Lattice Study of Critical Point and Roberge-Weiss Phase Transition in Heavy-Quark Region of QCD

Tatsuya Wada (Kyoto Univ./YITP)

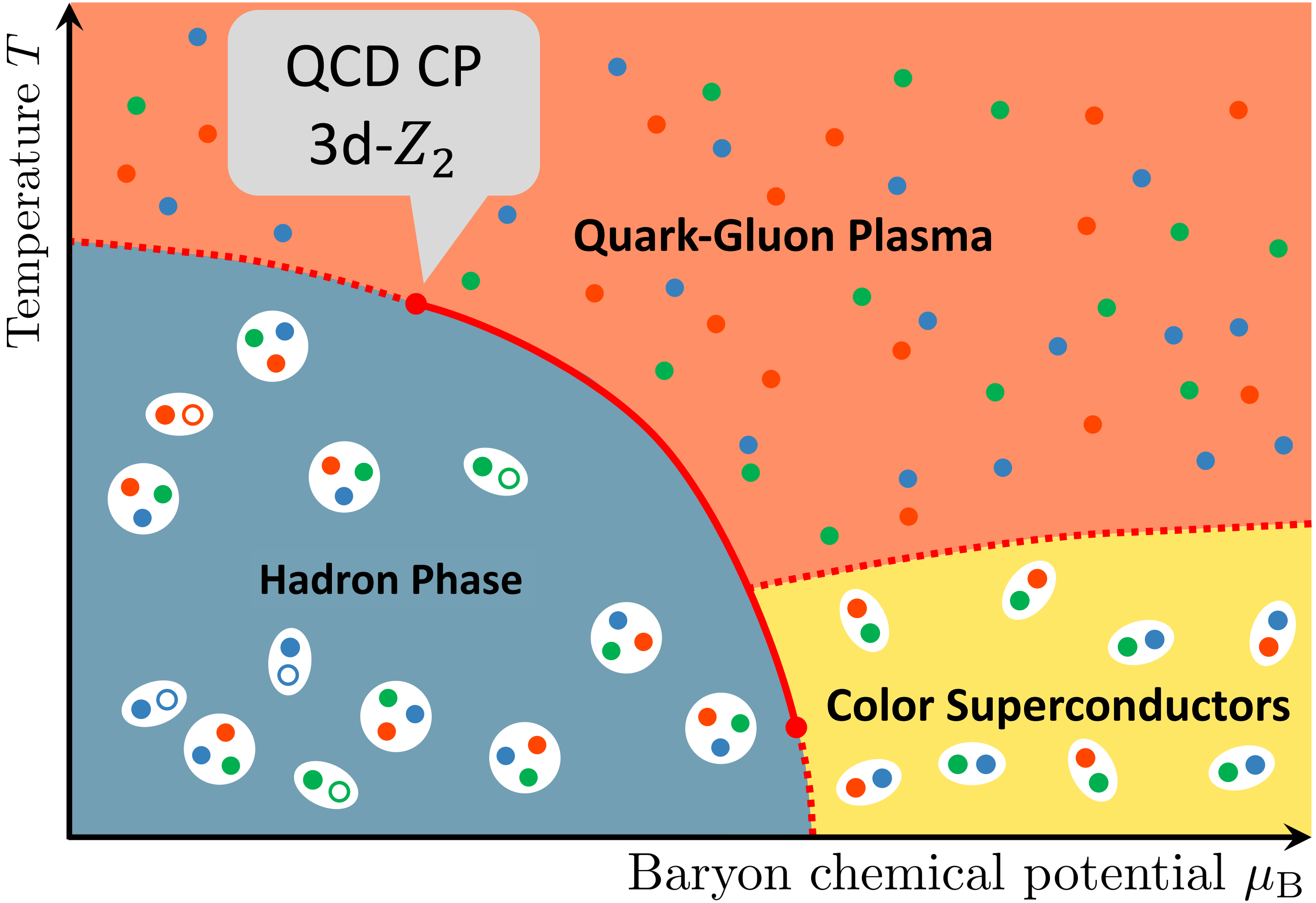
Collaborators : Masakiyo Kitazawa, Kazuyuki Kanaya (Tsukuba Univ.)

This talk is based on PRL 134, 162302, arXiv: 2508.20422[JPSJ, in press]

Institute of Physics, Academia Sinica

QCD Critical Point

QCD Phase diagram



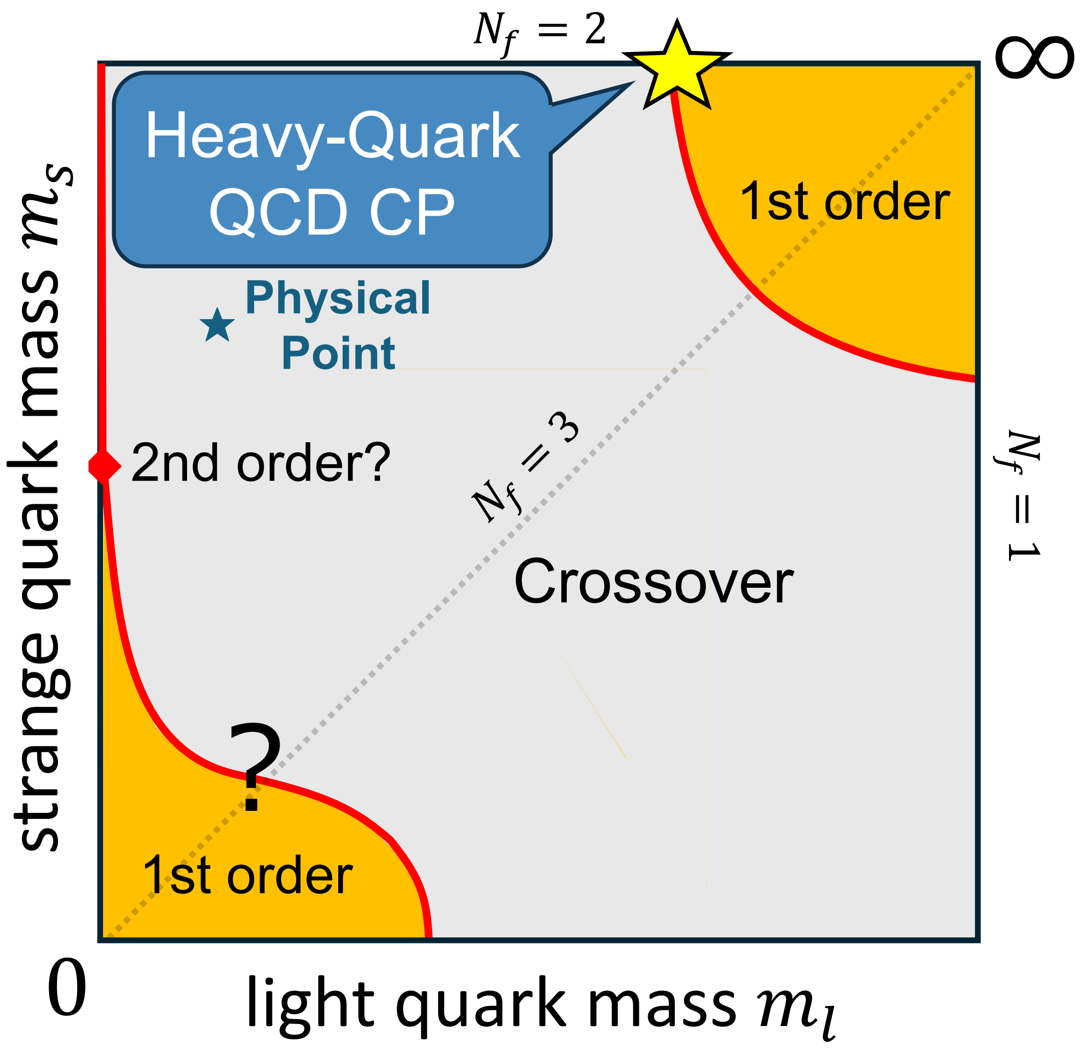
QCD Critical Point(CP)

- ◆Constraint to behavior of physical quantities
- ◆Classification by universality class

QCD is asymptotic-free theory
Non-perturbative in low energy.
Analytic approach is too difficult!

- ◆Effective Model Approach
  - ➢ NJL model etc.
- ◆FRG Approach
- ◆Numerical Approach
  - ➢ Monte Carlo simulation
  - ➢ Tensor Network
  - ➢ Quantum Computation

Columbia Plot



Columbia Plot :

The order of phase transition at $\mu = 0$

- $m \to m_{\text{phys}}$: $N_f = 2 + 1$ QCD
  → Crossover
- $m \sim 0$: Light quark sector
  → 1st or Crossover is controversial.
  (by Tsukuba group (〜2020))
- $m \to \infty$: Pure Yang-Mills theory
  → 1st order

Method

Lee-Yang Zero Approach

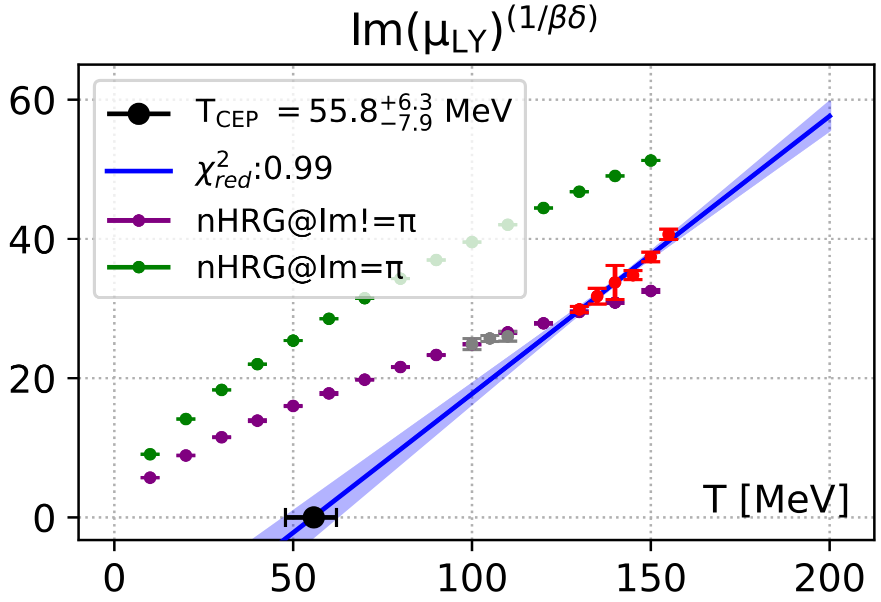
Recent progress: Lee-Yang zeros

Pure imaginary chemical potential

+ Pade approximation

+Lee-Yang zeros /edge singularity

- Clarke, *et al.*(2024)
- Basar (2024)
- Alexander, *et al.*(2025)

