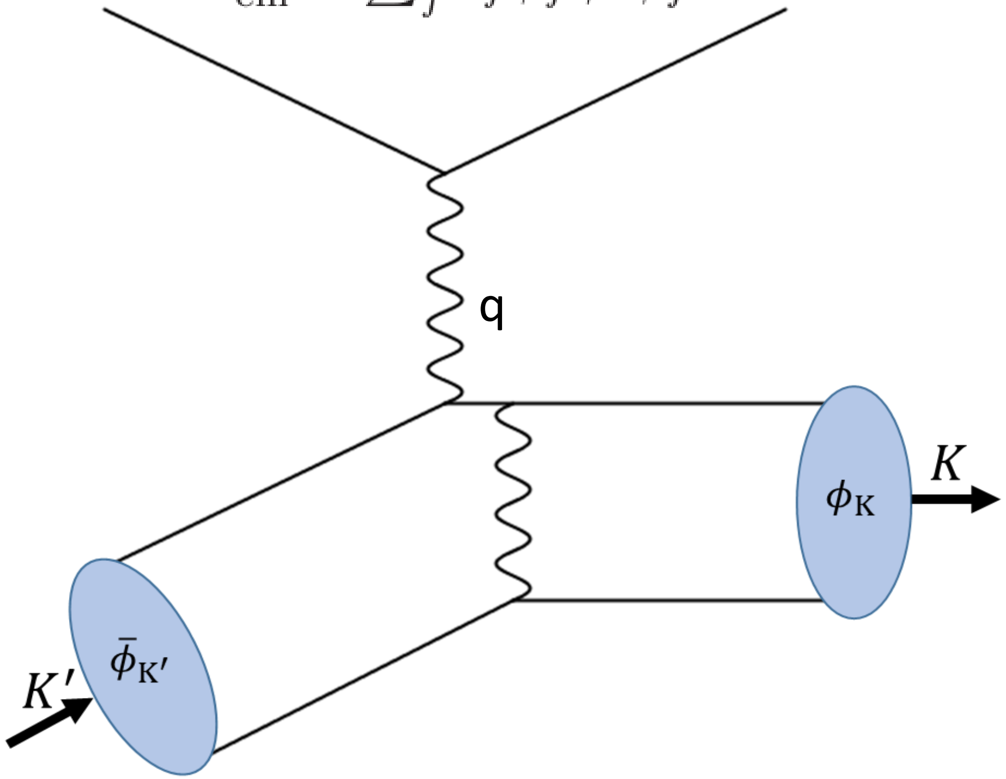
Kaon Electromagnetic Form Factors

◆ Kaon Electromagnetic Form Factors

electron-kaon scattering

$$\langle K^+(P')|J^\mu_{\rm em}|K^+(P)\rangle = F_K(Q^2)(P^\mu + P'^\mu)$$

$$J^\mu_{\rm em} = \sum_f e_f \bar{\psi}_f \gamma^\mu \psi_f$$



Key Motivations

The Internal Structure of Hadrons

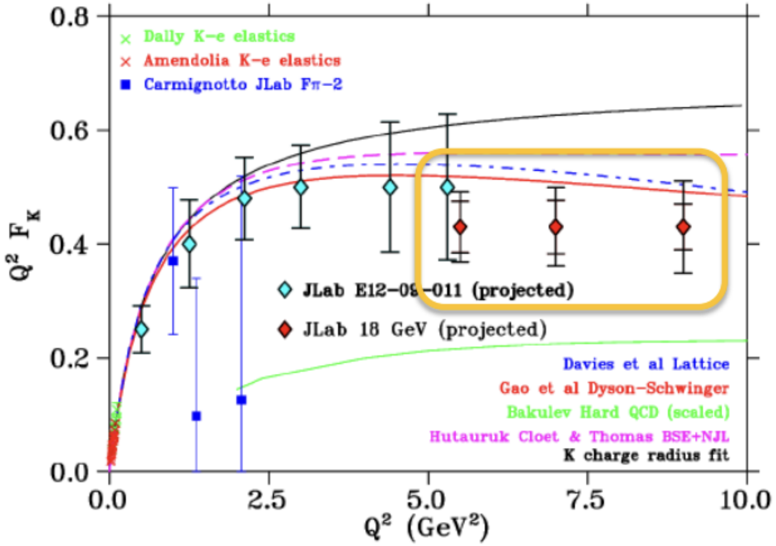
- how quarks and gluons are distributed inside the kaon

Test Predictions of QCD

Understand Flavor Symmetry Breaking

"electron-kaon elastics experiment data"

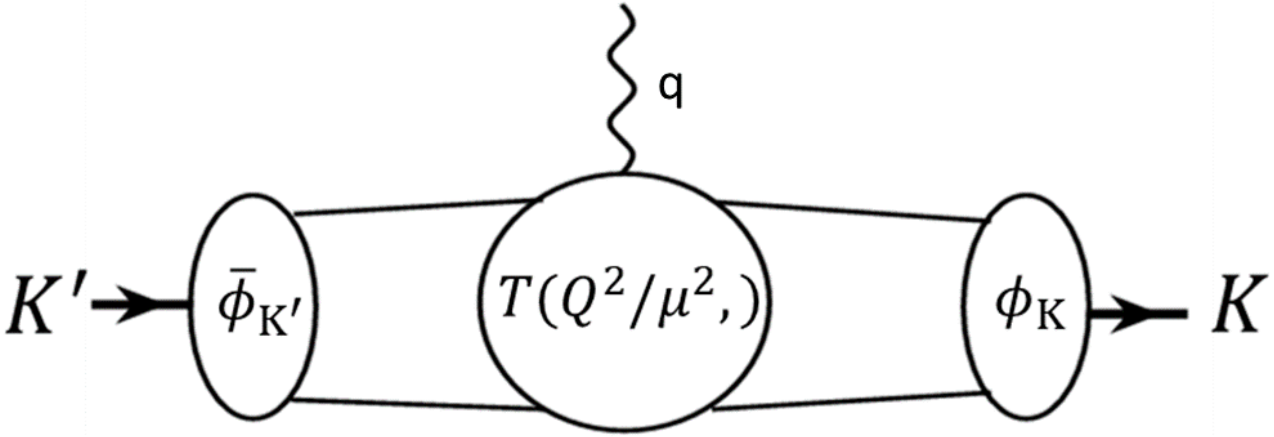
A. Accardi,…, T. Horn, et al., "Strong Interaction Physics at the Luminosity Frontier with 22 GeV electrons at Jefferson Lab", Eur. Phys. J. A **60** (2024) 9, 173



Copy from Tanja Horn's talk

Light-Cone Distribution Amplitudes (LCDAs)

QCD Factorization



$$F_K(Q^2) = \int dxdy\, \bar{\phi}_K(x,Q^2) T(x,y,Q^2) \phi_K(y,Q^2)$$

- Hard scattering kernel (T) calculable in perturbative QCD.

- Light-Cone Distribution Amplitudes (ϕ) encoding the non-perturbative structure.

→ Lattice QCD

Light-Cone Distribution Amplitudes (LCDAs)

Lattice QCD

Euclidean space

A non-perturbative approach to studying QCD on a Euclidean space

LCDAs

Minkowski space

- light cone, defined by $x^2 = 0$

Wick rotation $t \to -i\tau \quad \Rightarrow \quad \text{Minkowski} \to \text{Euclidean}$

$$x_E^2 = \tau^2 + |\vec{x}|^2 \geq 0$$

There is no notion of a light-cone in Euclidean space.

Lattice QCD

Directly

LCDA

➢ Direct calculation of light-cone objects is impossible on a Euclidean lattice

Light-Cone Distribution Amplitudes (LCDAs)

Operator Product Expansion

Light-Cone OPE

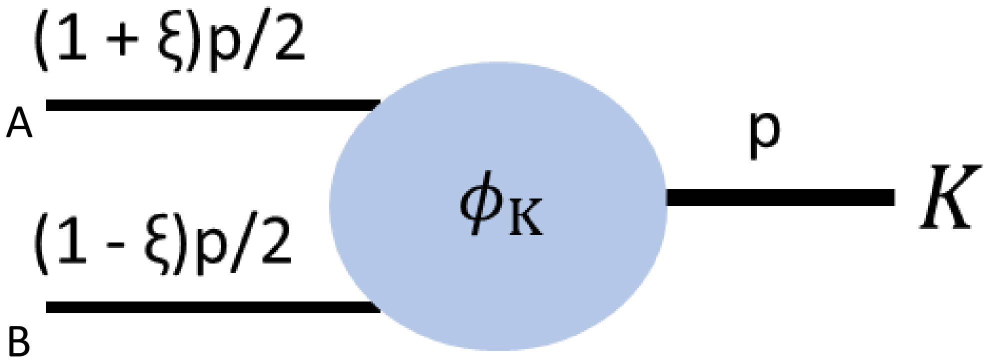
Gegenbauer OPE

The Moments of LCDA: Mellin moments

Heavy-quark Operator Product Expansion

Mellin moments provide the bridge between light-cone physics and Euclidean LQCD.

