Global structure of η′

Yuya Tanizaki (Yukawa Institute, Kyoto)

Based on the following works: 2201.06166 (with Mithat Ünsal), and 2402.04320, 2405.12402 (with Yui Hayashi) + on-going

QCD vacuum structure

Global symmetry: $\dfrac{\boxed{SU(N_f)_L \times SU(N_f)_R} \xrightarrow[\text{SSB}]{} SU(N_f)_V \quad \times U(1)_V \times \cancel{U(1)_A}}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f} \times \mathbb{Z}_2}$ explicitly broken by ABJ anomaly

Spontaneous chiral symmetry breaking via fermion-bilinear condensate:

$$\langle \bar{\Psi}_i \Psi_j \rangle = -\Lambda^3 \underbrace{e^{i\eta'/N_f}}_{\text{flavor-singlet } \eta'} \cdot \underbrace{U_{ij}}_{\pi, K, \eta}$$

One of the common chiral Lagrangian + η':

$$\mathcal{L} = \frac{F_\pi^2}{2} \mathrm{tr}(\partial_\mu U^\dagger \partial_\mu U) - \Lambda^3 \mathrm{tr}\left(M e^{i\frac{\eta'}{N_f}} U + c.c.\right) + WZW(U)$$

$$+ \frac{F_\pi'^2}{2}(\partial_\mu \eta')^2 - \underbrace{\# e^{-\frac{8\pi^2}{g^2}} \cos(\eta' + \theta)}_{\text{Kobayashi-Maskawa-'t Hooft vertex}}.$$

Main claim

$\boxed{\text{Global structure of } \eta' \text{ is more involved \& interesting !!}}$

4d QCD / YM on $\mathbb{R}^2 \times T^2_{\text{twist}}$

smoothly connected

$NL\Lambda \lesssim 1$

- Weakly-coupled
- Many expected features of 4d dynamics are obtained

$L\Lambda \sim O(1)$

- Confinement w/ strongly-coupled dynamics
- Difficult to solve

size of T^2

[YT, Ünsal '22]

- We solve QCD on $\mathbb{R}^2 \times T^2_{\text{twist}}$ with "controllable" approximation.

  No IR divergence at all.

  $\Longrightarrow$ chiral Lagrangian + η' is obtained.

Yang-Mills theory on $\mathbb{R}^2 \times T^2$ & 't Hooft flux

4d SU(N) YM : $\mathbb{Z}_N^{(1)}$ center symmetry

$$\mathbb{R}^2\times T^2 \Longrightarrow \begin{cases} \mathbb{Z}_N^{(1)} & : \text{Area vs Perimeter for 2d Wilson loops} \\ \mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)} & : \text{Conventional center symmetry for Polyakov loops } P_3, P_4 \end{cases}$$

Role of 't Hooft flux p

① 4d anomaly is maximally preserved in 2d effective theory

$$Z_{\theta+2\pi}[B] = e^{\frac{2\pi i}{N}\int \frac{1}{2} B\cup B} Z_\theta[B] \xRightarrow{T^2\text{-compact.}} Z_{\theta+2\pi}[B_{2d}] = e^{i\frac{2\pi}{N} p \int B_{2d}} Z_\theta[B_{2d}]$$

② Classical vacuum is unique & $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)}$ symmetric :

