Extraction of the Collins-Soper kernel on the lattice using complex directional Wilson lines

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with

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June 18, 2025

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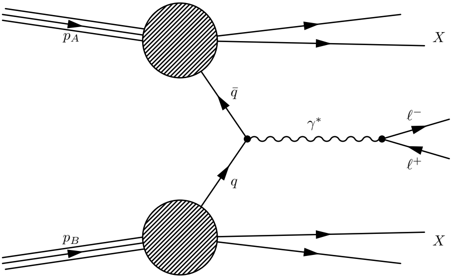
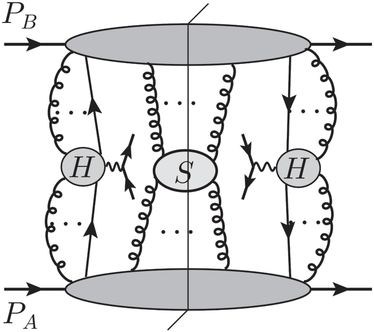
- $\Lambda_{\mathrm{QCD}} \lesssim |\vec{q}_\perp| \ll Q$
- $\vec{b}_\perp$ transverse dependence, conjugate to $\vec{q}_\perp$
- ζ_a, ζ_b, Collins-Soper (CS) scale related to rapidity divergences



Drell-Yan scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}Y\,\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}\left(Q^2, \mu\right) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} f_i\left(x_a, \vec{b}_\perp, \mu, \zeta_a\right) f_j\left(x_b, \vec{b}_\perp, \mu, \zeta_b\right) \times \left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

TMD factorization



Drell-Yan leading region [Collins, 2011]

- Leading region has contribution from soft momentum states
- Need to regulate rapidity divergences present in beam and soft functions
- ν, rapidity renormalization scale
- Form of rapidity scale depends on choice of scheme (e.g. Collins scheme)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}Y\,\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q,\mu) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$
$$\times S_i(b_\perp, \mu, \nu)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Soft function

Naive soft function:

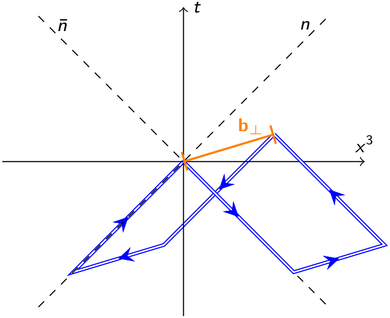
$$S\left(b_\perp, \epsilon\right) = \frac{1}{N_c} \langle 0| \operatorname{Tr} S_n^\dagger(\vec{b}_\perp) S_{\bar{n}}(\vec{b}_\perp) S_\perp(-\infty\bar{n}; \vec{b}_\perp, \vec{0}_\perp) S_{\bar{n}}^\dagger(\vec{0}_\perp) S_n(\vec{0}_\perp) S_\perp^\dagger(-\infty n; \vec{b}_\perp, \vec{0}_\perp) |0\rangle$$

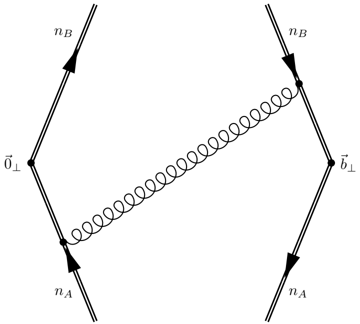
Soft Wilson line:

$$S_n\left(x\right) = P \exp\left\{ -ig \int_{-\infty}^{0} \mathrm{d}s n^\mu A_\mu(x + sn) \right\}$$