Light-Cone Distribution Amplitudes (LCDAs)



$$\langle\Omega|\, \bar{\psi}_A(z)\gamma_\mu\, \gamma_5\, W[z,-z]\psi_B(-z)|K^+(p)\rangle \;=\; i f_K\, p_\mu \int_{-1}^{1} d\xi\; e^{-i\xi\, p\cdot\, z}\, \phi_K(\xi,\mu^2)$$

- f_K is the pseudoscalar kaon decay constant and W is light-like Wilson line

LCDA can be interpreted as the probability amplitude for a collinear quark-antiquark pair with momentum fractions

$$\boxed{\begin{array}{l}\phi_K(\xi,\mu^2)\\ \neq \phi_K(-\xi,\mu^2)\end{array}}$$

Light-Cone Distribution Amplitudes (LCDAs)

◆ Gegenbauer Operator Product Expansion

$$\phi_K(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\sum_{n=0}^{\infty}\phi_n(\mu^2)\, C_n^{3/2}(\xi)$$

Gegenbauer moments: $\phi_n(\mu) = \dfrac{2(2n+3)}{3(n+1)(n+2)}\displaystyle\int_{-1}^{1} d\xi\, C_n^{3/2}(\xi)\,\phi_{\mathrm{K}}(\xi,\mu^2)$

Gegenbauer polynomials: $C_0^{3/2}(\xi) = 1, C_1^{3/2}(\xi) = 3\xi, C_2^{3/2}(\xi) = (-3+15\xi^2)/2$

◆ Mellin moments:

$$\langle\xi^n\rangle = \int_{-1}^{1} d\xi\, \xi^n \phi_{\mathrm{K}}(\xi,\mu^2)$$

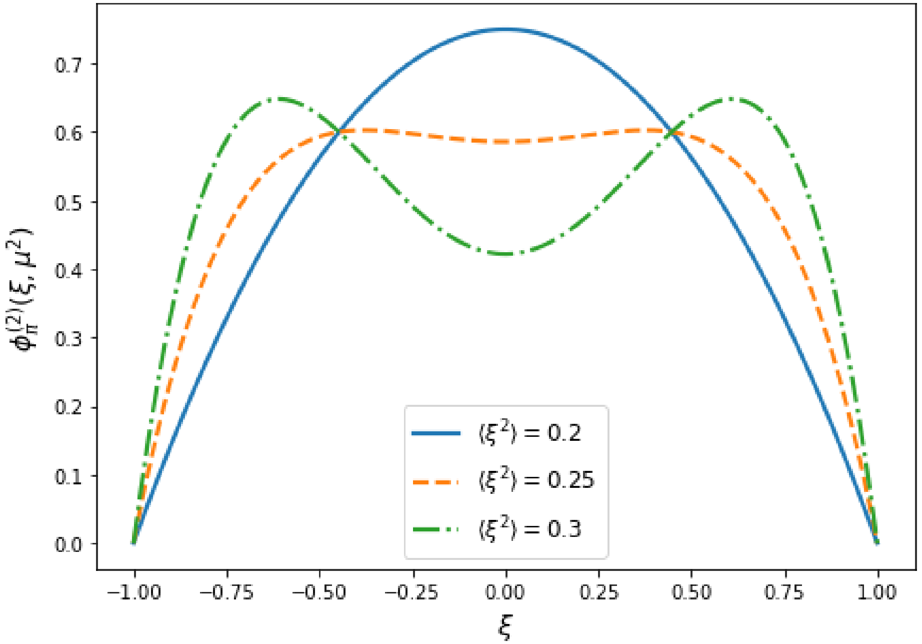
$$\phi_0 = \langle\xi^0\rangle = 1, \qquad \phi_1 = \frac{5}{3}(\langle\xi\rangle), \qquad \phi_2 = \frac{7}{12}(5\langle\xi^2\rangle - \langle\xi^0\rangle)$$

Mellin moments can be expressed as linear combinations of Gegenbauer moments.

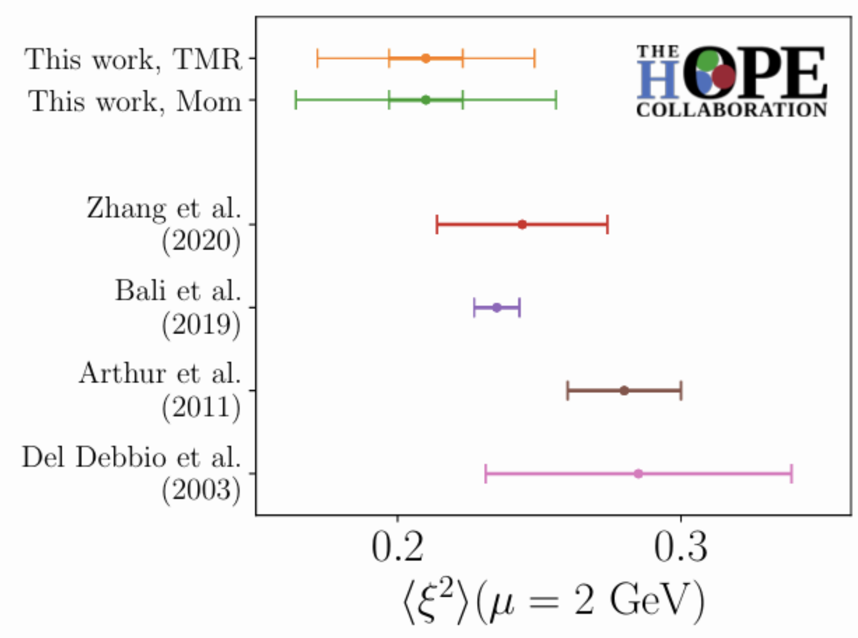
Light-Cone Distribution Amplitudes (LCDAs)

◆ Gegenbauer Operator Product Expansion

$$\phi_\pi^{(2)}(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\sum_{n=0,even}^{2} \phi_n(\mu^2)\, C_n^{3/2}(\xi)$$



single-humped or double-humped structure

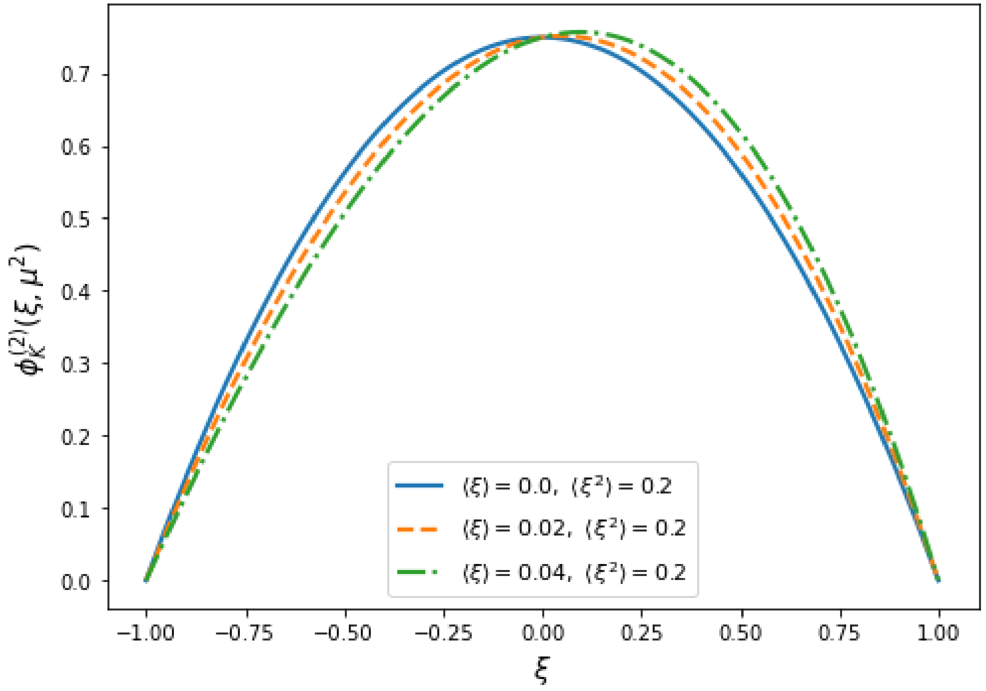


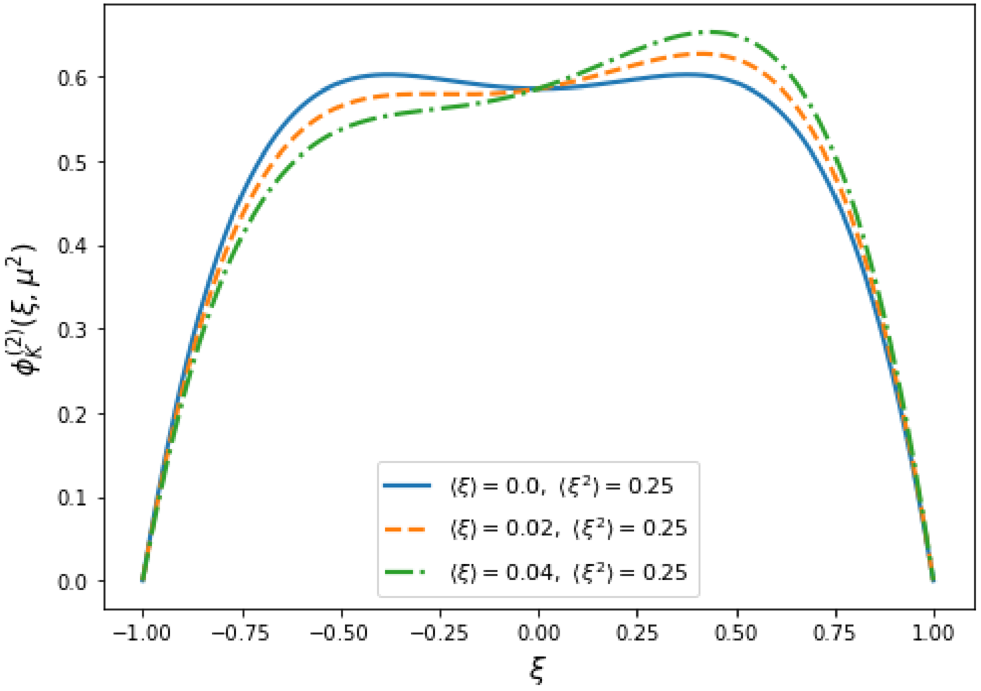
[1] William Detmold, Anthony V. Grebe, Issaku Kanamori, C.-J. David Lin, Robert J. Perry, & Yong Zhao. (2022). Parton physics from a heavy-quark operator product expansion: Lattice QCD calculation of the second moment of the pion distribution amplitude.