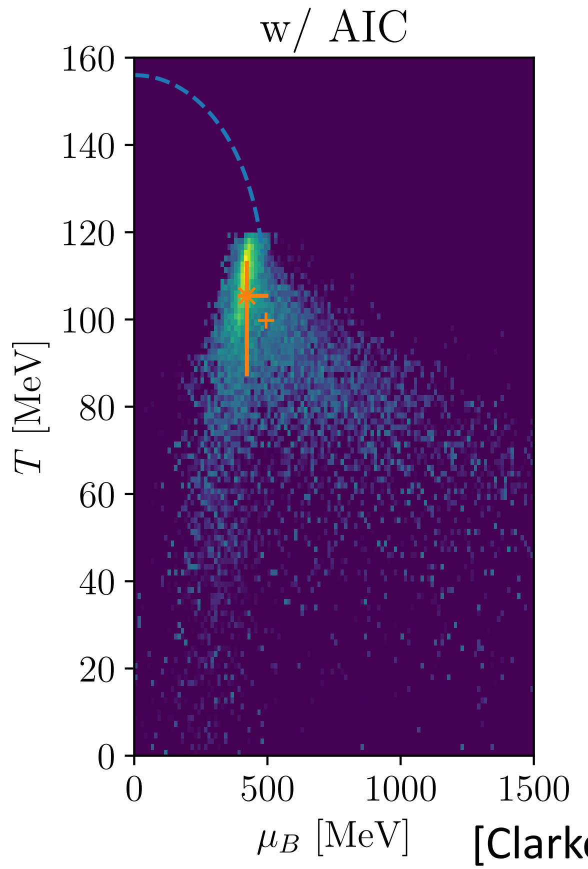


$T_c \sim 70\mathrm{MeV}$

or CP does not exist!

[Alexandar, *et al.*]

Estimation of location of QCD CP



[Clarke, *et al.*]

$$\begin{cases} \mu^{\mathrm{CEP}} = 422^{+80}_{-35}\mathrm{MeV} \\ T^{\mathrm{CEP}} = 105^{+8}_{-18}\mathrm{MeV} \end{cases}$$

Review : Lee-Yang Zero

Lee, Yang 1952

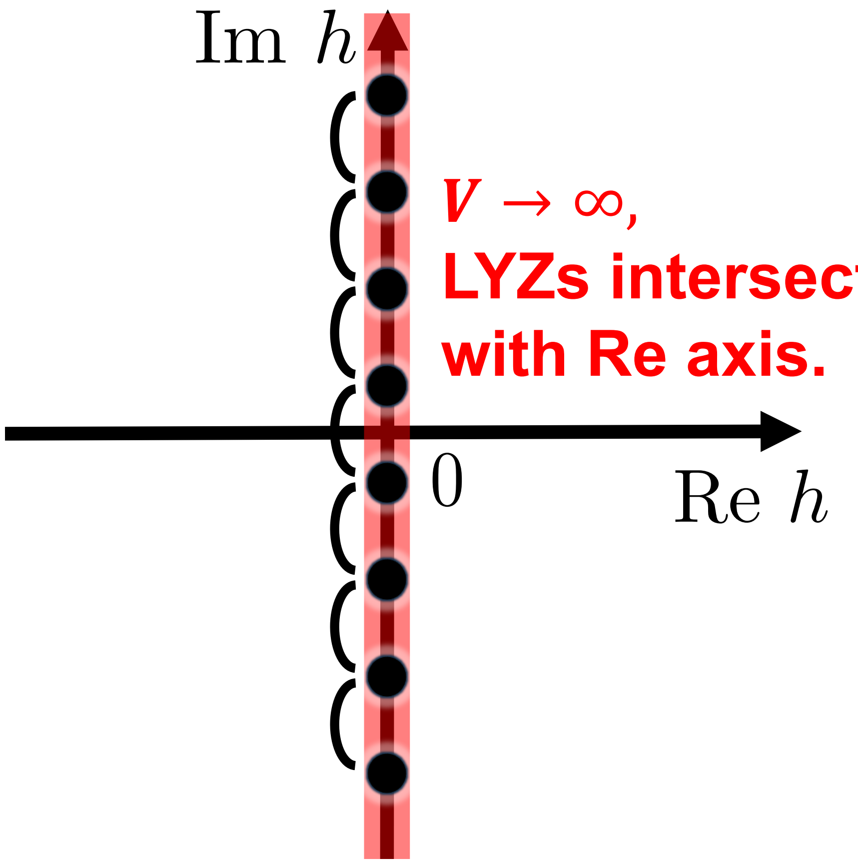
Lee-Yang Zeros

= Zeros of Z in Complex Parameter space

Ex.) d-dimensional Ising model $s_i = \pm 1$

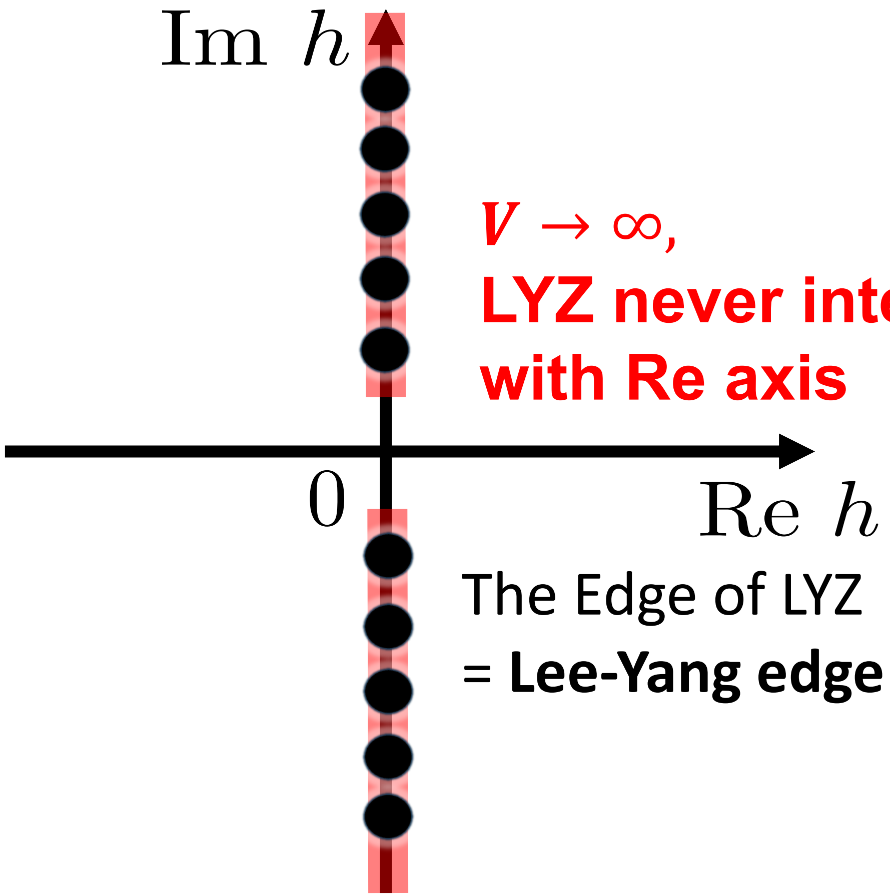
$$Z(t,h,L) = \mathrm{Tr}\exp\left(\frac{1}{T}\sum_{\langle i,j\rangle} s_i s_j + \frac{h}{T}\sum_{i}^{L^d} s_i\right)$$

$t < 0$:1st order



$V \to \infty$,
LYZs intersect with Re axis.

$t > 0$: Crossover



$V \to \infty$,
LYZ never intersect with Re axis

The Edge of LYZ
= **Lee-Yang edge singularity**

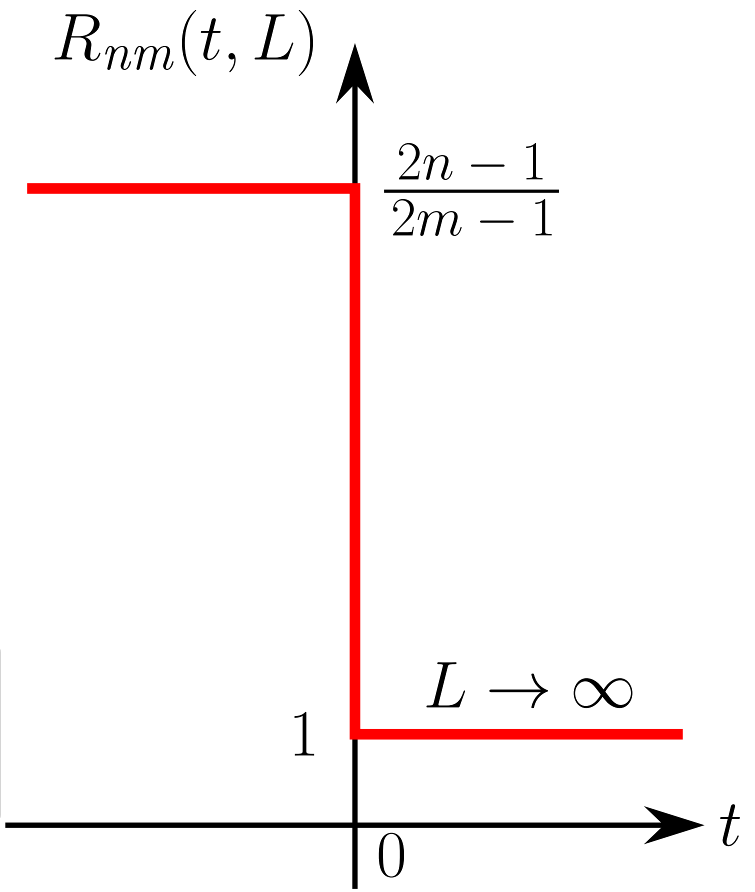
Lee-Yang Zero Ratio

TW, Kitazawa, Kanaya (2025)

LYZ Ratio

$$R_{nm}(t,L) \equiv \frac{h^{(n)}(t,L)}{h^{(m)}(t,L)}$$

$$R_{nm}(t,L) \xrightarrow[L\to\infty]{} \begin{cases} \dfrac{2n-1}{2m-1} & (t<0) \\ 1 & (t>0) \end{cases}$$



◆ $t < 0$: 1st order phase transition

The distance between LYZ and Re-axis is 1:3:5… in $V \to \infty$

→ Ratio = $(2n-1)/(2m-1)$

◆ $t > 0$: Crossover

All LYZ accumulate at LYES in $V \to \infty$

→ Ratio = 1

How is the behavior in finite volume?