Lightlike vectors:

$$n = (1, 0, 0, 1), \qquad \bar{n} = (1, 0, 0, -1)$$
$$n^2 = 0, \qquad \bar{n}^2 = 0, \qquad n \cdot \bar{n} = 2$$





$$\int \frac{\mathrm{d}k^+\mathrm{d}k^-}{(2\pi)^2}\frac{1}{k^+k^- - k_\perp^2 - m^2 + i0}\frac{1}{k^- - i0}\frac{1}{k^+ + i0}$$

$$= \int \frac{\mathrm{d}\alpha}{(2\pi)^2}\frac{1}{\alpha - k_\perp^2 - m^2 + i0}\frac{1}{\alpha - i0}\int_{-\infty}^{\infty}\mathrm{d}y$$

$$k^- = n\cdot k, \qquad k^+ = \bar{n}\cdot k, \qquad k^\pm = k^0 \pm k^3$$

$$\alpha = k^+k^-, \quad y = \frac{1}{2}\ln\left(\frac{k^-}{k^+}\right)$$

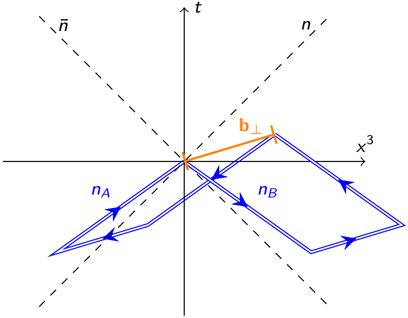
$$y \to \pm\infty$$

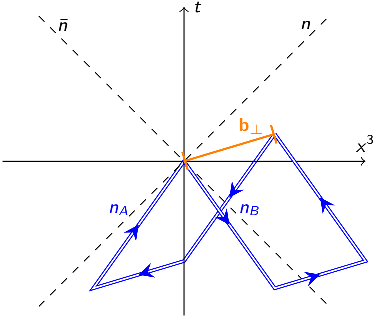
Spacelike Wilson lines:

$$n_A \equiv n - e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B} n$$

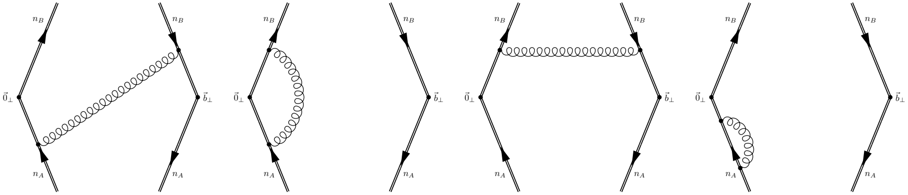
Timelike Wilson lines:

$$n_A \equiv n + e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} + e^{+y_B} n$$





One loop result in Minkowski space



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One-loop result in Collins schemes [Collins, 2011]:

$$S(b_\perp, \epsilon, y_A, y_B)$$
$$= 1 + \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 - 2|y_A - y_B|\frac{1+e^{2(y_B-y_A)}}{1-e^{2(y_B-y_A)}}\right\} + \mathcal{O}(\alpha_s^2)$$

Evolution kernel:

$$\gamma_{\mu}^{q}(\mu,\zeta) = \frac{\mathrm{d} f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)}{\mathrm{d}\log\mu}$$

Collins-Soper (CS) kernel:

$$K_{CS}(b_\perp, \mu) = \frac{\mathrm{d}\log f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)}{\mathrm{d}\log\sqrt{\zeta}}$$

$$S_q\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{2K_{CS}(b_\perp,\mu)(y_A - y_B)}\left(1 + \mathcal{O}\left(e^{-2(y_A - y_B)}\right)\right)$$

Relevance to lattice extraction of TMDPDFs

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = C_q\left(x\tilde{P}^z, \mu\right) \exp\left[\frac{1}{2} K_{CS}\left(\mu, b_\perp\right) \log \frac{\tilde{\zeta}}{\zeta}\right] f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)$$
$$\times \left\{1 + \mathcal{O}\left(\frac{1}{(x\tilde{P}^z b_\perp)^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{(x\tilde{P}^z)^2}\right)\right\}$$

[Ebert, *et. al.*, 2019], [Ebert, *et. al*, 2022]

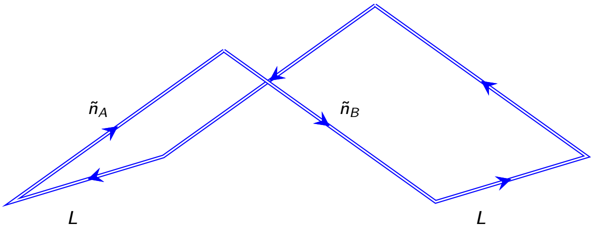
- Perturbative matching kernel: C_q
- quasi-CS scale: $\tilde{\zeta} = x^2 m_h^2 e^{2(y_{\tilde{P}} + y_B - y_n)}$
- quasi-TMDPDF: $\tilde{f}_q$
- TMDPDF: f_q
- CS kernel: K_{CS}