Light-Cone Distribution Amplitudes (LCDAs)

◆ Gegenbauer Operator Product Expansion

$$\phi_K^{(2)}(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\sum_{n=0}^{2} \phi_n(\mu^2)\, C_n^{3/2}(\xi)$$





The Kaon LCDA asymmetry is clearly visible when $\langle\xi\rangle>0$.

Light-Cone Distribution Amplitudes (LCDAs)

◆ Light-Cone OPE

$$\boxed{\bar{\psi}(0)\gamma^{+}\mathcal{W}(0,z^{-})\,\psi(z^{-}) = \sum_{n=0}^{\infty}\frac{(z^{-})^{n}}{n!}\,\bar{\psi}(0)\gamma^{+}(iD^{+})^{n}\psi(0)}$$

$$\langle 0|\underbrace{\left[\,\bar{\psi}_A\,\gamma^{\{\mu_0}\gamma_5\left(i\overleftrightarrow{D}^{\mu_1}\right)\ldots\left(i\overleftrightarrow{D}^{\mu_n\}}\right)\psi_B \;-\; trace\right]}_{\substack{\text{local \& twist-two} \\ O^{\mu_0\mu_1\ldots\mu_n}}}|K^{+}(p)\rangle$$

twist = dim - spin

$$= f_K\,\langle\xi^{n}\rangle\left[\,p^{\mu_0}\,p^{\mu_1}\ldots\,p^{\mu_n}\;-\;trace\right]$$

$\{\cdots\}$ denotes total symmetrization of the Lorentz indices

Each Mellin moment is related to a matrix element of a local, twist-two operator in the light-cone OPE.

Heavy-quark Operator Product Expansion (HOPE)

Limitations of Traditional OPE on Lattice:

Each Mellin moment corresponds to a **local twist-2 operator**
Higher moments → higher spin → more derivatives.

On the lattice:
The loss of symmetry on the lattice leads to severe **operator mixing, power divergences** and **renormalization challenges**, especially for high-spin (high-moment) operators.

→ Traditional OPE infeasible beyond the first few Mellin moments

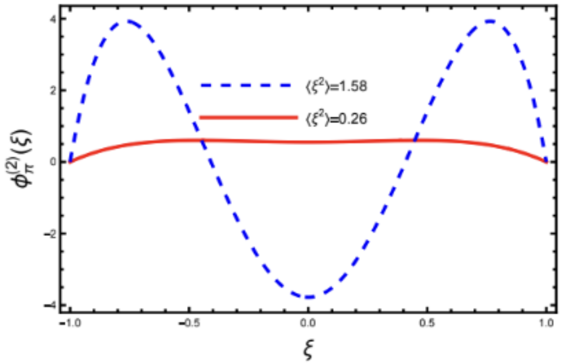
The lattice regularization breaks the full rotation group SO(4)

For operators of spin n > 4 they mix with lower dimension operators and the mixing coefficients contain power divergences.

Early lattice results

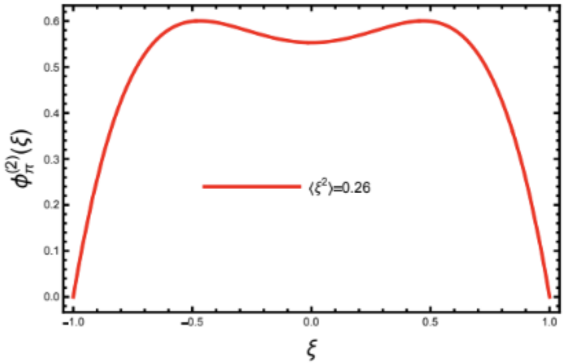
- $\langle \xi^2 \rangle = 1.58 \pm 0.23$

S. Gottlieb and A. Kronfeld, Phys. Rev. D33 (1986)



- $\langle \xi^2 \rangle = 0.26 \pm 0.13$

G. Martinelli and C. Sachrajda, Phys. Lett. B190 (1987)



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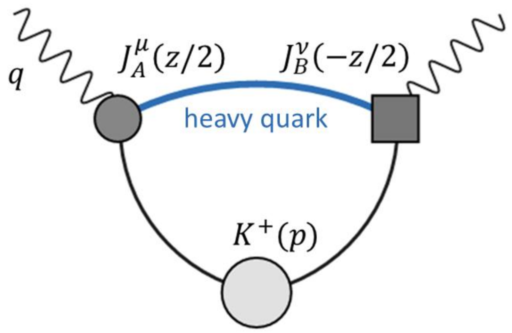
the short-distance operator product expansion (OPE) of a current-current correlator



Heavy-quark Operator Product Expansion (HOPE)

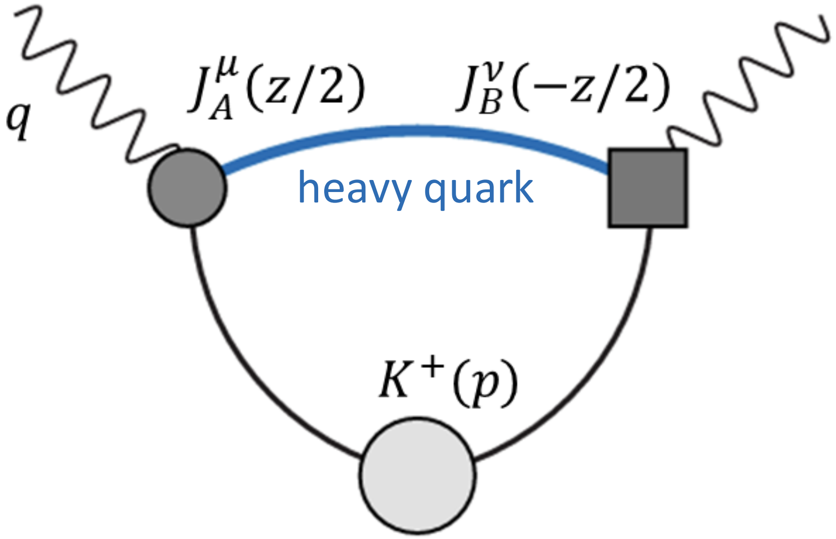
No power-divergent operator mixing

Enables extraction of higher-spin

continuum limit extrapolate

The moments can be extracted through fitting correlator

Heavy-quark Operator Product Expansion (HOPE)



◆ Hadronic Tensor

$$V^{\mu\nu}(p,q) = \int d^4z\, e^{iq\cdot z} \langle \Omega | T\{ J_A^\mu(z/2) J_B^\nu(-z/2) \} | K^+(p) \rangle$$

$$J_A^\mu(z) = \bar{\Psi}(z)\, \gamma^\mu \gamma_5\, \psi_A(z) + \bar{\psi}_A(z)\, \gamma^\mu \gamma_5\, \Psi(z)$$

Ψ is the fictitious valence heavy quark

HOPE allows extraction of matrix elements with **Wilson coefficients** & **Mellin moment**.

$$T\{J^\mu(z/2)\, J^\nu(-z/2)\} = \sum_{i,n} \frac{z^{\mu_1} \ldots z^{\mu_n}}{n!} C_i^{\mu\nu\mu_1\cdots\mu_n}(z^2, \mu^2)\, \mathcal{O}_{i,\mu_1\cdots\mu_n}(\mu)$$

Heavy-quark Operator Product Expansion (HOPE)

LQCD calculations:
3-point correlation functions
→ extract hadronic tensor
$V^{[\mu\,\nu]}(q,p)$

Fitting

QCD perturbation theory:
One-loop Wilson coefficients $C_W^{(n)}$

$$V^{[\mu\,\nu]}(q,p) = \frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho\,p_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2, \mu^2, m_\Psi) \langle \xi^n \rangle \left(\frac{\tilde{\omega}}{2}\right)^n$$

LQCD calculations

QCD perturbation theory

$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \tilde{\omega} = (2\,p \cdot q)/\tilde{Q}^2$

Fit parameters:
- $\langle \xi^n \rangle$ – Mellin moments
- f_K – kaon decay constant
- m_Ψ – fictitious heavy quark mass

Analysis Method & Technics

- Generalized Eigenvalue Problem (GEVP)
  - extract energy spectra from correlation matrices.

- Akaike Information Criterion (AIC)
  - select the one that best balances goodness of fit and model simplicity.

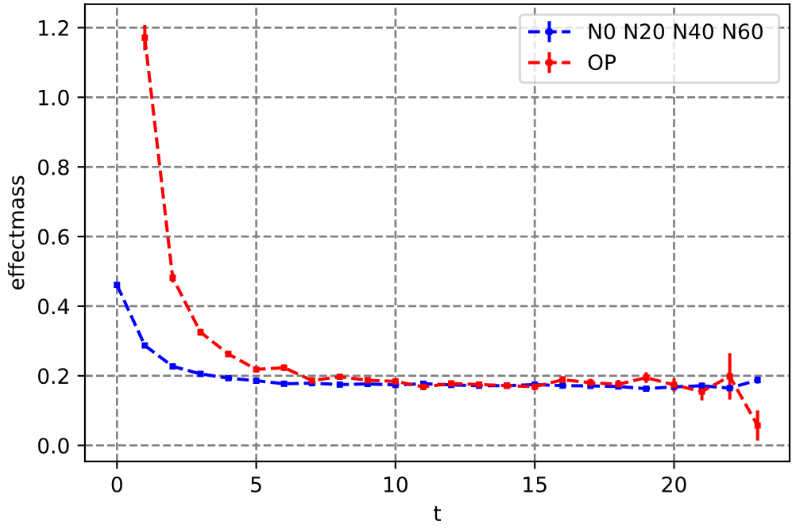
- Shrinkage Covariance Estimator
  - improves condition number, stability, and invertibility of covariance matrices.