Finite Size Scaling (FSS)

FSS of Free Energy $\beta F_{\text{sing}}(t, h, L^{-1}) = -\log Z(t, h, L^{-1})$

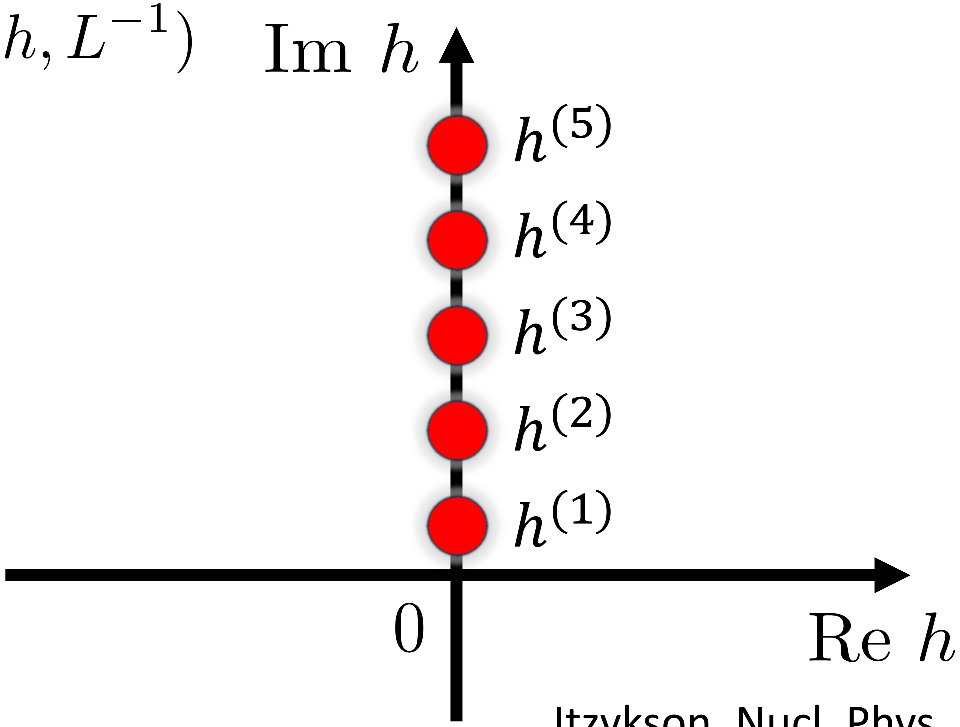
$$F_{\text{sing}}(t, h, L^{-1}) = \tilde{F}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

FSS of Partition function

$$Z_{\text{sing}}(t, h, L^{-1}) = \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

Factorization of Partition function

$$Z_{\text{sing}}(t, h, L^{-1}) = \prod_{n=1}^{N} (h - h^{(n)}(t, L^{-1}))$$



Itzykson, Nucl. Phys. **220** (198

LYZ function

$$h = h_{\text{LY}}^{(n)}(t, L^{-1})$$

FSS of LYZ

$$L^{y_h} h_{\text{LY}}^{(n)}(t, L^{-1}) = \tilde{h}_{\text{LY}}^{(n)}(L^{y_t} t)$$

We can treat finite volume using FSS in the vicinity of the CP!

LYZR near CP

TW, Kitazawa, Kanaya (2025)

FSS & Taylor Expansion near the CP

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t, L^{-1}) = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$

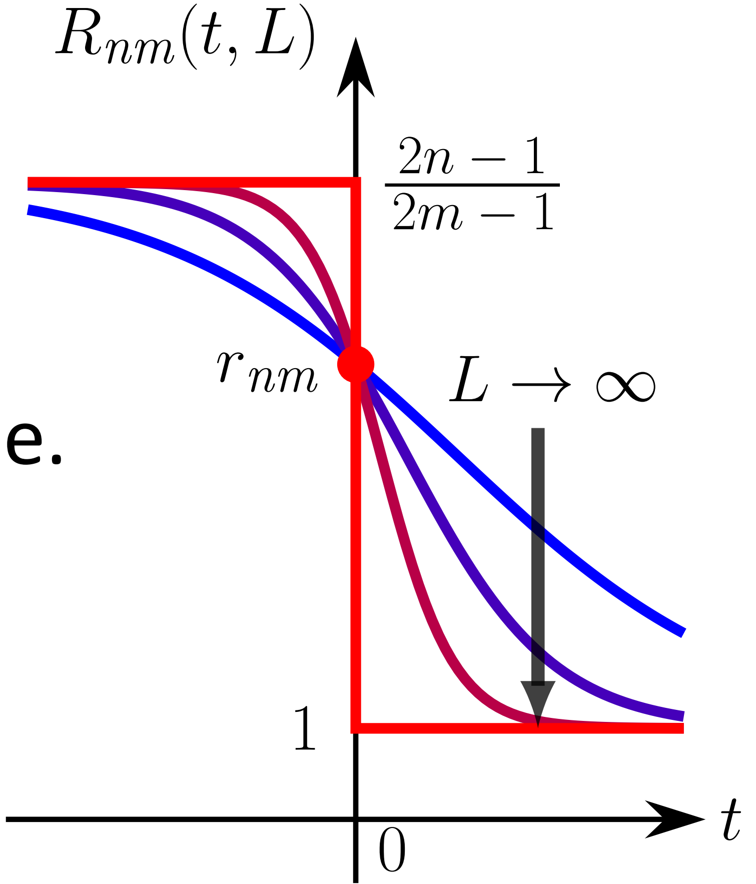
$$R_{nm}(t, L) \equiv \frac{h^{(n)}(t, L)}{h^{(m)}(t, L)} = \frac{X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)}{X_m + Y_m L^{y_t} t + \mathcal{O}(t^2)}$$

$$\boxed{R_{nm}(t, L) = \left(r_{nm} + c_{nm} L^{y_t} t + \mathcal{O}(t^2)\right)}$$

$\boldsymbol{R_{nm}(t = 0)}$ does not depend on the volume.

→ Useful to locate CP

Cf. Similar to Binder Cumulant $B_4(t = 0) = b_4$



Crossing Point = Critical Point

Lattice Simulation in HQ-QCD

Lattice Simulation

- Wilson Fermion + Hopping parameter expansion → Heatbath method

$$\kappa \sim \frac{1}{2am} \ \text{Hopping parameter}$$

- Lattice size : $N_s^3 \times (N_t = 4)$

  Aspect Ratio $\boldsymbol{LT = N_s/N_t = 8, 9, 10, 12}$

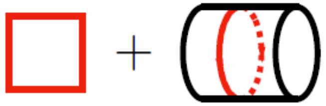
- # measurement = 6×10^5

  →**Reweighting method**

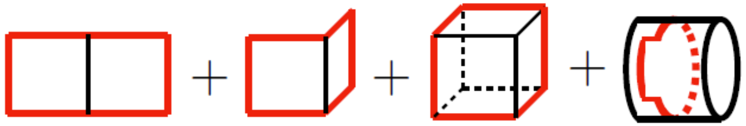
- For fixed $\beta \in \mathbb{R}$, search LYZ **complex λ-plane**

$$\lambda \equiv 64 N_c N_f \kappa^4$$

$S_G \sim$ 

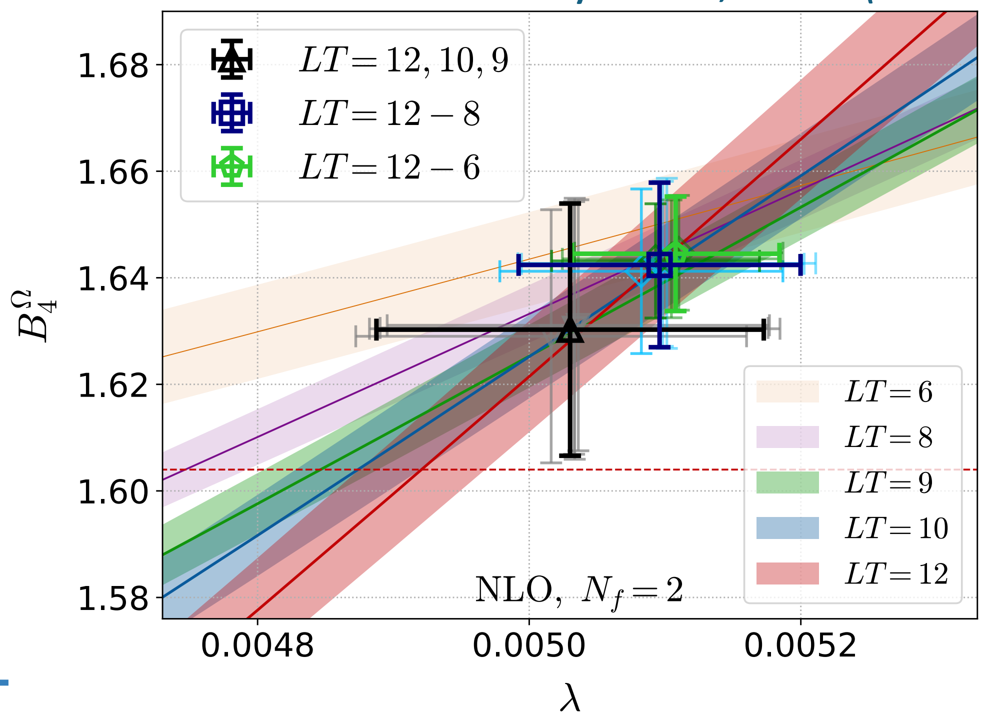
$S_{\mathrm{LO}} \sim$ 

Generation of Config.