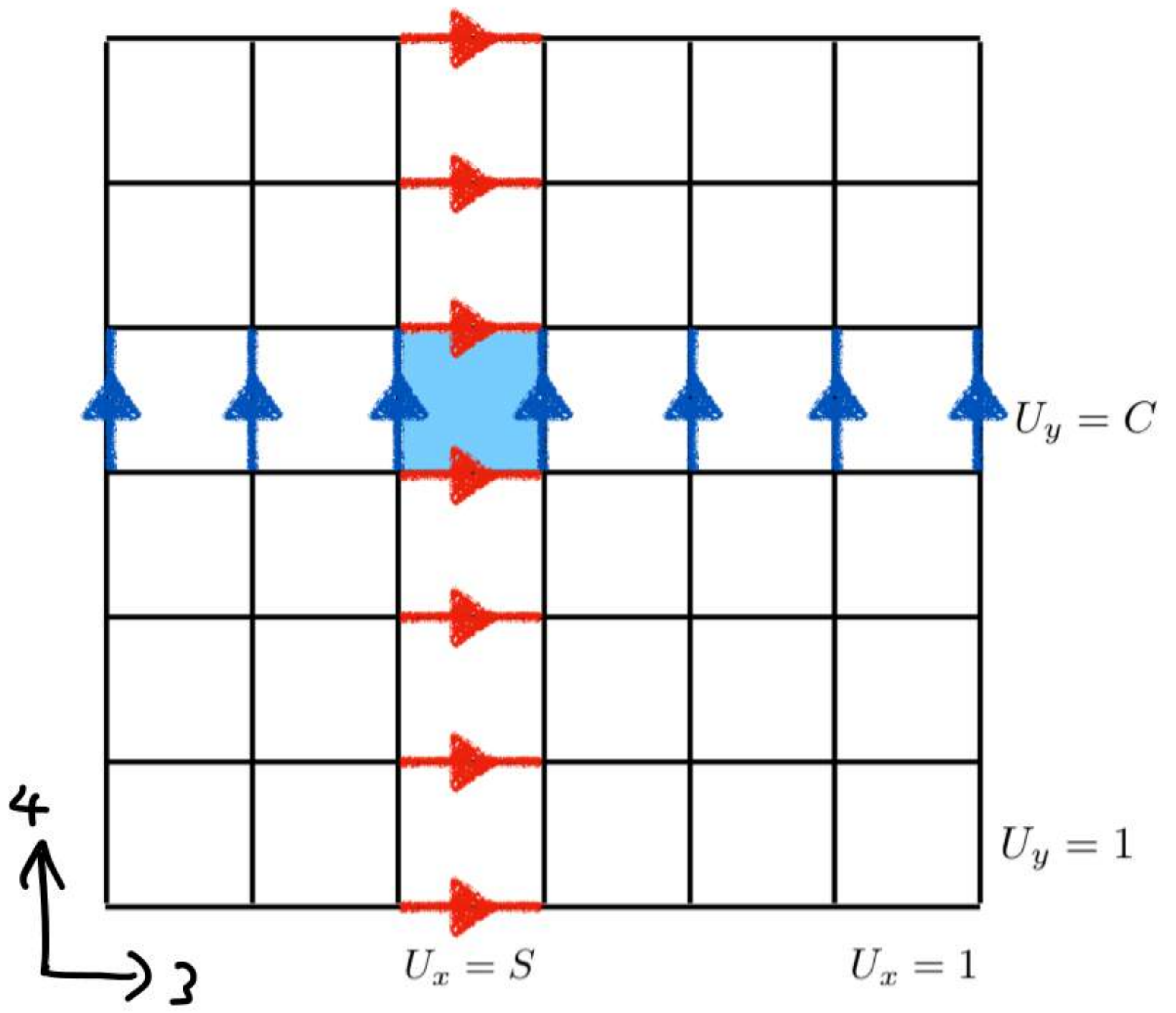
$$P_3\, P_4 = e^{\frac{2\pi i}{N} p}\, P_4 P_3$$

③ Classical vacuum "violates" $\mathbb{Z}_N^{(1)}$ but semiclassically restored as

4d instanton $\textcircled{I_4}$ $\xRightarrow{\mathbb{R}^2 \times T^2_{p\text{-twist}}}$  N center-vortex constituents.

[Gonzalez-Arroyo, Montero '98]

't Hooft flux & Classical vacuum



Lattice action

$$S_W[U_\ell, B] = -\frac{1}{g^2}\sum_p \left(e^{-iB_p}\,\mathrm{tr}[U_p] + e^{iB_p}\,\mathrm{tr}[U_p^\dagger] \right).$$

$$B_p = \begin{cases} \frac{2\pi}{N} & \text{(for the plaquette indicated with light blue)} \\ 0 & \text{(otherwise)} \end{cases}$$

We can minimize this action by setting

$$U_\ell = \begin{cases} S = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \\ C = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \\ \mathbb{1} \end{cases}$$

[* Any classical minimum is gauge equivalent to this one]

$$\Rightarrow \quad P_3 = S \quad , \quad P_4 = C .$$

This configuration completely preserves $\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