Analysis Method & Technics

➢Generalized Eigenvalue Problem (GEVP)

- Helps reduce contamination from higher excited states at short time separations.
- Multiple operators in the correlation matrix improves statistical precision

$$C_{ij}(t) = \langle 0|\mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)|0\rangle \qquad C(t)\, v_n(t,t_0) = \lambda_n(t,t_0)\, C(t_0)\, v_n(t,t_0)$$

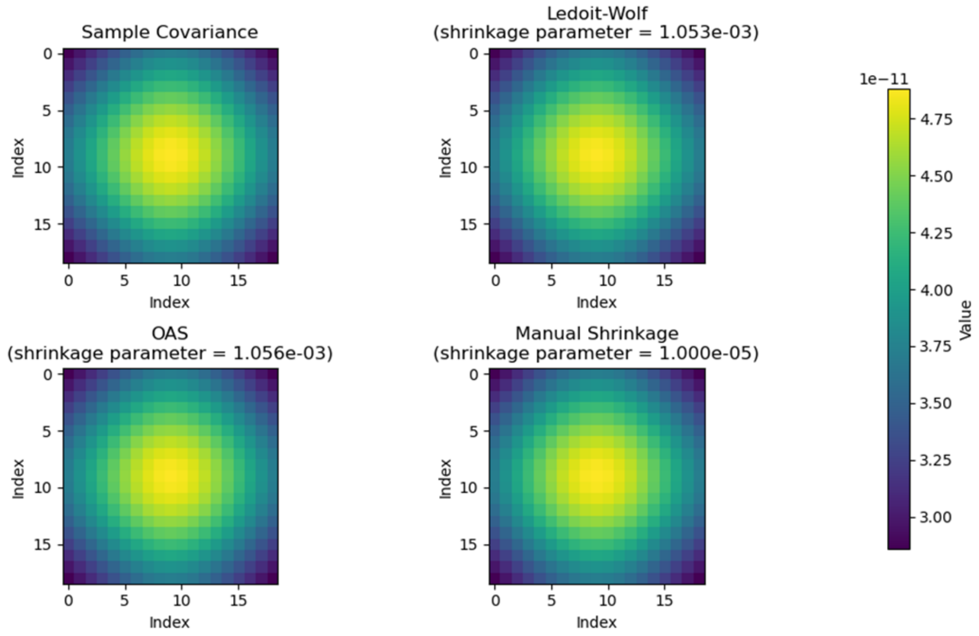


Isolates excited states more cleanly than single-operator

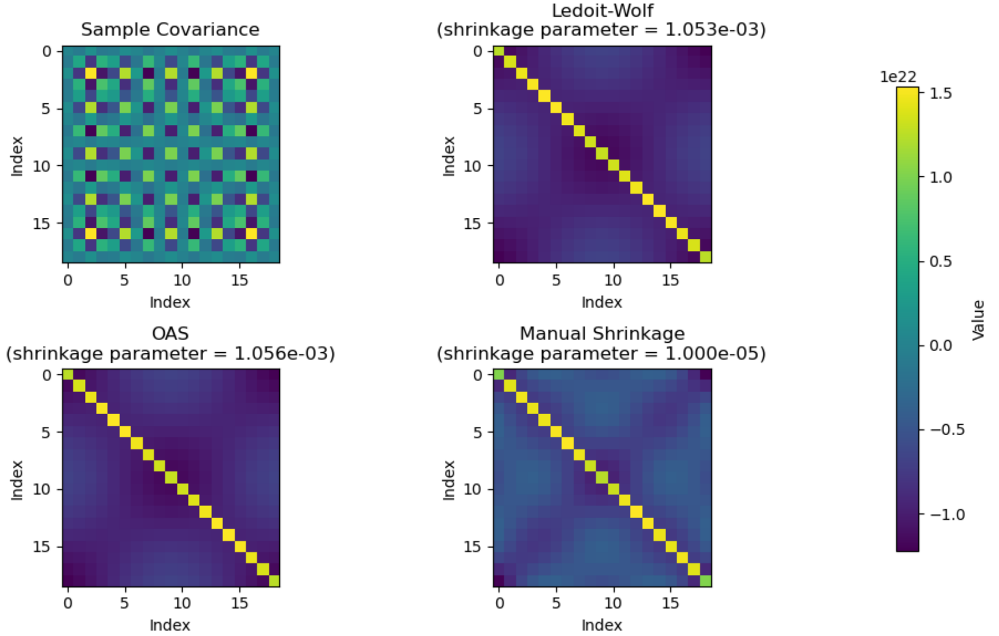
➢Shrinkage Covariance Estimator

Ledoit-Wolf Shrinkage $\boxed{\hat{\Sigma}_{\mathrm{LW}} = (1-\lambda)S + \lambda T}$

Where **S is the sample covariance** and **T is the identity scaled** by average variance.
The optimal shrinkage parameter $\lambda*$ minimizes mean squared error (MSE) to the covariance.



inv



improves stability and invertibility

Hadronic Tensor

◆ Fourier transform

$$V_{ij}^{\mu\nu}(p,q) = \int_0^\infty d\tau\, e^{i\tau q_4}\, R_{ij}^{\mu\nu}(\tau, \mathbf{p}, \mathbf{q})$$

- Separated by **real** and **image** parts

$$\mathrm{Re}\, V_{ij,K^+}^{[\mu\nu]}(p,q) = \int_0^\infty dt\, \sin(q_4 t)\cdot \mathrm{Im}\left[R_{ij,K^+}^{[\mu\nu]}(t;\mathbf{p},\mathbf{q}) - R_{ij,K^+}^{[\mu\nu]}(-t;\mathbf{p},\mathbf{q})\right]$$

$$\mathrm{Im}\, V_{ij,K^+}^{[\mu\nu]}(p,q) = \int_0^\infty dt\, \cos(q_4 t)\cdot \mathrm{Im}\left[R_{ij,K^+}^{[\mu\nu]}(t;\mathbf{p},\mathbf{q}) + R_{ij,K^+}^{[\mu\nu]}(-t;\mathbf{p},\mathbf{q})\right]$$

γ_5-Hermiticity $\boxed{R_{ij,K^+}^{\mu\nu}(t;\mathbf{p},\mathbf{q}) = -R_{ji,K^+}^{\nu\mu}(-t;-\mathbf{p},\mathbf{q})}$

$$\mathrm{Re}\, V_{ij,K^+}^{[\mu\nu]}(p,q) = \int_0^\infty dt\, \sin(q_4 t)\, \mathrm{Im}\left[R_{ij,K^+}^{[\mu\nu]}(t;\mathbf{p},\mathbf{q}) - R_{ji,K^+}^{[\mu\nu]}(t;-\mathbf{p},\mathbf{q})\right]$$

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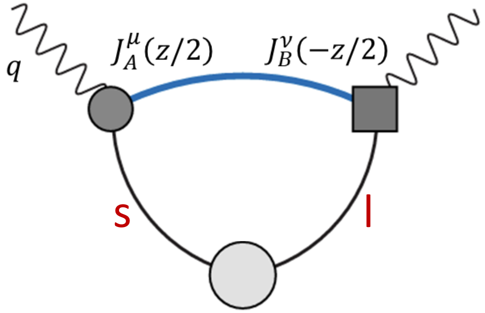
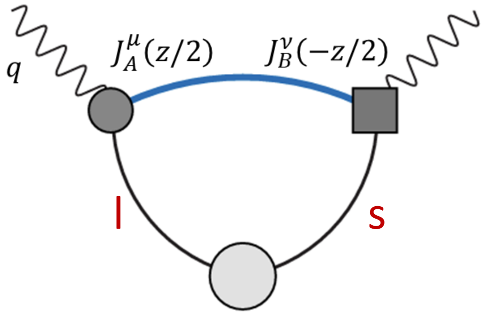
(a)



(b)



Hadronic Tensor

◆ Fourier transform

$$\mathrm{Re}\, V^{[\mu\nu]}_{ij,K^+}(p,q) = \int_0^\infty dt\, \sin(q_4 t)\cdot \mathrm{Im}\left[R^{[\mu\nu]}_{ij,K^+}(t;\mathbf{p},\mathbf{q}) - R^{[\mu\nu]}_{ij,K^+}(-t;\mathbf{p},\mathbf{q})\right]$$

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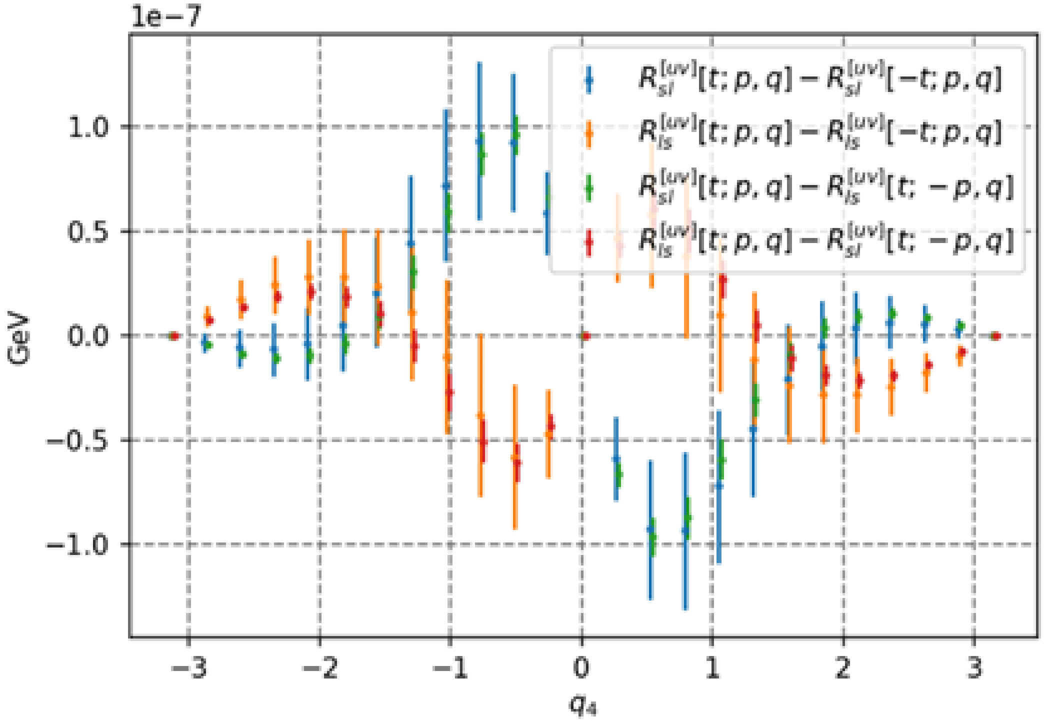


Statistical Noise Cancellation by Symmetrization

Correlated difference

→ needs **delicate cancellation**

→ works better if terms are **highly correlated**

→ use t>0 and t<0 at **same current insertion times**

→ enhanced stability.