$S_{\mathrm{NLO}} \sim$  Reweighting method

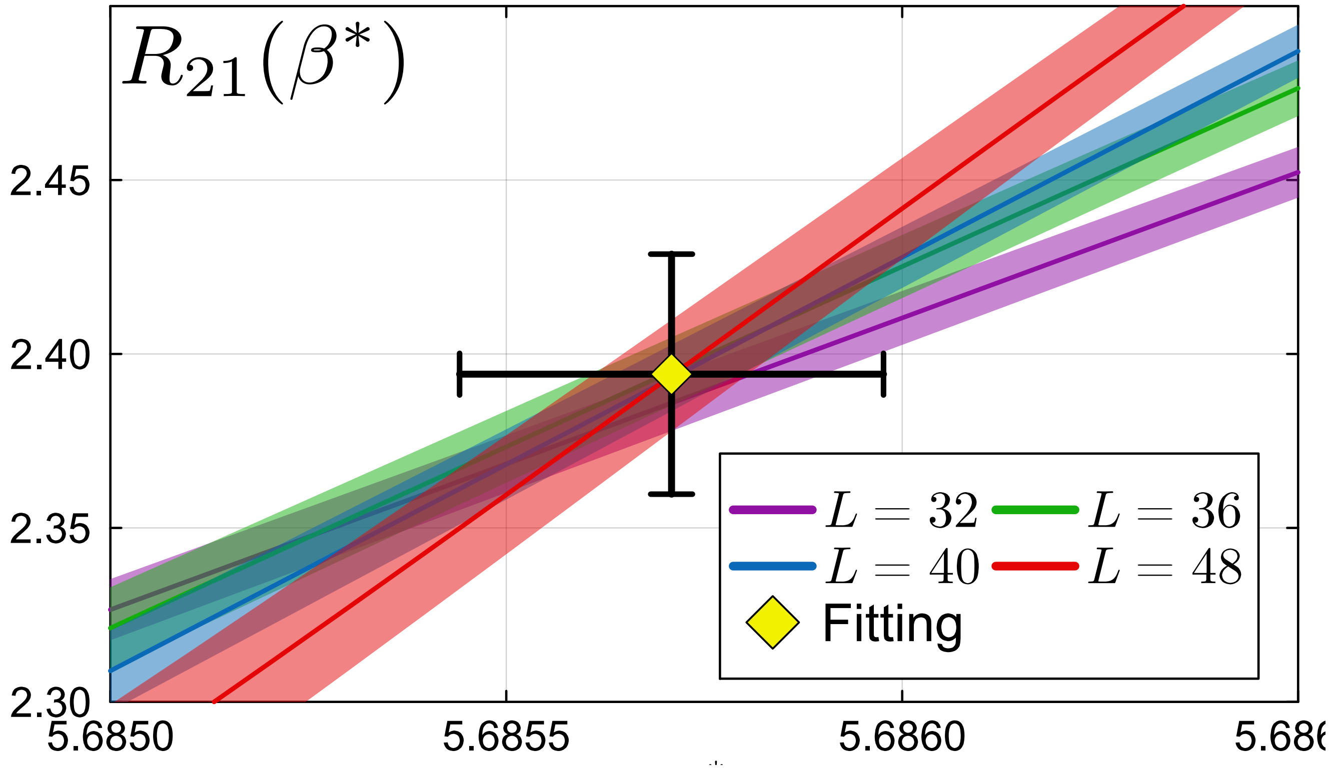
Binder Cumulant analysis

Kiyohara, *et al. (2021)*



Heavy-Quark QCD CP

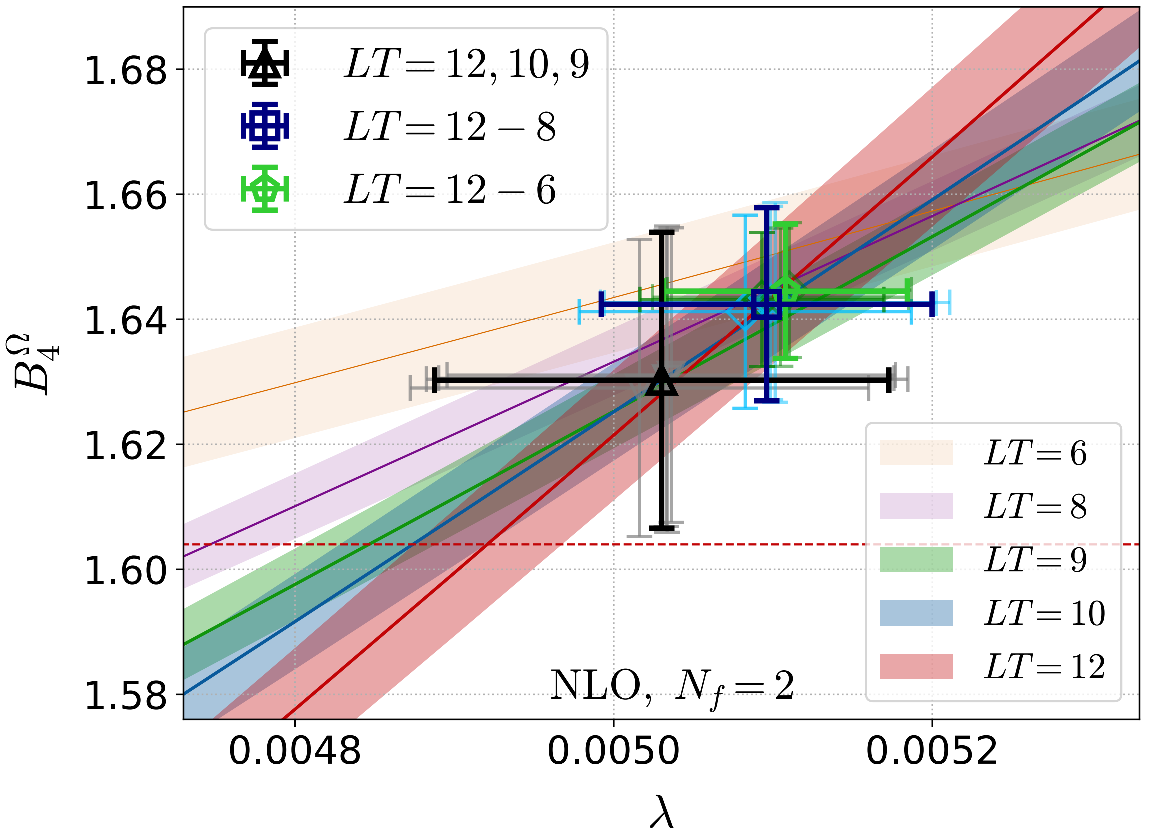
$2^{nd}/1^{st}$ LYZ Ratio



$$\beta^* = \beta - \lambda/4$$

Binder Cumulant analysis

Kiyohara, *et al. (2021)*

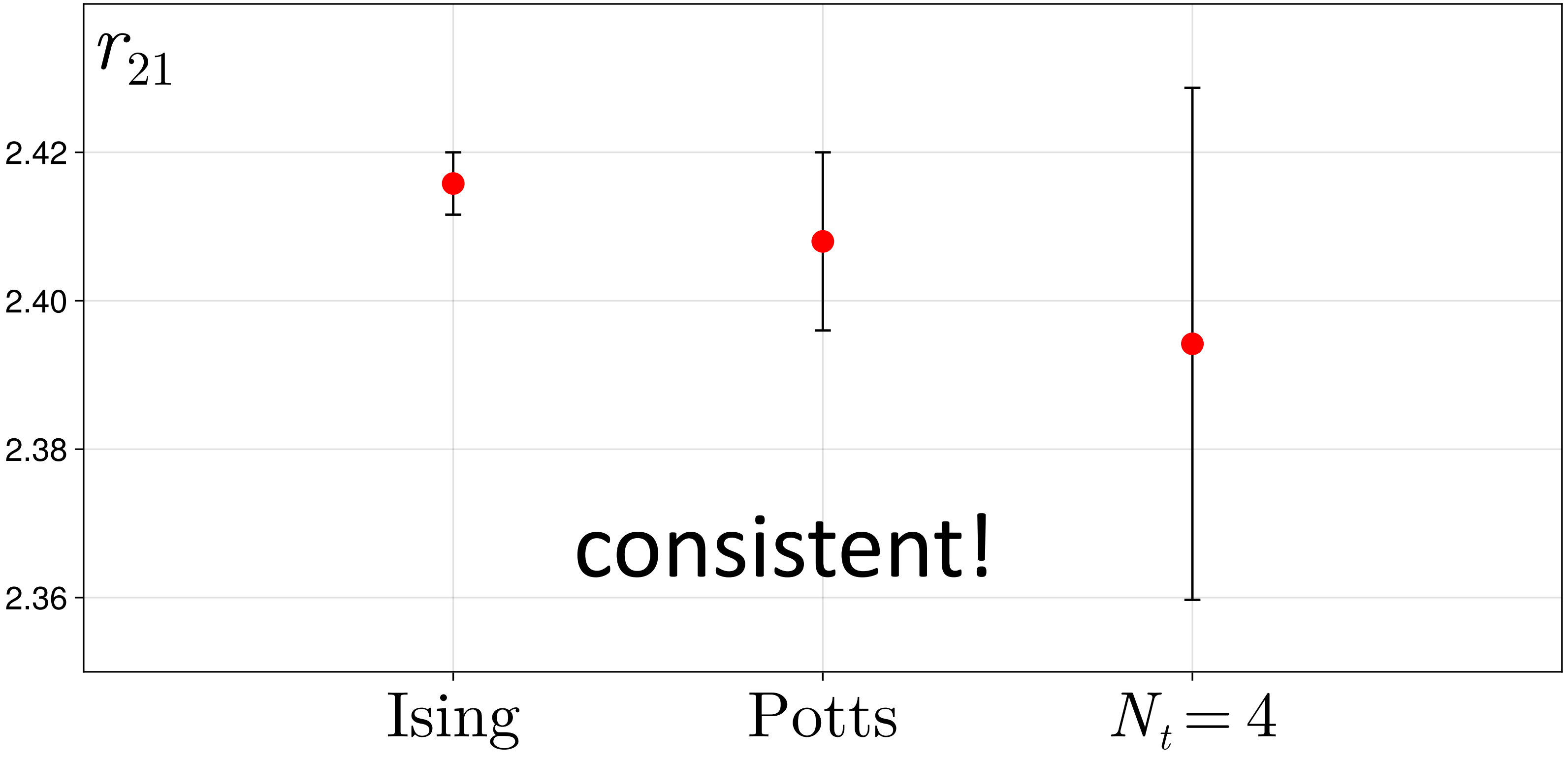


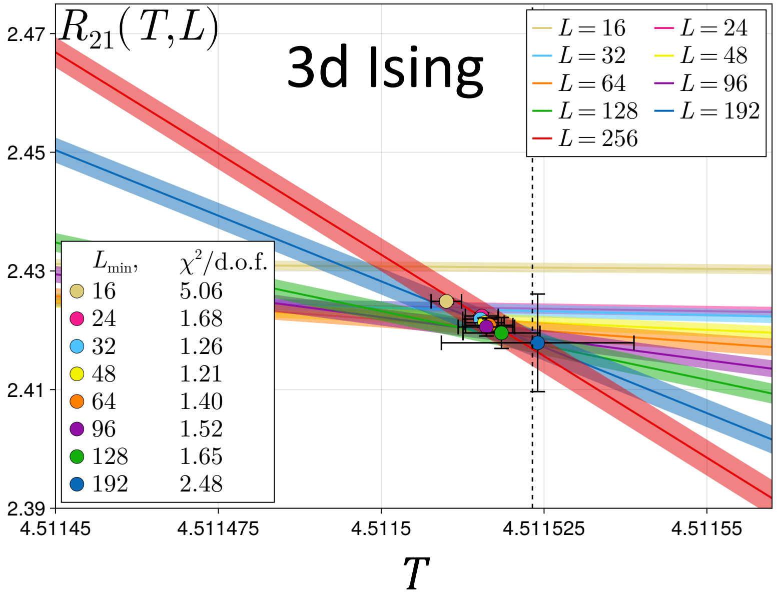
Binder : $\beta^* = 5.68579(26), \lambda = 0.00503(14)(2)$
LYZR21 : $\beta^* = 5.68571(27), \lambda = 0.00472(14)$