Outline

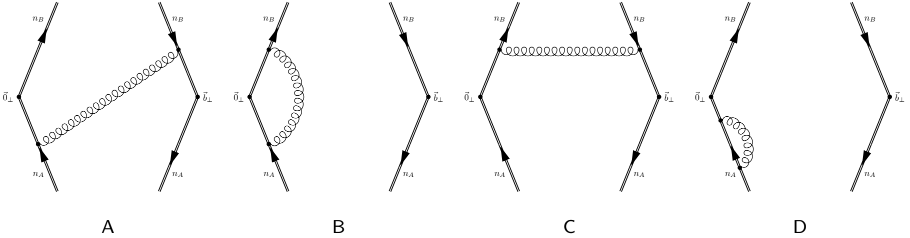
In Euclidean space

Euclidean space directional vectors with purely imaginary time components

$$\tilde{n}_A = \left(in_A^0, 0, 0, n_A^3\right), \quad \tilde{n}_B = \left(in_B^0, 0, 0, n_B^3\right)$$



Soft function in Euclidean space at one loop



Calculation in coordinate space at one loop:

$$S^{(1)}\left(b_\perp, \epsilon, r_a, r_b\right) = \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 + \log\left|\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right| \frac{r_a r_b + 1}{r_a + r_b}\right\}$$

$$|r_{a,b}| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

$$r_a = \frac{n_A^3}{n_A^0} = \frac{1+e^{-2y_A}}{1-e^{-2y_A}}, \quad r_b = \frac{n_B^3}{n_B^0} = \frac{1+e^{2y_B}}{1-e^{2y_B}}$$

Finite L computation at one loop

For $L \to \infty$ and $r_a, r_b \to 1\,(y_A, -y_B \to \infty)$:

$$
\begin{aligned}
& S_{\text{ratio}}\,(b_\perp, a, r_a, r_b, L) \\
& \quad = 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)}\log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right)\right)\log\left(\frac{b_\perp^2}{a^2}\right) \\
& \quad + \mathcal{O}\left(\frac{b_\perp^2, a^2}{L^2(r_{a,b} - 1)}, \frac{b_\perp^2, a^2}{L^2}, (r_{a,b} - 1), \frac{b_\perp}{a}, \frac{L}{a}, \frac{L}{b_\perp}, \alpha_s^2\right)
\end{aligned}
$$

Tension between $r_{a,b} \to 1$ and $\frac{1}{r_{a,b}-1}$

Compare with large rapidity expansion of Soft factor:

$$
S_C\,(b_\perp, y_A - y_B, \mu) = S_I\,(b_\perp, \mu)\, e^{2\gamma_\zeta^q(b_\perp,\mu)(y_A - y_B)}\left(1 + \mathcal{O}\left(e^{-2(y_A - y_B)}\right)\right)
$$

$$
\begin{gathered}
r_{a,b} = 1.1 \\
y_A, -y_B \simeq 1.52 \\
e^{-2(y_A - y_B)} \simeq 0.0023
\end{gathered}
$$

Two methods for removing UV and power divergences.

- Method 1: Ratio

$$\frac{\tilde{S}\left(b_\perp, a, r_a, r_b, L\right)}{\sqrt{\tilde{S}\left(b_\perp, a, r_a, -r_a, L\right)\tilde{S}\left(b_\perp, a, -r_b, r_b, L\right)}} = \frac{\tilde{S}_{\mathrm{bfly}}}{\sqrt{\tilde{S}_A\tilde{S}_B}}$$

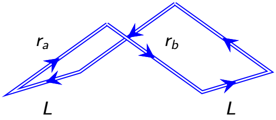
This method gives the soft function as well as the CS kernel