Hadronic Tensor

- Separated by **even** and **odd** Mellin moments

$$V^{[\mu\,\nu]}(q,p) \;=\frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\;q_\rho\;p_\sigma}{\tilde{Q}^2}f_K\sum_{n=0}^{\infty}C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

$\tilde{\omega}=(2\,p\cdot q)/\tilde{Q}^2$
$\tilde{\omega}^{2n}$: even function of q
$\tilde{\omega}^{2n+1}$: odd function of q

up to order $\tilde{\omega}^3$

$$V^{[\mu\nu]}(q,p)=\frac{-2i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho p_\sigma}{\tilde{Q}^2}\,f_K\left[C_W^{(0)}+C_W^{(1)}\langle\xi\rangle\tilde{\omega}+C_W^{(2)}\langle\xi^2\rangle\tilde{\omega}^2+C_W^{(3)}\langle\xi^3\rangle\tilde{\omega}^3+\mathcal{O}(\tilde{\omega}^4)\right]$$

- Antisymmetric (odd under q → −q) – **Even Mellin Moments**

$$\frac{1}{2}(V_q^{\mu\nu}-V_{-q}^{\mu\nu})=\frac{-2i\,\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma f_K}{\tilde{Q}^2}\left[C_W^{(0)}+C_W^{(2)}\langle\xi^2\rangle\tilde{\omega}^2+\mathcal{O}(\tilde{\omega}^4)\right]$$

- Symmetric (even under q → −q) – **Odd Mellin Moments**

$$\frac{1}{2}(V_q^{\mu\nu}+V_{-q}^{\mu\nu})=\frac{-2i\,\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma f_K}{\tilde{Q}^2}\left[C_W^{(1)}\langle\xi\rangle\tilde{\omega}+C_W^{(3)}\langle\xi^3\rangle\tilde{\omega}^3+\mathcal{O}(\tilde{\omega}^5)\right]$$

Hadronic Tensor

◆ Separated by **even** and **odd** Mellin moments

- Symmetric (even under q → −q) – **Even Mellin Moments**

$$\frac{1}{2}(V_q^{\mu\nu} - V_{-q}^{\mu\nu}) = \frac{-2i\,\epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma f_K}{\tilde{Q}^2}\left[\boxed{C_W^{(0)}} + \boxed{C_W^{(2)} \langle \xi^2 \rangle \tilde{\omega}^2} + \mathcal{O}(\tilde{\omega}^4)\right]$$

Leading order of real part

$$\tilde{\omega}(p,q) = 2\frac{iE_K q_4 + \mathbf{p}\cdot\mathbf{q}}{\tilde{Q}^2}$$

Leading order of Image part

This dataset shows a **strong signal**, allowing us to **fit stable parameters** ($\boldsymbol{f_K}$, $\mathbf{m}_\Psi$)**.**

Hadronic Tensor

Separated by **even** and **odd** Mellin moments

- Symmetric (even under q → −q) – **Even Mellin Moments**

$$\frac{1}{2}(V_q^{\mu\nu} - V_{-q}^{\mu\nu}) = \frac{-2i\,\epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma f_K}{\tilde{Q}^2}\left[\boxed{C_W^{(0)}} + \boxed{C_W^{(2)} \langle \xi^2 \rangle \tilde{\omega}^2} + \mathcal{O}(\tilde{\omega}^4)\right]$$

Leading order of real part

$$\tilde{\omega}(p,q) = 2\frac{iE_K q_4 + \mathbf{p}\cdot\mathbf{q}}{\tilde{Q}^2}$$

Leading order of Image part

This dataset shows a **strong signal**, allowing us to **fit stable parameters (** $\boldsymbol{f_K}$**,** $\mathbf{m_\Psi}$**).**

Step 1
– Fit from Imaginary Part (Even Moments):
→ extract the parameters $\boldsymbol{f_K}$, $\mathbf{m_\Psi}$

Step 2
– Fit from Real Part (Even Moments):
With $\boldsymbol{f_K}$ and $\mathbf{m_\Psi}$ fixed from **Step 1**
→ fit the **second Mellin moment** $\langle \xi^2 \rangle$

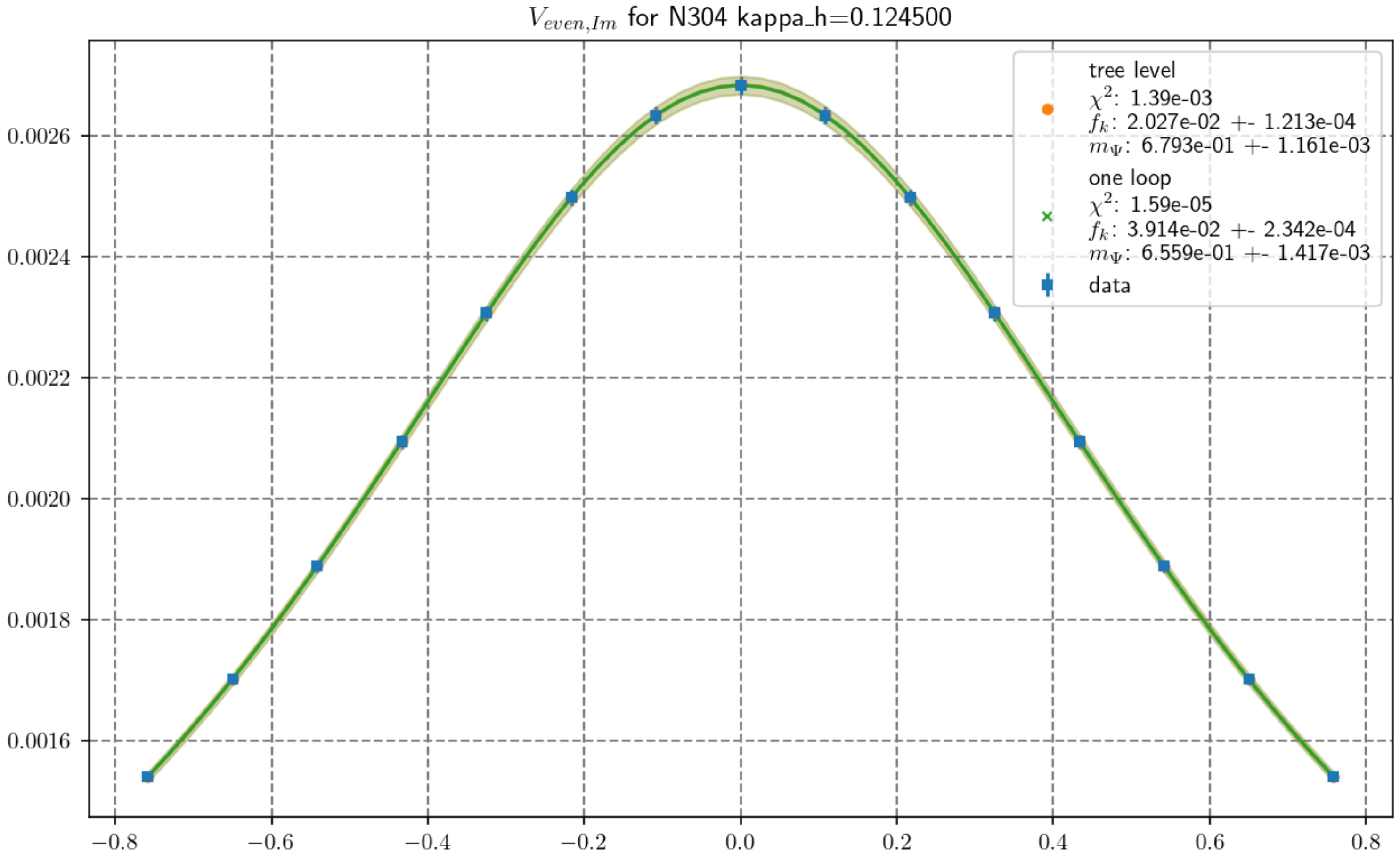
⋮

fit the **first and third Mellin moment**

One Loop OPE Fitting

Imaginary Part

$$\frac{1}{2}(V_q^{\mu\nu} - V_{-q}^{\mu\nu}) = \frac{-2i\,\epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma f_K}{\tilde{Q}^2}\left[C_W^{(0)} + C_W^{(2)}\langle\xi^2\rangle\tilde{\omega}^2 + \mathcal{O}(\tilde{\omega}^4)\right]$$