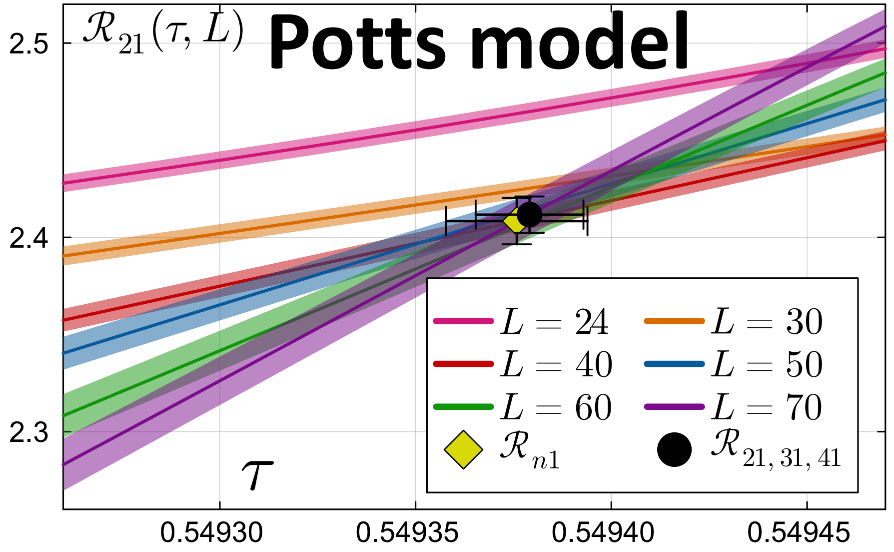
Consistent Result!

Comparing the r_{21}

TW, Kitazawa, Kanaya (2025)



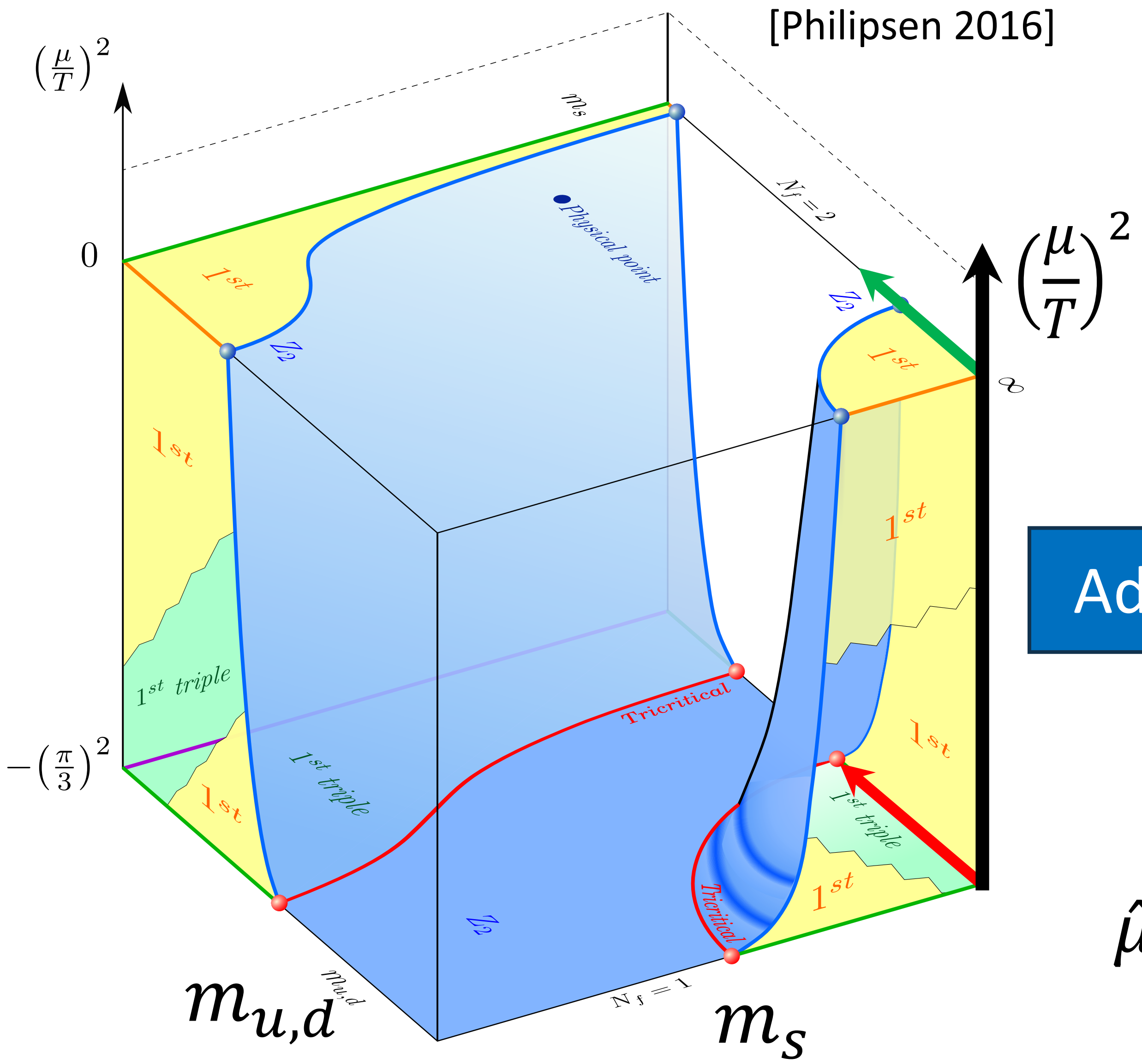


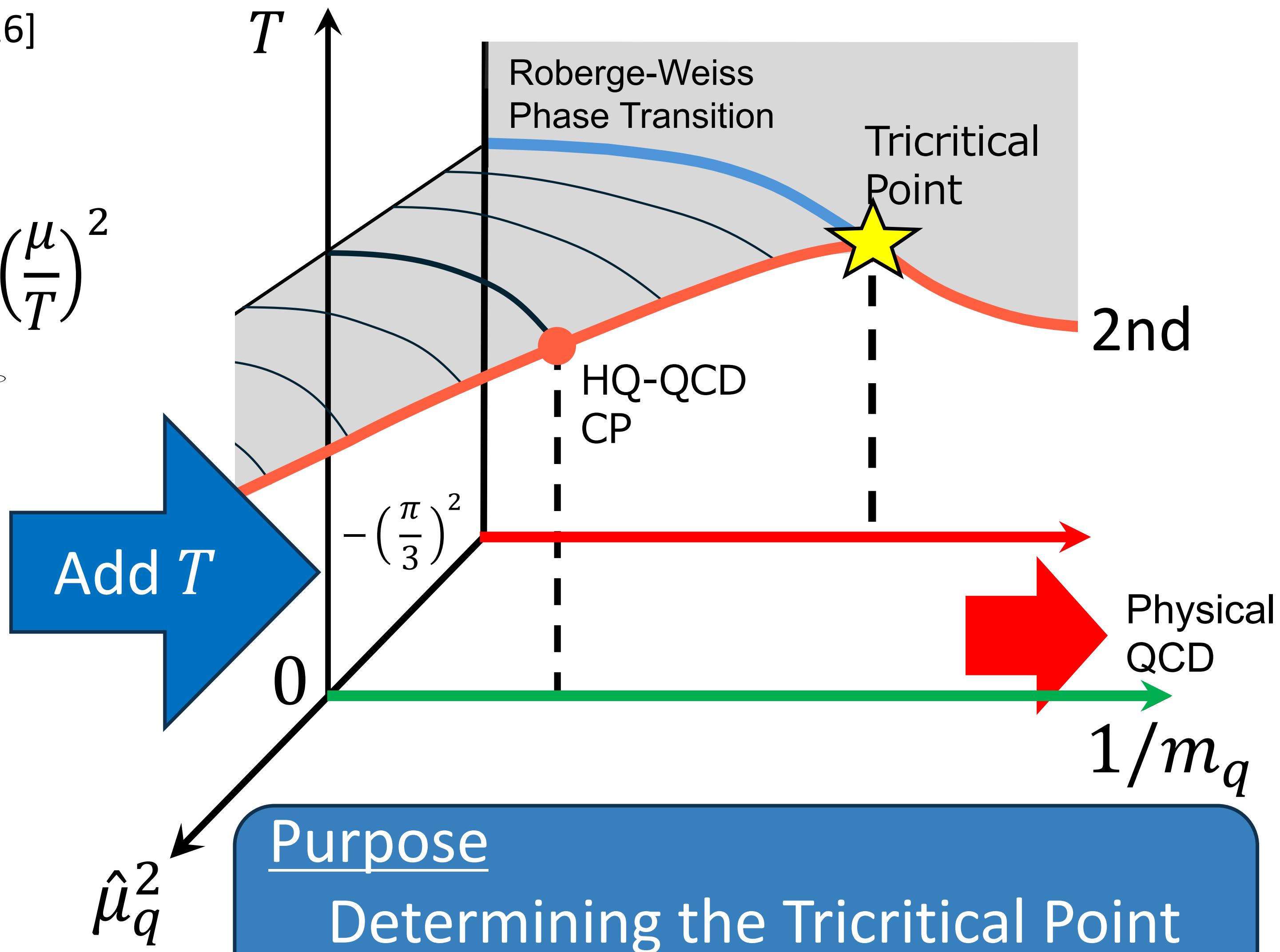


※Ising Result is extrapolated to infinite volume.
The others are not.