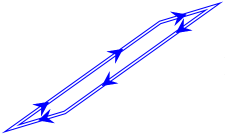
- Method 2: Double ratio

$$\frac{\tilde{S}\left(b_{\perp,1}, a, r_1, r_1, L\right)}{\tilde{S}\left(b_{\perp,2}, a, r_1, r_1, L\right)} \Bigg/ \frac{\tilde{S}\left(b_{\perp,1}, a, r_2, r_2, L\right)}{\tilde{S}\left(b_{\perp,2}, a, r_2, r_2, L\right)} = \frac{\tilde{S}^{1,1}_{\mathrm{bfly}}\tilde{S}^{2,2}_{\mathrm{bfly}}}{\tilde{S}^{2,1}_{\mathrm{bfly}}\tilde{S}^{1,2}_{\mathrm{bfly}}}$$

We can only obtain the CS kernel from this method

$$\tilde{S}_{\mathrm{ratio}}\left(b_\perp, a, r_a, r_b, L\right) =$$





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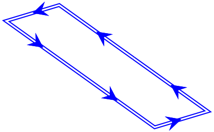


$$= \frac{\tilde{S}\left(b_\perp, a, r_a, r_b, L\right)}{\sqrt{\tilde{S}\left(b_\perp, a, r_a, -r_a, L\right)\tilde{S}\left(b_\perp, a, -r_b, r_b, L\right)}}$$

Ratio

- For large lattice time, τ:

$$\tilde{S}_{\text{ratio}}\left(b_\perp, r_a, r_b, a, \tau\right) = \tilde{S}_{\text{lat}}\left(b_\perp, y_A, y_B, a\right) + \mathcal{O}\left(\frac{b_\perp^2, a^2}{\tau^2(r_{a,b}-1)}, \frac{b_\perp^2, a^2}{\tau^2}, (r_{a,b}-1)\right)$$

- Construct matching between lattice and continuum renormalization schemes

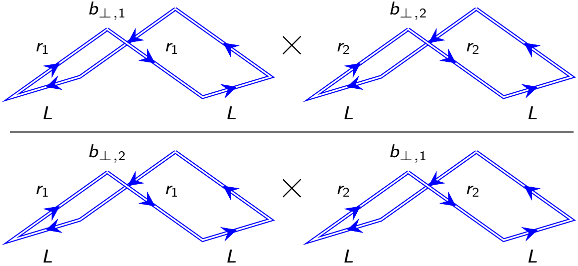
$$S\left(b_\perp, y_A, y_B, \mu\right) = C(y_A, y_B, \mu, a) \times \tilde{S}_{\text{lat}}\left(b_\perp, y_A, y_B, a\right)$$

- Get CS kernel from soft function

$$S\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{2K_{CS}(b_\perp,\mu)(y_A-y_B)}\left(1 + \mathcal{O}\left(e^{-2(y_A-y_B)}\right)\right)$$

Double ratio

$$\tilde{S}_{\mathrm{double}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2, L\right) =$$



$$= \frac{\tilde{S}\left(b_{\perp,1}, a, r_1, r_1, L\right)}{\tilde{S}\left(b_{\perp,2}, a, r_1, r_1, L\right)} \Bigg/ \frac{\tilde{S}\left(b_{\perp,1}, a, r_2, r_2, L\right)}{\tilde{S}\left(b_{\perp,2}, a, r_2, r_2, L\right)}$$

Double ratio

- For large lattice time:

$$\tilde{S}_{\mathrm{double}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2, \tau\right) = \frac{\tilde{S}_{\mathrm{lat}}\left(b_{\perp,1}, r_1, r_1\right)}{\tilde{S}_{\mathrm{lat}}\left(b_{\perp,2}, r_1, r_1\right)} \Bigg/ \frac{\tilde{S}_{\mathrm{lat}}\left(b_{\perp,1}, r_2, r_2\right)}{\tilde{S}_{\mathrm{lat}}\left(b_{\perp,2}, r_2, r_2\right)} + \mathcal{O}\left(\frac{b_1^2 - b_2^2}{\tau^2}\left(\frac{1}{r_1 - 1} - \frac{1}{r_2 - 1}\right), a^2, \frac{r_{1,2} - 1}{r_{1,2} + 1}\right)$$