- **Tree Level:**

  $f_K = 0.1632 \pm 0.00098\,\mathrm{GeV}$

  $m_\Psi = 2.735 \pm 0.00468\,\mathrm{GeV}$

- **One Loop:**

  $f_K = 0.1576 \pm 0.00094\,\mathrm{GeV}$

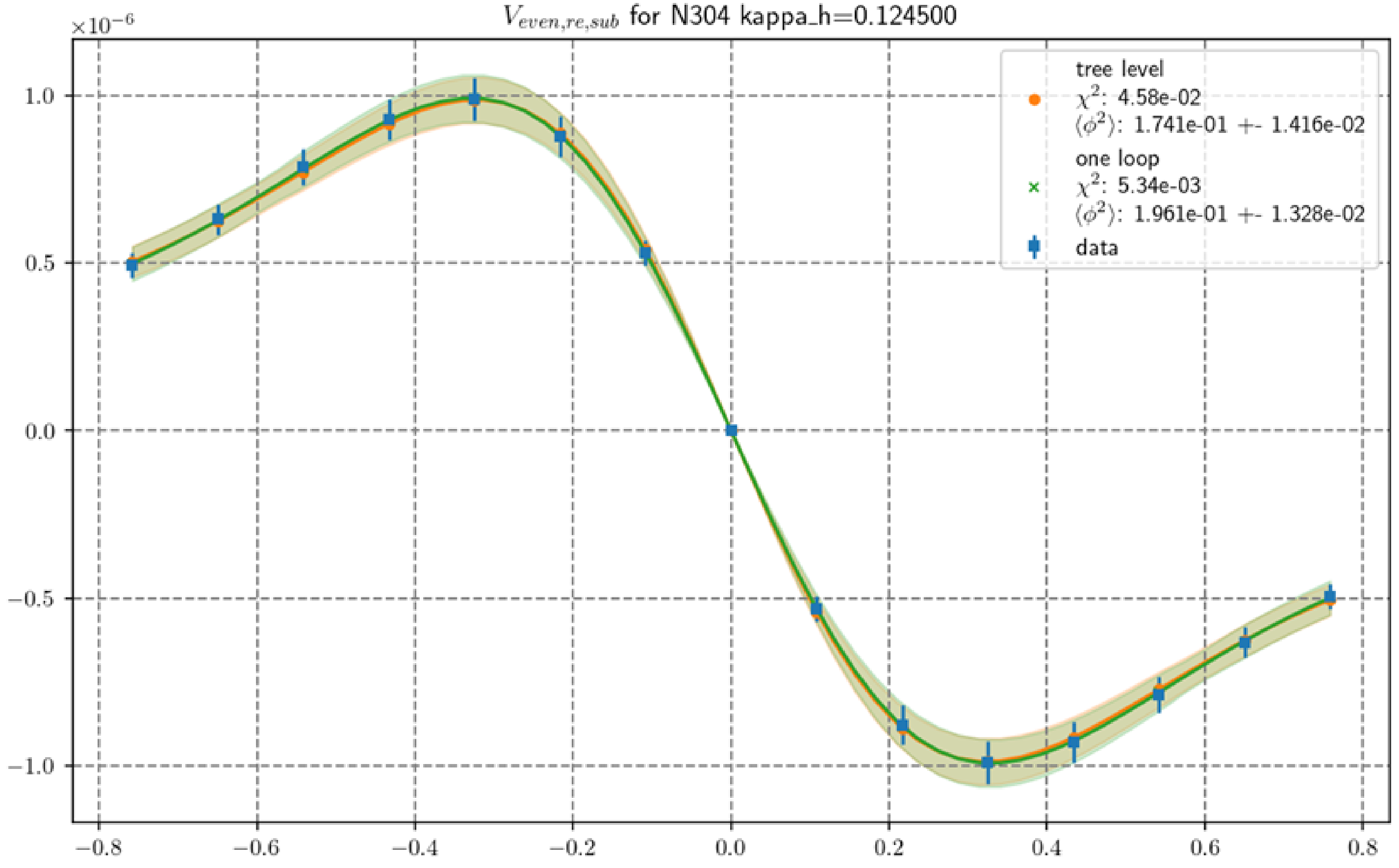
  $m_\Psi = 2.641 \pm 0.00571\,\mathrm{GeV}$

One Loop OPE Fitting

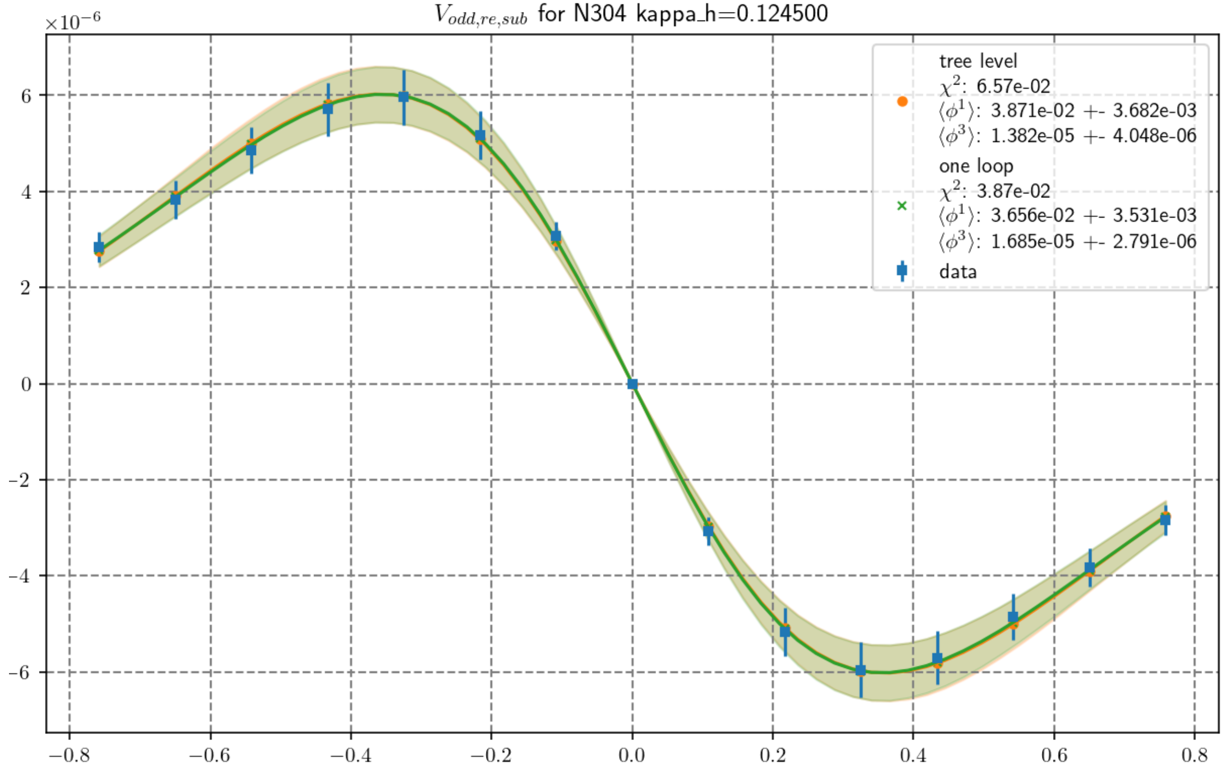
Real Part

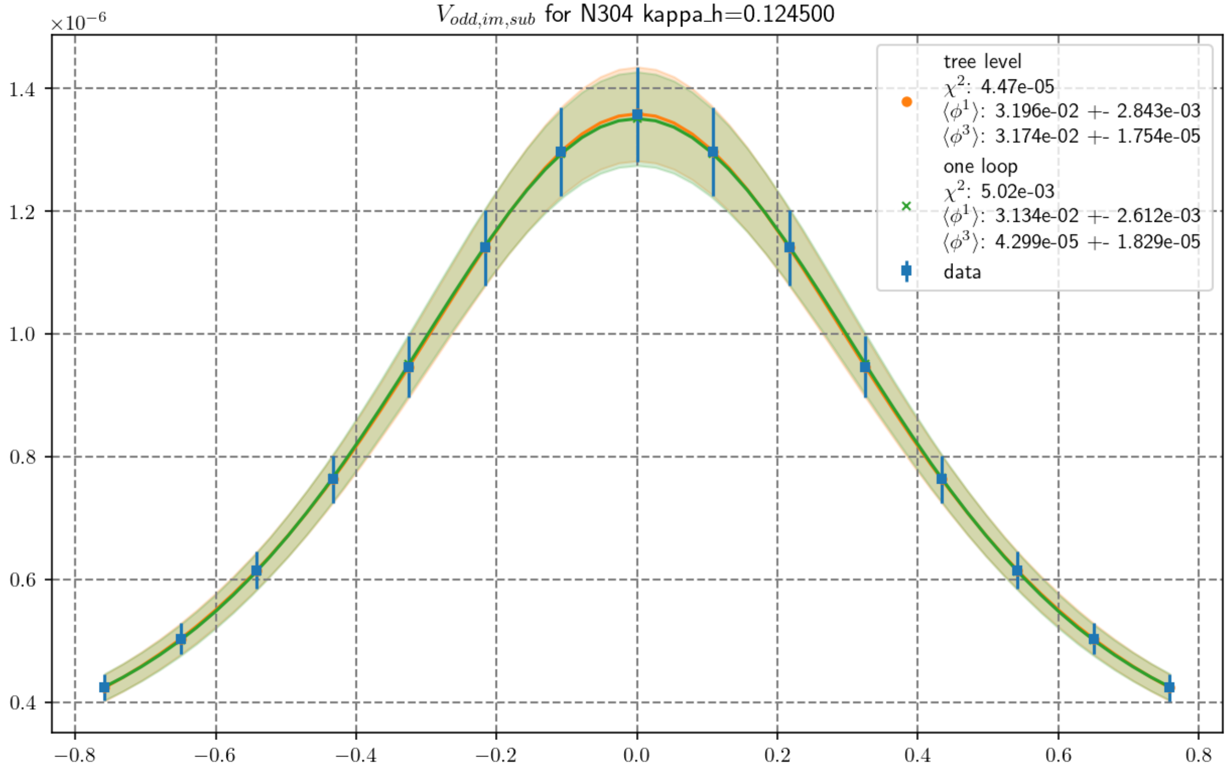
$$\frac{1}{2}(V_q^{\mu\nu} - V_{-q}^{\mu\nu}) = \frac{-2i\,\epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma f_K}{\tilde{Q}^2}\left[C_W^{(0)} + C_W^{(2)}\langle\xi^2\rangle\tilde{\omega}^2 + \mathcal{O}(\tilde{\omega}^4)\right]$$