Columbia Plot at non-zero μ

3d-Columbia Plot

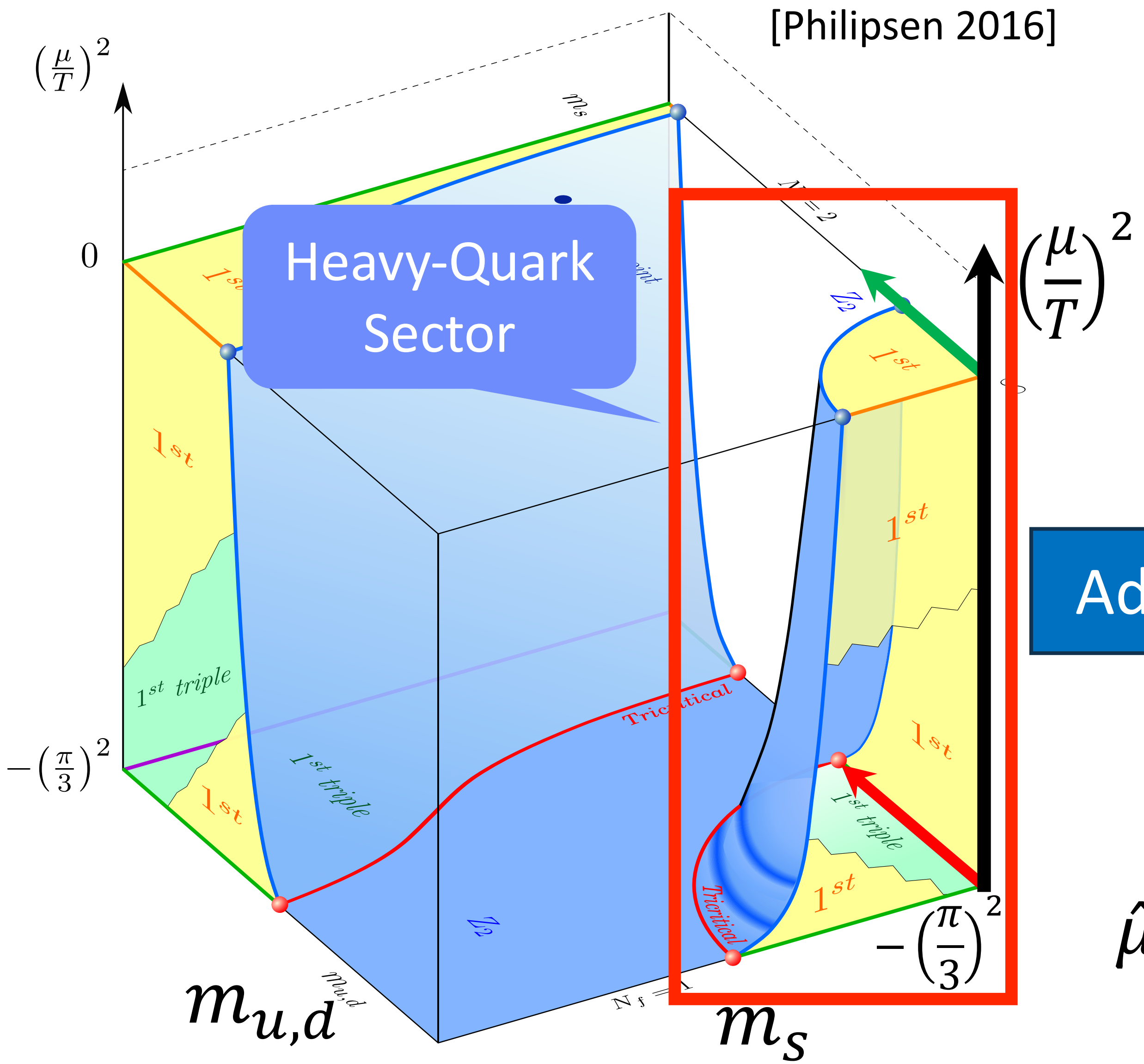




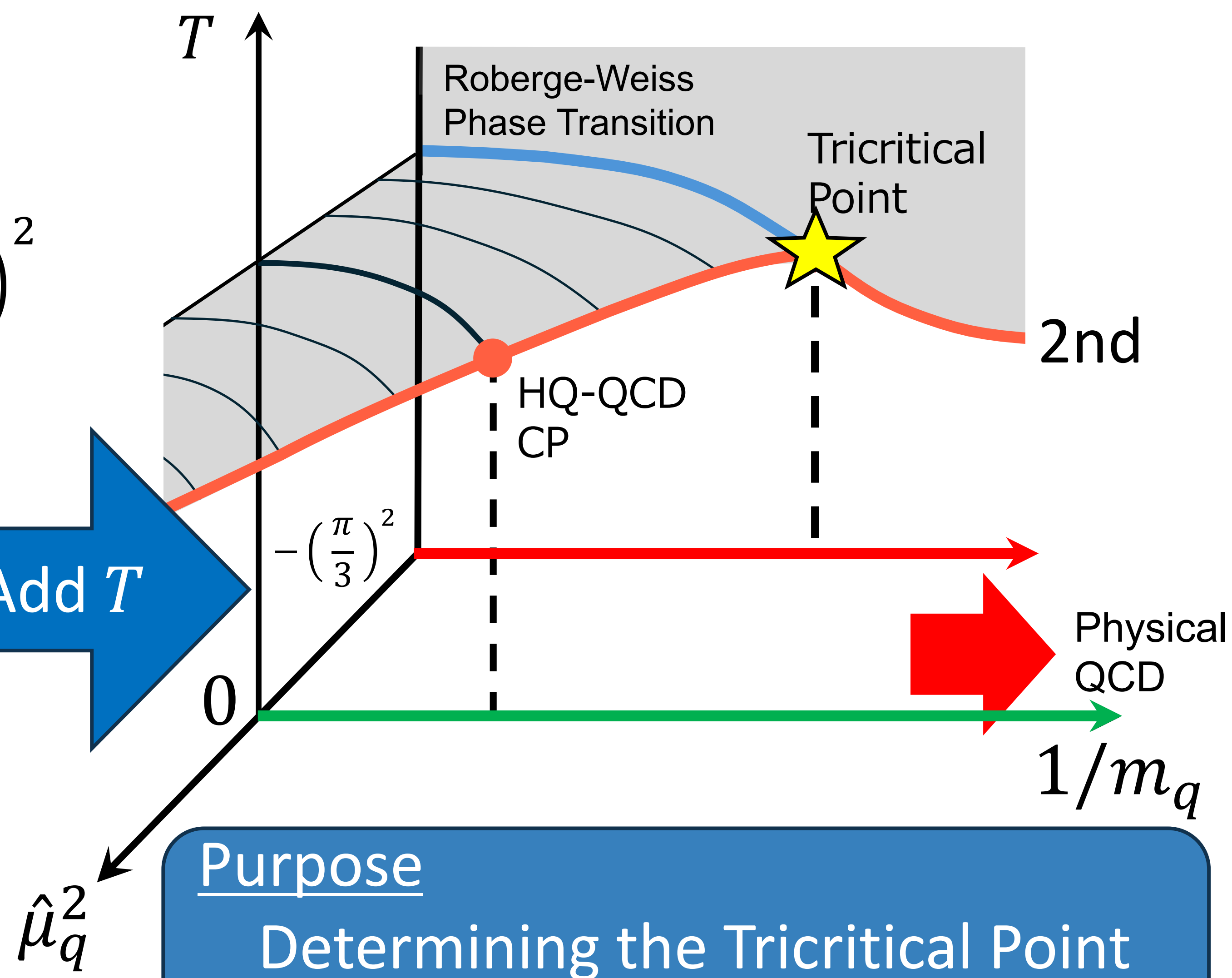
<u>Purpose</u>

Determining the Tricritical Point at $\mu = i\pi/3$ by LYZ approach

3d-Columbia Plot



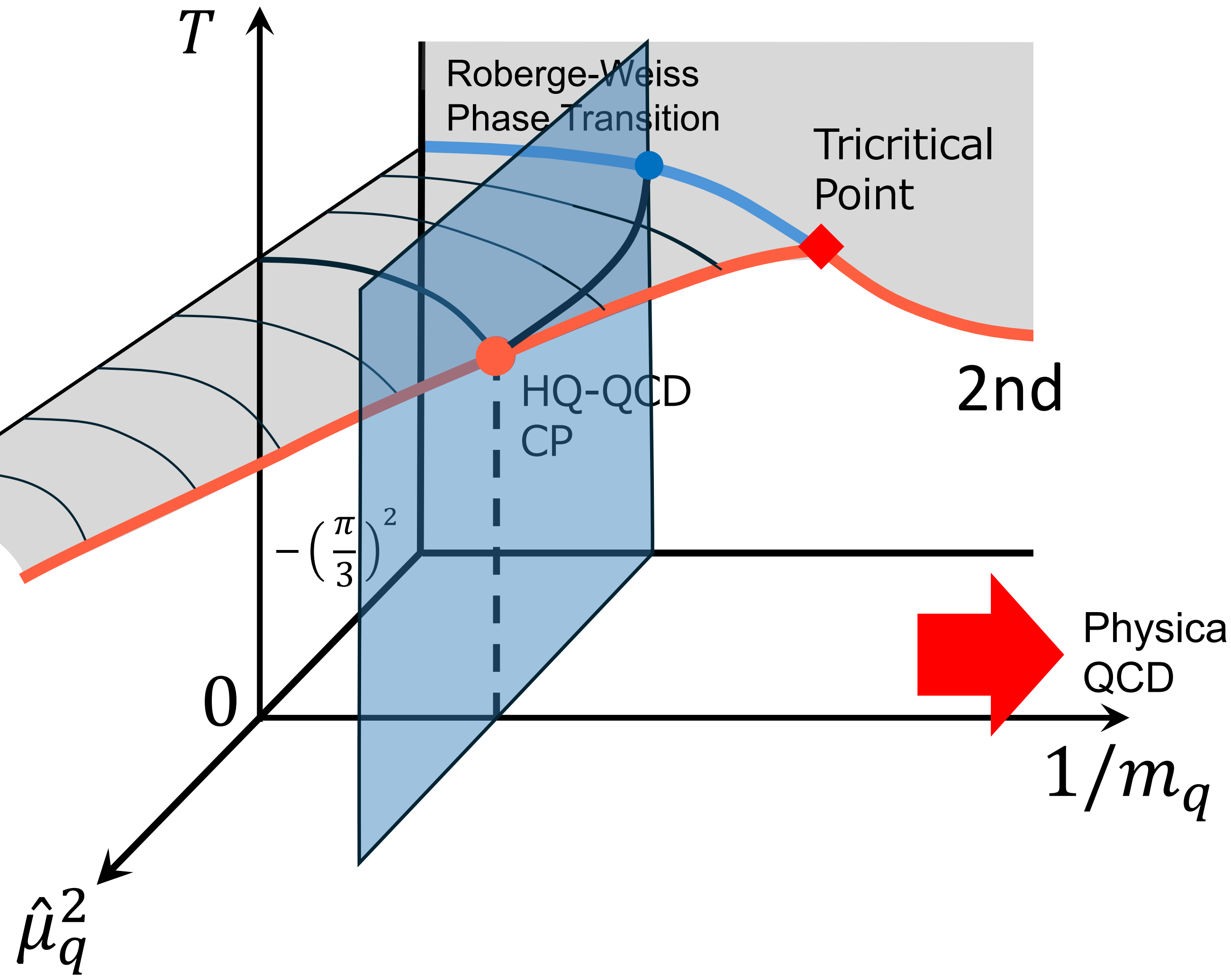
Add T



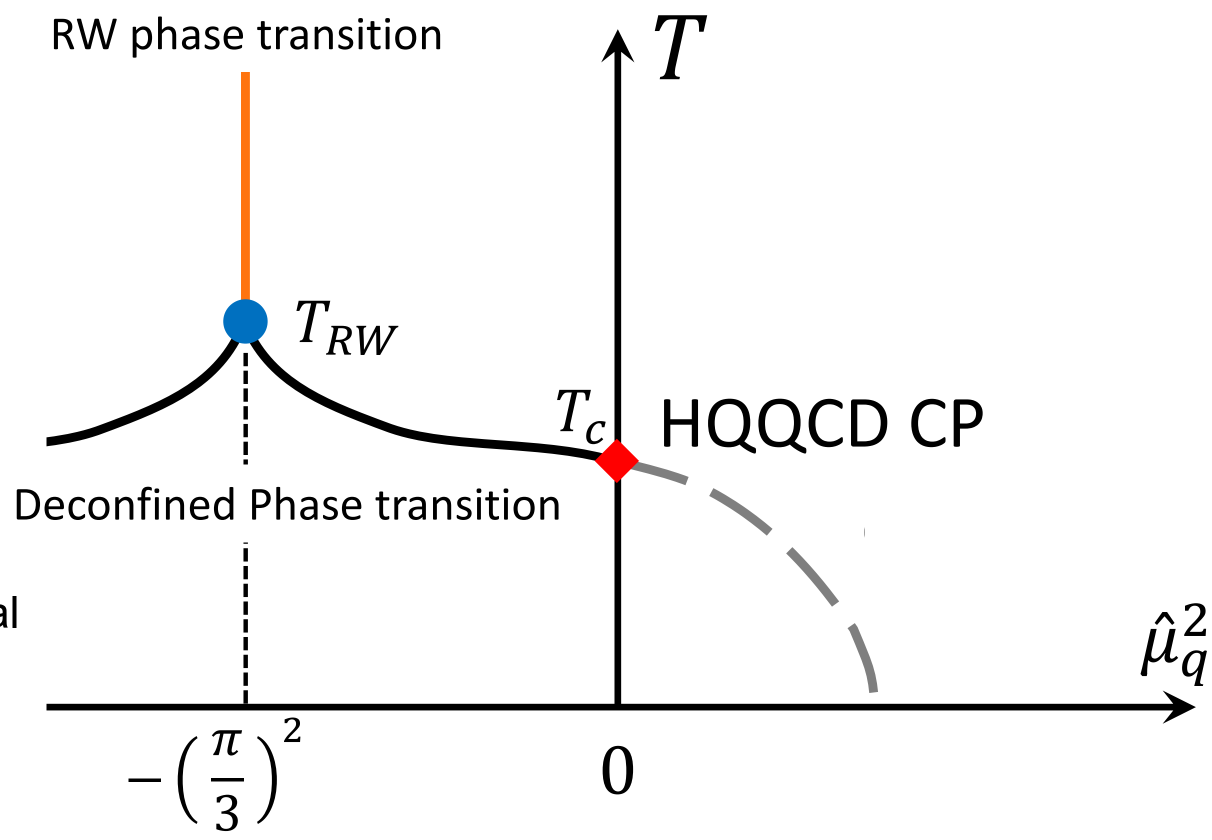
Purpose

Determining the Tricritical Point at $\mu = i\pi/3$ by LYZ approach

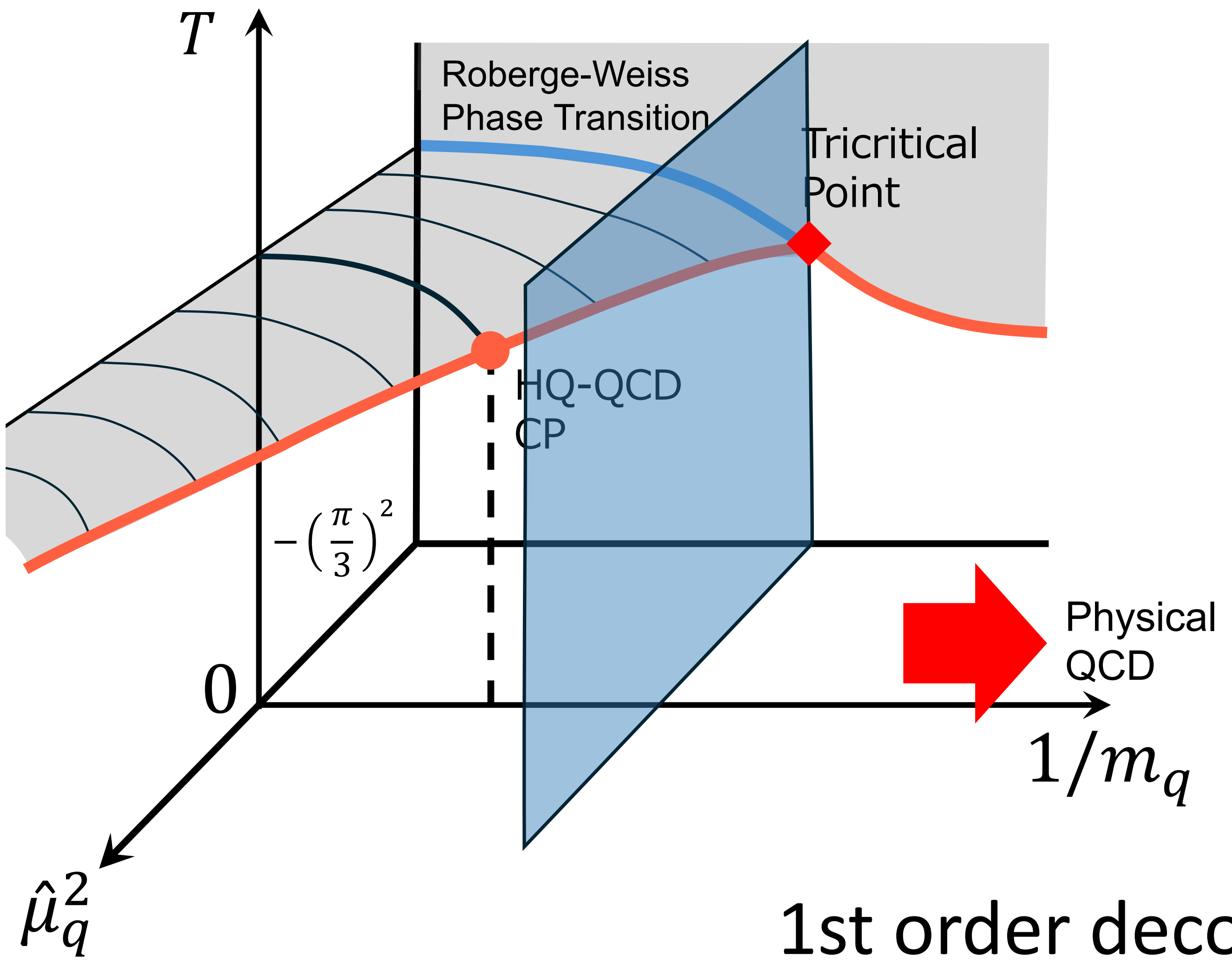
Columbia Plot at non-zero μ

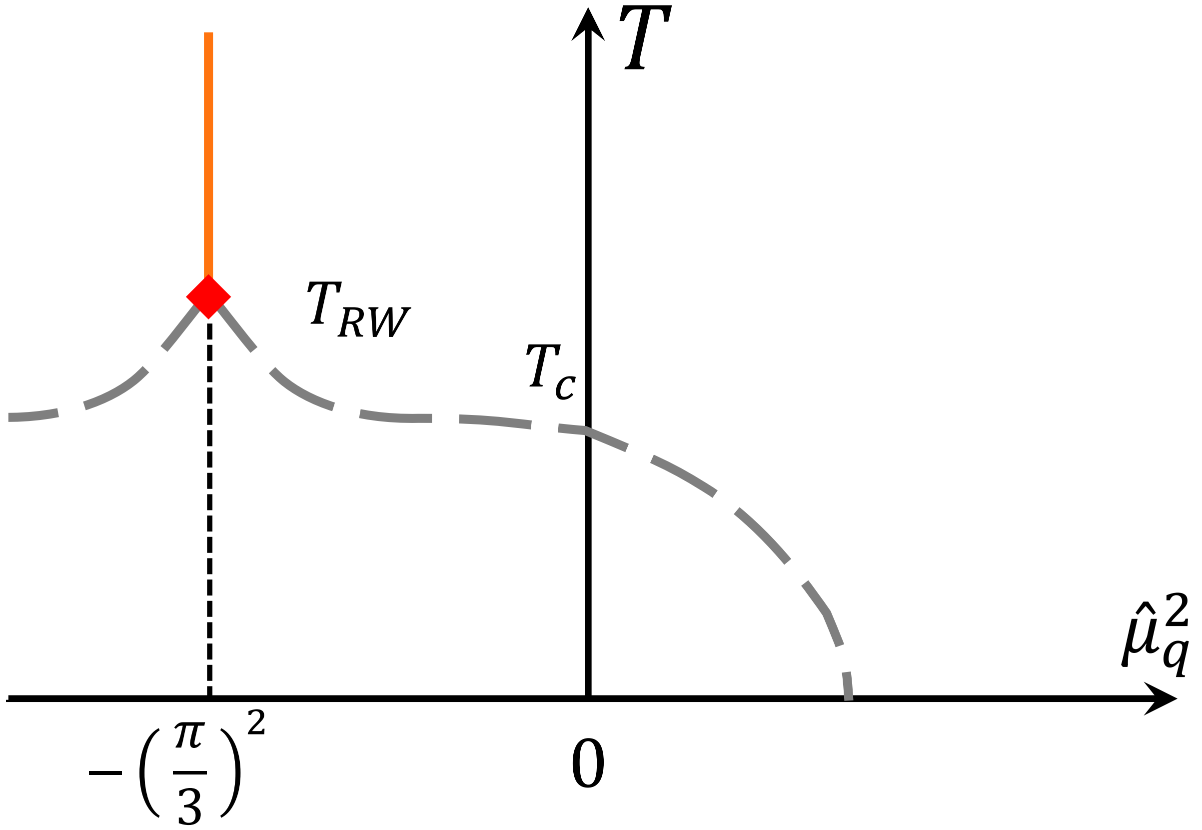


Physical
QCD



Columbia Plot at non-zero μ

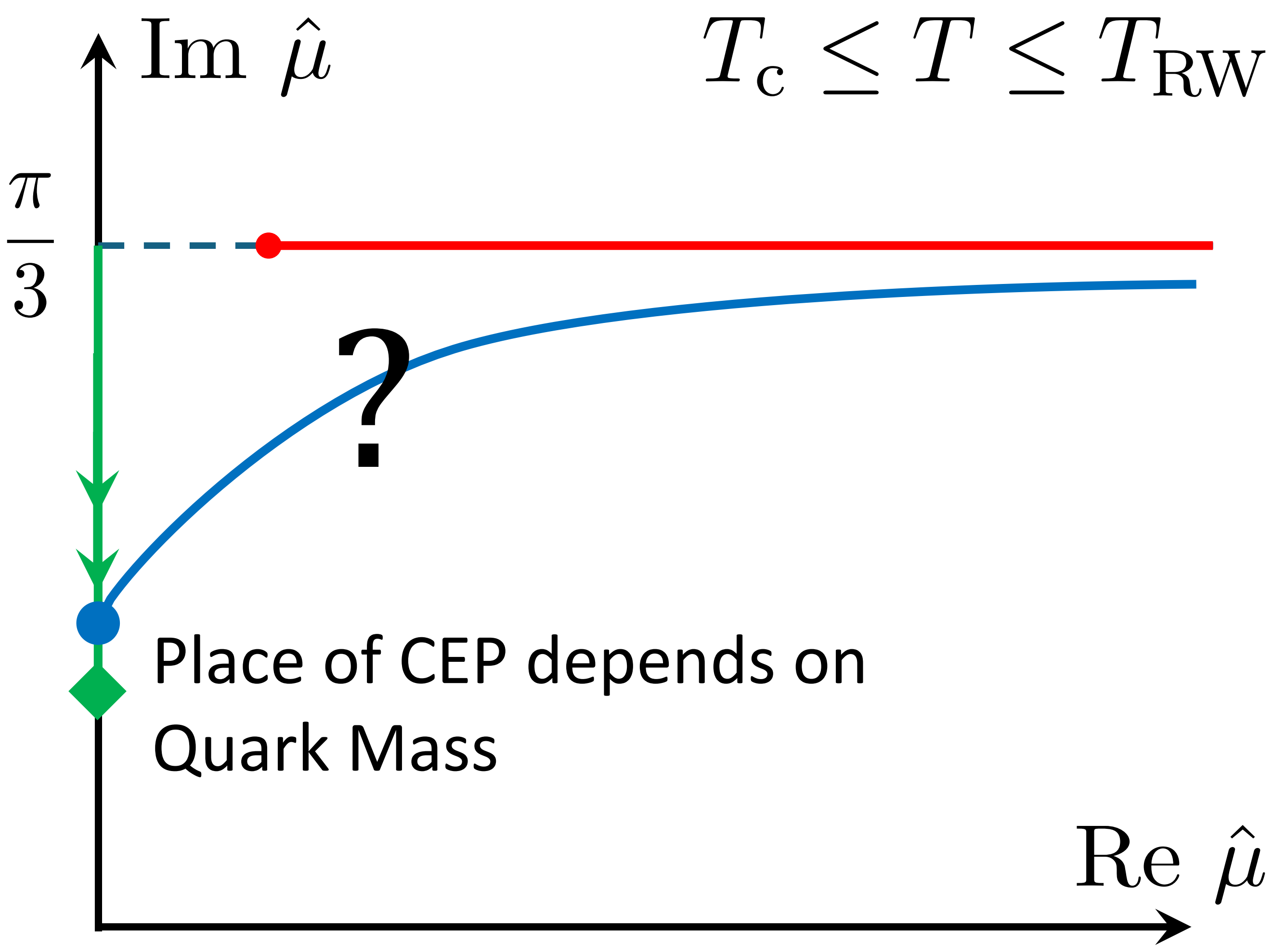