- Match between lattice and continuum renormalization schemes

$$S_{\mathrm{double}}\left(b_{\perp,1}, b_{\perp,2}, \mu, r_1, r_2\right) = C\left(r_1, r_2, \mu, a\right) \times \tilde{S}_{\mathrm{double}}\left(b_{\perp,1}, b_{\perp,2}, a, r_1, r_2\right)$$

- We can then extract the CS kernel relative to it's value at another value of $b_\perp$:

$$K_{\mathrm{CS}}\left(b_{\perp,1}, \mu\right) = K_{\mathrm{CS}}\left(b_{\perp,2}, \mu\right) + \frac{\frac{1}{2}\log\left(S_{\mathrm{double}}\left(b_{\perp,1}, b_{\perp,2}, \mu, r_1, r_2\right)\right)}{\log\left(\frac{r_1+1}{r_1-1} \Big/ \frac{r_2+1}{r_2-1}\right)}$$

Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional ‘fermions’ that live along the path:

$$P \exp\left\{-ig\int_{s_i}^{s_f} \mathrm{d}s n^\mu A_\mu(y(s))\right\}$$
$$= Z_\psi^{-1}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\psi\bar{\psi}\exp\left\{ig\int_{s_i}^{s_f}\mathrm{d}s\bar{\psi}i\partial_s\psi - \bar{\psi}n\cdot A\psi\right\}$$

[Gervais, Nevau 1980], [Aref’eva 1980]

Auxiliary field propagator:

$$in\cdot DH_n(y) = \delta(y) \xrightarrow{\text{Euclidean space}} -i\tilde{n}\cdot D_E H_{\tilde{n}}(y) = \delta(y), \qquad \tilde{n} = (in_0, \vec{n})$$

Meaningful solution only obtained with a UV cutoff [Aglietti, *et. al.* 1992], [Agglietti, 1994]

Butterfly and rectangle factors

- Large τ form of butterfly and rectangle loops from UV cutoff effects:

$$\tilde{S}_{\mathrm{bfly}} = \tilde{S}\left(b_\perp, a, r_a, r_b, \tau\right) \overset{\tau\to\infty}{\sim} e^{2\pi\tau(r_a+r_b)/a}/\tau^4$$

$$\tilde{S}_{\mathrm{A}} = \tilde{S}\left(b_\perp, a, r_a, -r_a, \tau\right) \overset{\tau\to\infty}{\sim} e^{4\pi(\tau r_a - iz)/a}/\tau^4$$

$$\tilde{S}_{\mathrm{B}} = \tilde{S}\left(b_\perp, a, -r_b, r_b, \tau\right) \overset{\tau\to\infty}{\sim} e^{4\pi(\tau r_b + iz)/a}/\tau^4$$

- Expect to see real and imaginary contributions to $\tilde{S}_{A,B}$
- Combined factor $\sqrt{\tilde{S}_A \tilde{S}_B}$, should be purely real
- In theory, these cutoff effects cancels in the ratio
- The double ratio only concerns the 'butterfly' factor, so the cutoff effects easily cancel

Outline

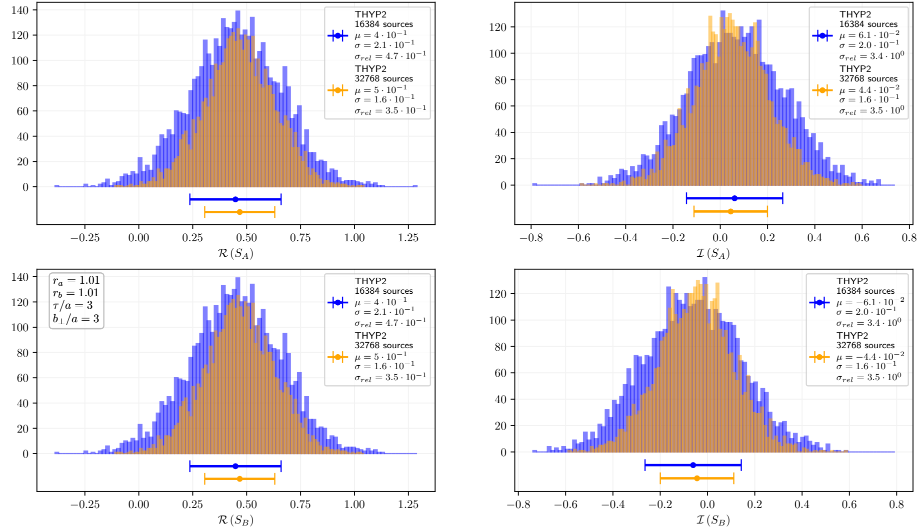
1. Motivation
2. Theoretical motivation
3. Lattice implementation
4. Results so far