- Tree Level:
  $\langle\xi^2\rangle = 0.1741 \pm 0.0142$
- One Loop:
  $\langle\xi^2\rangle = 0.1961 \pm 0.0133$



One Loop OPE Fitting





Real Part

- Tree Level
  $\langle\xi^1\rangle = 0.03871 \pm 0.00368$
  $\langle\xi^3\rangle = 1.382 \times 10^{-5} \pm 4.048 \times 10^{-6}$
- One Loop
  $\langle\xi^1\rangle = 0.03656 \pm 0.00353$
  $\langle\xi^3\rangle = 1.685 \times 10^{-5} \pm 2.791 \times 10^{-6}$

Imaginary Part

- Tree Level
  $\langle\xi^1\rangle = 0.03196 \pm 0.00284$
  $\langle\xi^3\rangle = 3.174 \times 10^{-2} \pm 1.754 \times 10^{-5}$
- One Loop
  $\langle\xi^1\rangle = 0.03134 \pm 0.00261$
  $\langle\xi^3\rangle = 4.299 \times 10^{-5} \pm 1.829 \times 10^{-5}$

first Mellin moment is comparable

Continuum Limit Extrapolation

$$\langle \xi^2 \rangle (a, m_\Psi) = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2 \boxed{+ E a^2 m_\pi^2}$$

contaminated by both lattice artifacts and higher-twist corrections

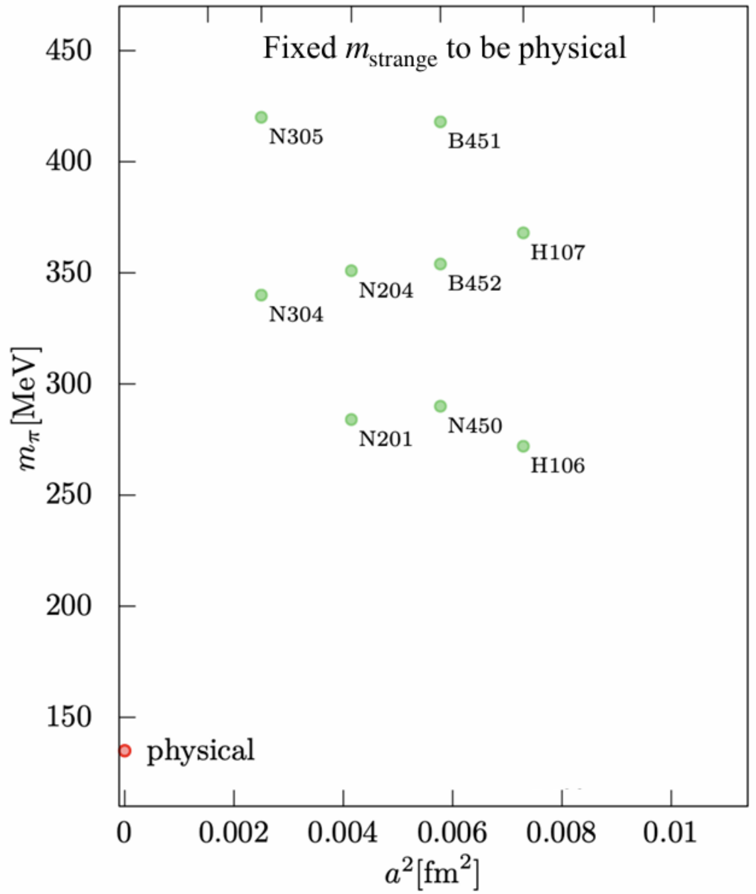
where $\langle \xi^2 \rangle$, A, B, C, and D are the fit parameters

$a = 0.049 fm$

f_K (GeV)	0.1576 ± 0.00094
m_Ψ (GeV)	2.641 ± 0.00571
$\langle \xi^1 \rangle$ (Im)	0.03656 ± 0.00353
$\langle \xi^1 \rangle$ (Re)	0.03134 ± 0.00261
$\langle \xi^2 \rangle$	0.1961 ± 0.0133

• • •

⋮



Planned our calculations on these CLS ensembles

Continuum Limit Extrapolation

$$\langle \xi^2 \rangle (a, m_\Psi) = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2 \boxed{+ Ea^2 m_\pi^2}$$

contaminated by both lattice artifacts and higher-twist corrections

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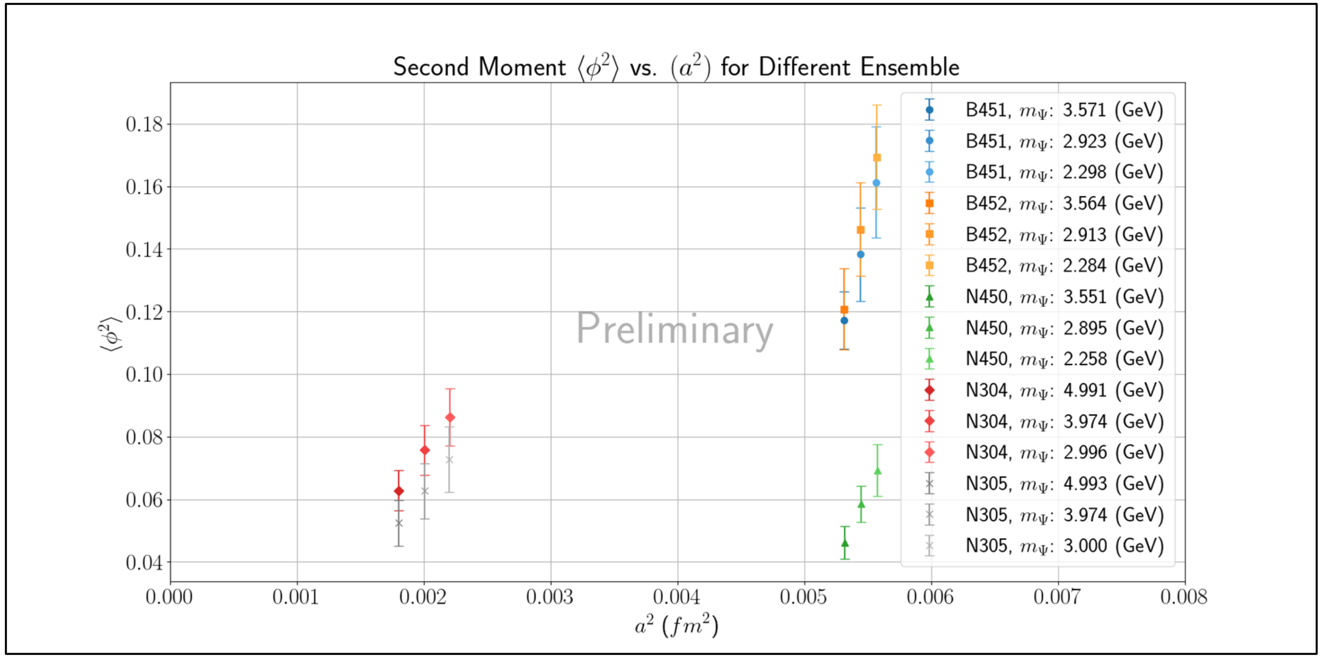

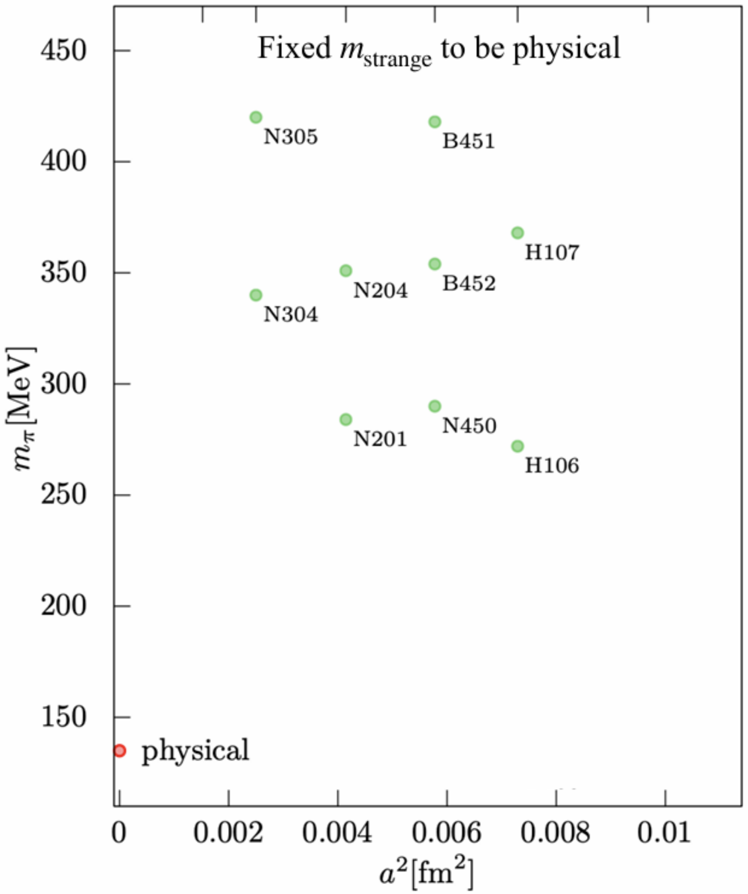

Planned our calculations on these CLS ensembles

Summary

Summary

- We use the **Heavy-Quark Operator Product Expansion (HOPE)** to access the **moments of kaon LCDAs** from **Lattice QCD** calculations.
- The extracted **second moment is comparable** to recent lattice studies, validating the method.
- This framework connects **Euclidean Lattice calculations to light-cone physics** via perturbative OPE and matching.