1st order deconf. phase transition lines disappear.

Behavior of LYZ

$V \to \infty$



Series 1

Roberge Weiss Phase transition

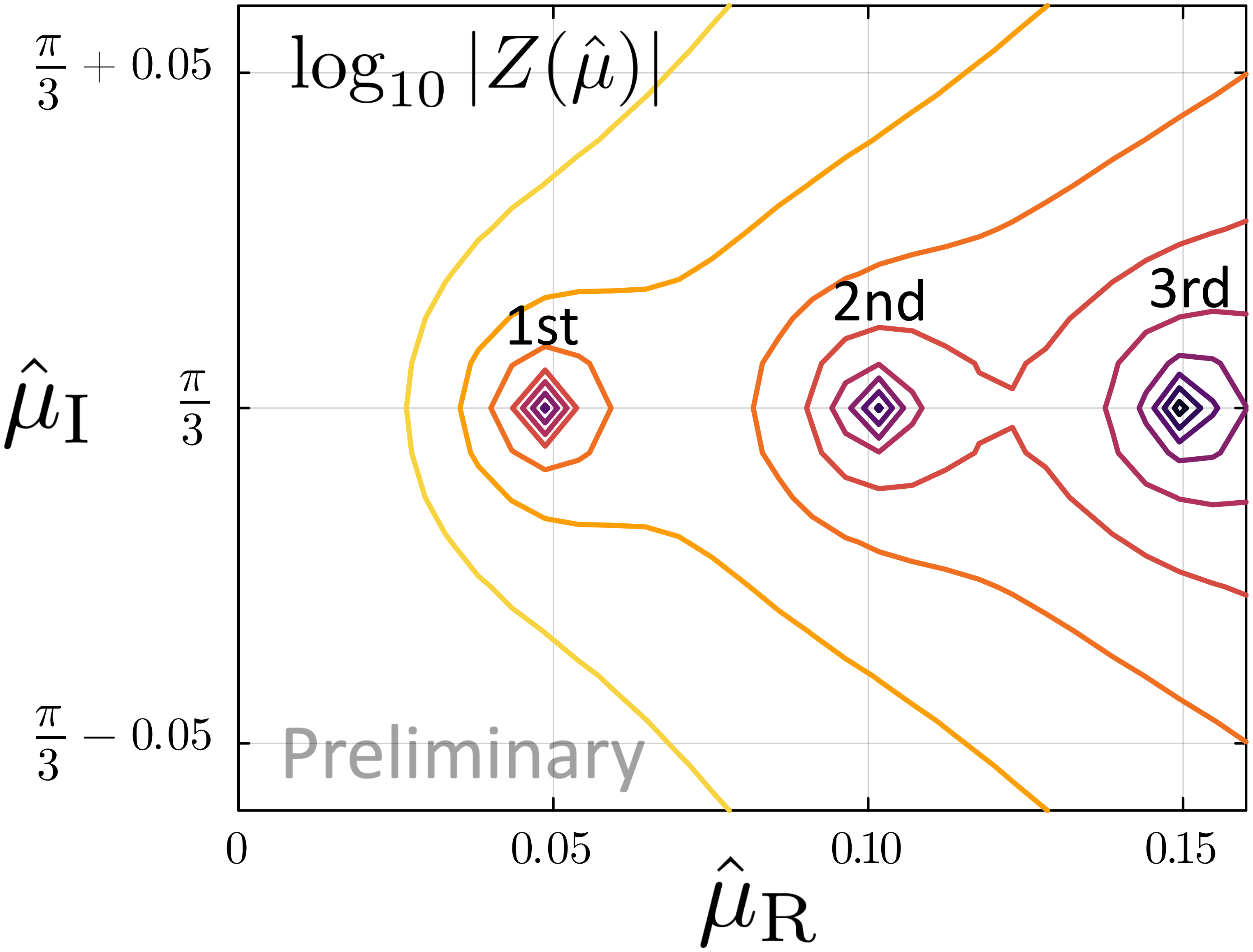
Red line extends to the Re-axis.
Due to the Symmetry of Polyakov loop,
LYZ exists only at $\mu_I = \pi/3$.

Series 2

Deconfined Phase transition

The edge of the LYZ are predictable,
the zero lines extend in ℂ-plane
remains unclear

Contour of Partition Function



$N_x = 24, N_t = 4$
$\kappa \simeq 0.107$
$\beta = 5.6571$

1st-3rd LYZ are determined

The generation of configurations and the implementation of the LYZR method are ongoing.

Summary

- ✓ We detect the critical point in Heavy-Quark QCD using Lee-Yang zero Ratio method. The location of CP is consistent with Binder cumulant method.
- ✓ We conjecture the behavior of Lee-Yang zeros in the non-zero chemical potential.
- ✓ We make the configuration at $\mu = i\pi/3$ and detect the LYZ from Roberge Weiss Phase transition.

Future Works

- ☐ Calculating the Lee-Yang zero Ratio at $\mu = i\pi/3$ and detect the tricritical point in the Heavy-Quark region.
- ☐ Detect Critical end point in the imaginary and real μ
  →They give insight into QCD at m_{phys}
- ☐ Final Goal = Application to 2+1 flavor QCD
  → QCD critical point (We need 2nd LYZ)