Details of lattice calculation

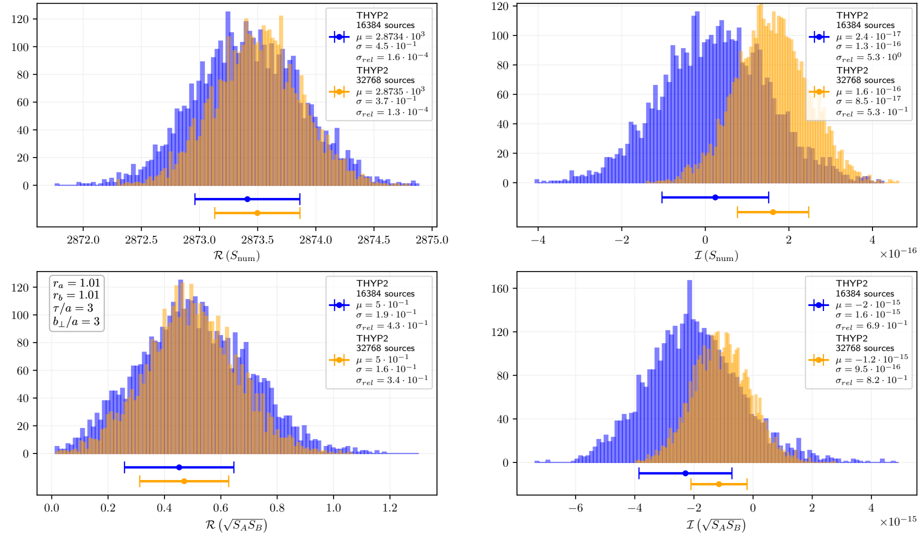
- Use auxiliary field definition of the Wilson line
  [Gervais, Nevau 1980], [Aref'eva 1980], [Aglietti, *et. al.* 1992], [Agglietti, 1994], [Horgan, *et. al.*, 2009]
- Using $N_f = 2 + 1$ flavor PACS-CS configurations
- non-perturbatively $\mathcal{O}(a)$-improved Wilson quark action and Iwasaki gauge action
- $32^3 \times 64$ lattice with $a = 0.0907(13)$ fm

  [PACS-CS '09, '11]
- 400 configurations

Denominator loops, THYP 2 steps



Numerator and combined denominator, THYP 2 steps



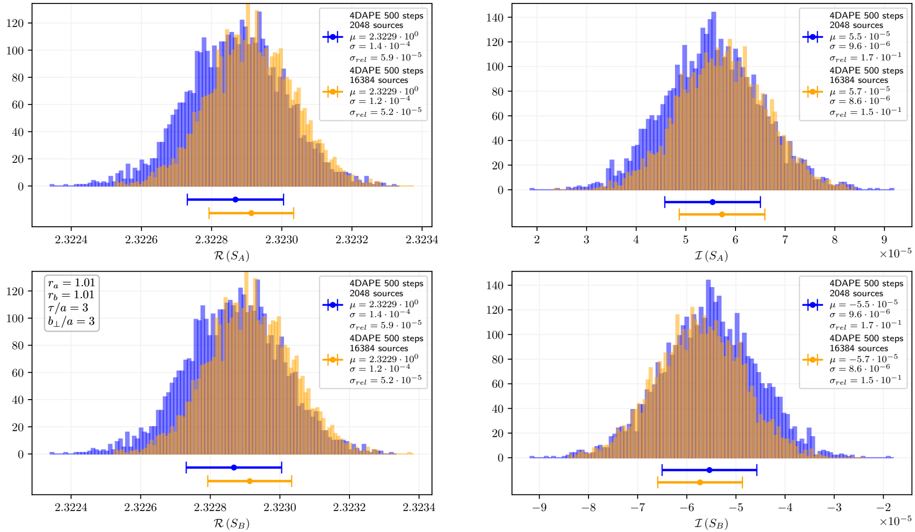
- 500 steps of 4D APE smearing:

$$U'_\mu(n) = (1-\alpha)U_\mu(n) + \frac{\alpha}{6}\sum_{\nu\neq\mu} C_{\mu\nu}(n)$$