Future Work

- Implement one-loop Wilson coefficient fitting in **time-momentum** space
- Incorporate **more lattice ensembles** to improve continuum and chiral control

Heavy-quark Operator Product Expansion (HOPE)

LQCD calculations: → extract hadronic tensor $V^{[\mu\,\nu]}(q,p)$

$$C_2(\tau,\mathbf{p}) = \int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} \langle 0|\mathcal{O}_K(\tau,\mathbf{x})\,\mathcal{O}_K^\dagger(0,\mathbf{0})|0\rangle = \langle 0|\mathcal{O}_K(\tau,\mathbf{p})\,\mathcal{O}_K^\dagger(0,\mathbf{p})|0\rangle$$

2-point correlator is saturated with the contribution of the lowest-lying hadronic state

$$C_2(\tau,\mathbf{p}) \sim \frac{|Z_K(\mathbf{p})|^2}{2E_K(\mathbf{p})}\left[e^{-E_K(\mathbf{p})\tau} + e^{-E_K(\mathbf{p})(T-\tau)}\right]$$

→ extract: **Overlap factor Z_K** & **Kaon energy E_K**

$$C_3^{\mu\nu}(\tau_e,\tau_m;\mathbf{p}_e,\mathbf{p}_m) = \int d^3x_e\, d^3x_m\, e^{i\mathbf{p}_e\cdot\mathbf{x}_e} e^{i\mathbf{p}_m\cdot\mathbf{x}_m} \langle 0|\mathcal{T}\left[J_A^\mu(\tau_e,\mathbf{x}_e)\,J_B^\nu(\tau_m,\mathbf{x}_m)\,\mathcal{O}_K^\dagger(0)\right]|0\rangle$$

$$C_3^{\mu\nu}(\tau_e,\tau_m;\mathbf{p}_e,\mathbf{p}_m) \sim R^{\mu\nu}(\tau;\mathbf{p},\mathbf{q})\,\frac{Z_K(\mathbf{p})}{2E_K(\mathbf{p})}\,e^{-E_K(\mathbf{p})(\tau_e+\tau_m)/2}$$

$$R^{\mu\nu}(\tau,\mathbf{p},\mathbf{q}) = \int \frac{dq_4}{(2\pi)} e^{-iq_4\tau} V^{\mu\nu}(p,q)$$

Heavy-quark Operator Product Expansion (HOPE)

$$t^{\mu\nu}(q;m_\Psi) = \int d^4z\, e^{iq\cdot z}\, \mathcal{T}\left\{J_A^\mu(z/2)\, J_B^\nu(-z/2)\right\} \qquad t^{\mu\nu}_{\Psi,\psi} = \overline{\psi}\gamma^\mu \frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{(i\overleftrightarrow{D}+q)^2 + m_\Psi^2}\gamma^\nu\psi$$

$$\frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{(i\overleftrightarrow{D}+q)^2 + m_\Psi^2} = -\frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{Q^2 + \overleftrightarrow{D}^2 - m_\Psi^2}\sum_{n=0}^{\infty}\left(\frac{-2i\, q\cdot\overleftrightarrow{D}}{Q^2 + \overleftrightarrow{D}^2 - m_\Psi^2}\right)^n$$

[Appendix a.]

HOPE allows extraction of matrix elements with Wilson coefficients.

$$T\{J^\mu(z/2)\, J^\nu(-z/2)\} = \sum_{i,n}\frac{z^{\mu_1}\cdots z^{\mu_n}}{n!}\, C_i^{\mu\nu\mu_1\cdots\mu_n}(z^2,\mu^2)\, \mathcal{O}_{i,\mu_1\cdots\mu_n}(\mu)$$

$$V^{[\mu\,\nu]}(q,p) = \frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\, q_\rho\, p_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \tilde{\omega} = (2\,p\cdot q)/\tilde{Q}^2$

$$
\begin{aligned}
t^{\{\mu\nu\}}(q;m_\Psi) = \frac{4i}{\tilde{Q}^2}\sum_{\substack{n=0\\ \text{even}}}^{\infty}\frac{(2q_{\mu_1})\dots(2q_{\mu_n})}{\tilde{Q}^{2n}}C_{1,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)\mathcal{O}^{\mu\nu\mu_1\dots\mu_n}_{n+2,V}(\mu) \\
+ ig^{\mu\nu}\sum_{\substack{n=2\\ \text{even}}}^{\infty}\frac{(2q_{\mu_1})\dots(2q_{\mu_n})}{\tilde{Q}^{2n}}C_{2,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)\mathcal{O}^{\mu_1\dots\mu_n}_{n,V}(\mu) \\
- \frac{2}{\tilde{Q}^2}g^{\mu\nu}m_\Psi\sum_{\substack{n=0\\ \text{even}}}^{\infty}\frac{(2q_{\mu_1})\dots(2q_{\mu_n})}{\tilde{Q}^{2n}}C_{3,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)\hat{\mathcal{O}}^{\mu_1\dots\mu_n}_{n}(\mu) \\
- \frac{4i}{\tilde{Q}^2}q^{\{\mu}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{(2q_{\mu_1})\dots(2q_{\mu_n})}{\tilde{Q}^{2n}}C_{4,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)\mathcal{O}^{\nu\}\mu_1\dots\mu_n}_{n+1,V} + \text{higher twist}(\mu)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}^{\mu_1\dots\mu_n}_{n,V} &= \overline{\psi}\gamma^{\{\mu_1}(i\overleftrightarrow{D}^{\mu_2})\dots(\overleftrightarrow{D}^{\mu_n\}})\psi - \text{tr}, \\
\mathcal{O}^{\mu_1\dots\mu_n}_{n,A} &= \overline{\psi}\gamma^{\{\mu_1}\gamma_5(i\overleftrightarrow{D}^{\mu_2})\dots(i\overleftrightarrow{D}^{\mu_n\}})\psi - \text{tr}, \\
\hat{\mathcal{O}}^{\mu_1\dots\mu_n}_{n} &= \overline{\psi}(i\overleftrightarrow{D}^{\{\mu_1})(i\overleftrightarrow{D}^{\mu_2})\dots(i\overleftrightarrow{D}^{\mu_n\}})\psi - \text{tr},
\end{aligned}
$$

$$\langle\Omega|\,\mathcal{O}^{\mu_1\ldots\mu_n}_{n,A}\,|M(\mathbf{p})\rangle = f_M\,\left\langle\xi^{n-1}\right\rangle_M(\mu^2)[p^{\mu_1}p^{\mu_2}\ldots p^{\mu_n}-\mathrm{tr}],$$

$$\langle H(\mathbf{p},\mathbf{s})|\,\mathcal{O}^{\mu_1\ldots\mu_n}_{n,V}\,|H(\mathbf{p},\mathbf{s})\rangle = 2a^H_{n,V}(\mu^2)[p^{\mu_1}\ldots p^{\mu_n}-\mathrm{tr}],$$

$$\langle H(\mathbf{p},\mathbf{s})|\,\mathcal{O}^{\mu_1\ldots\mu_n}_{n,A}\,|H(\mathbf{p},\mathbf{s})\rangle = 2a^H_{n,A}(\mu^2)[s^{\{\mu_1}p^{\mu_2}\ldots p^{\mu_n\}}-\mathrm{tr}],$$

$$\langle H(\mathbf{p},\mathbf{s})|\,\hat{\mathcal{O}}^{\mu_1\ldots\mu_n}_{n}\,|H(\mathbf{p},\mathbf{s})\rangle = 2b^H_{n}(\mu^2)[p^{\mu_1}\ldots p^{\mu_n}-\mathrm{tr}],$$

$$\begin{aligned}T^{\{\mu\nu\}}(p,q) &= \langle\pi(\mathbf{p})|\,t^{\{\mu\nu\}}(q;m_\Psi)\,|\pi(\mathbf{p})\rangle \\ &= \frac{i}{\tilde{Q}^2}\Bigg(4p^\mu p^\nu\sum_{\substack{n=0\\ \text{even}}}^{\infty}\tilde{\omega}^n C_{1,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)2a^\pi_{n+2,V}(\mu^2)+\tilde{Q}^2 g^{\mu\nu}\sum_{\substack{n=2\\ \text{even}}}^{\infty}\tilde{\omega}^n C_{2,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)2a^\pi_{n,V}(\mu^2) \\ &+2ig^{\mu\nu}m_\Psi\sum_{\substack{n=0\\ \text{even}}}^{\infty}\tilde{\omega}^n C_{3,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)2b^\pi_{n}(\mu^2)-2(p^\mu q^\nu+q^\mu p^\nu)\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\tilde{\omega}^n C_{4,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)2a^\pi_{n+1,V}(\mu^2)\Bigg)\end{aligned}$$

$$T^{\{30\}}(p,q) = 8i\frac{p^3p^0}{\tilde{Q}^2}\sum_{n=0,\text{even}}^{\infty}\tilde{\omega}^n C_{1,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)a^\pi_{n+2,V}(\mu^2)-4i\frac{p^0q^3+p^3q^0}{\tilde{Q}^2}\sum_{n=1,\text{odd}}\tilde{\omega}^n C_{4,n}(\tilde{Q}^2,m_\Psi^2,\mu^2)a^\pi_{n+1,V}(\mu^2)$$