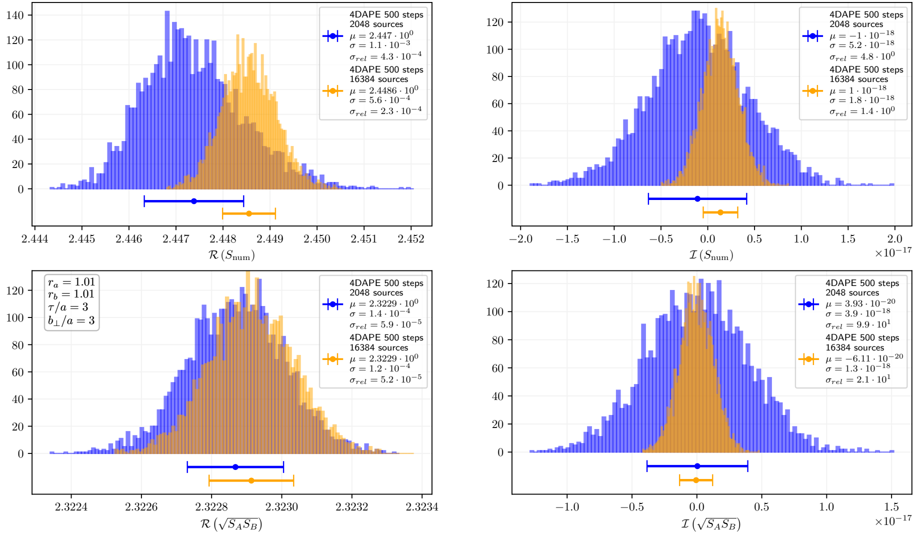
$$\begin{aligned} C_{\mu\nu}(n) = {} & U_\nu(n)U_\mu(n+\hat{\nu})U_\nu(n+\hat{\mu})^\dagger \\ & + U_\nu(n-\hat{\nu})^\dagger U_\mu(n-\hat{\nu})U_\nu(n-\hat{\nu}+\hat{\mu}) \end{aligned}$$

- Expect this to smooth out fluctations due to UV divergence in auxiliary propagator

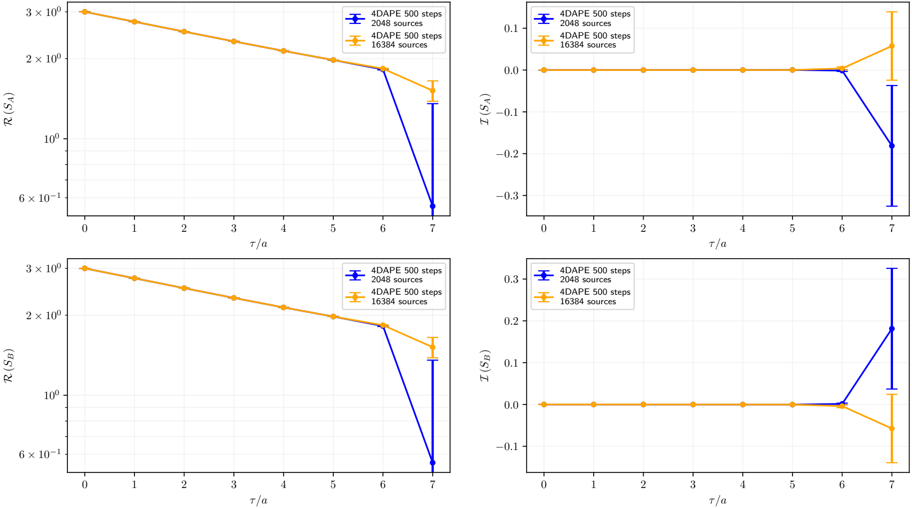
Denominator, APE 500 steps



Numerator and combined denominator, APE 500 steps

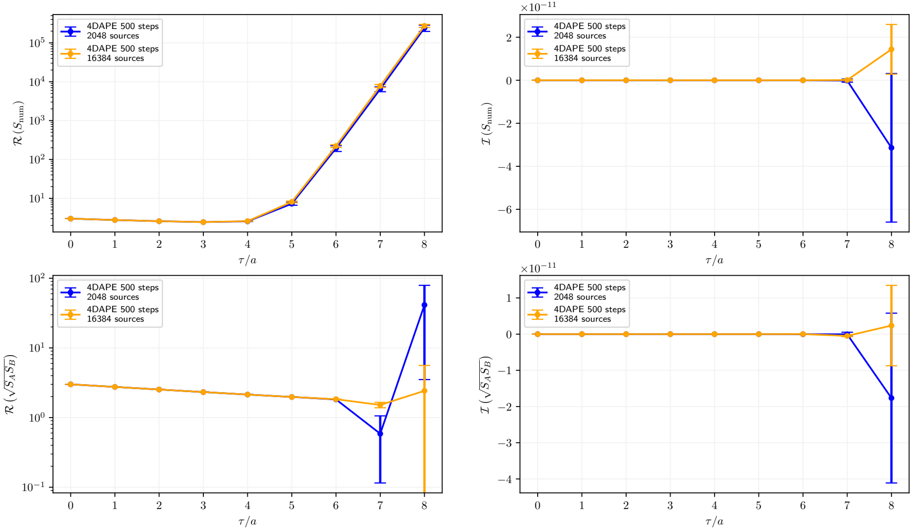


Denominator, APE 500 steps



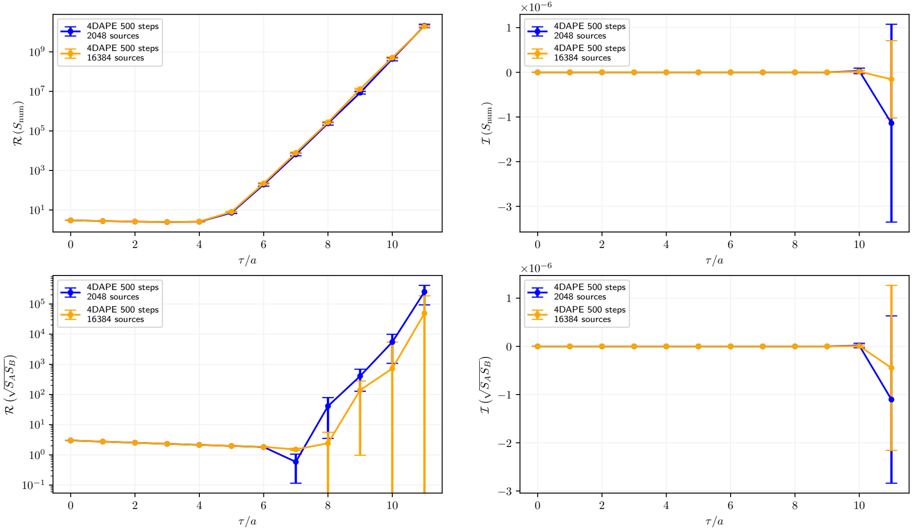
- Exponential decrease up to $\tau/a = 6$, then cutoff effects begin to dominate along with large fluctuations due to phase factor

Numerator and combined denominator, APE 500 steps



- Top row: 'Butterfly' shows decrease up to $\tau/a = 4$ before cutoff effects dominate
- Bottom row: Combined denominator still suffers from complex phase in large τ region
- No plateau in region of τ with suppression of cutoff effects

Numerator and combined denominator, APE 500 steps



- Euclidean space calculation of soft function has a direct mapping to Minkowski space result
- High precision for ‘numerator‘ factor, so double ratio method looks promising
- Improvements in precision with increased statistics, but poor signal for denominator factors at $\sim$ 30000 sources
- 4D APE smearing gives significant improvements at 500 smearing steps, but we don’t see a plateau for ratio method at $\tau = 7$
- Measure on finer lattices, and use gradient flow for ratio method
- Currently working on an extraction using double ratio

Thank you!

Group members

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