2d $\mathbb{Z}_N^{(1)}$ symmetry & center vortex

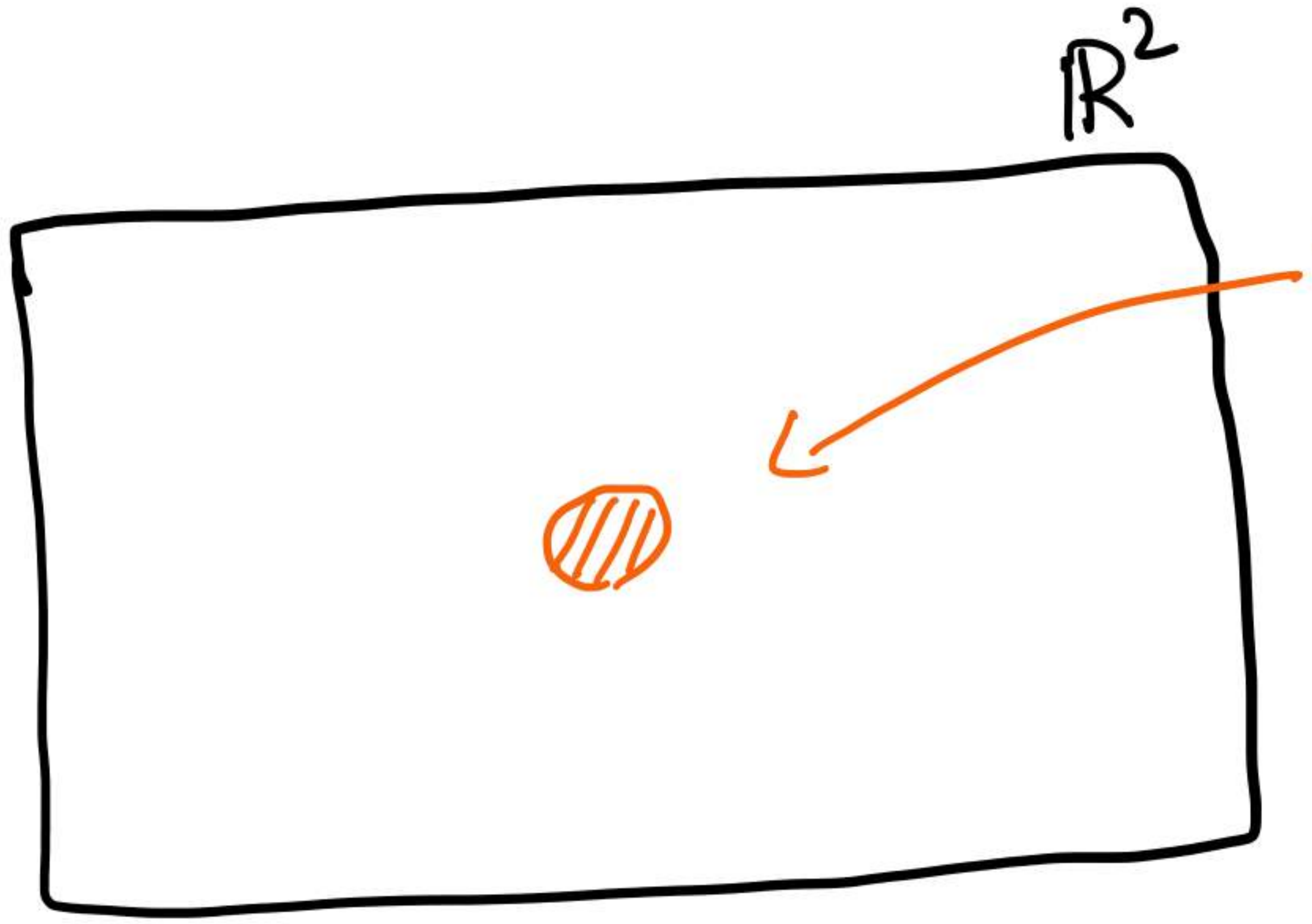
- For 2d effective theory,

  $P_3 = S$ and $P_4 = C$ behave as Adjoint Higgs with orthogonal VEV:

$$SU(N) \xrightarrow{\text{Higgs}} \mathbb{Z}_N .$$

$\Rightarrow$ 2d 1-form symmetry is spontaneously broken at classical vacua.

- Key player for the Area law: Center vortex = Fractional instanton



Vortex Soliton

= Center vortex

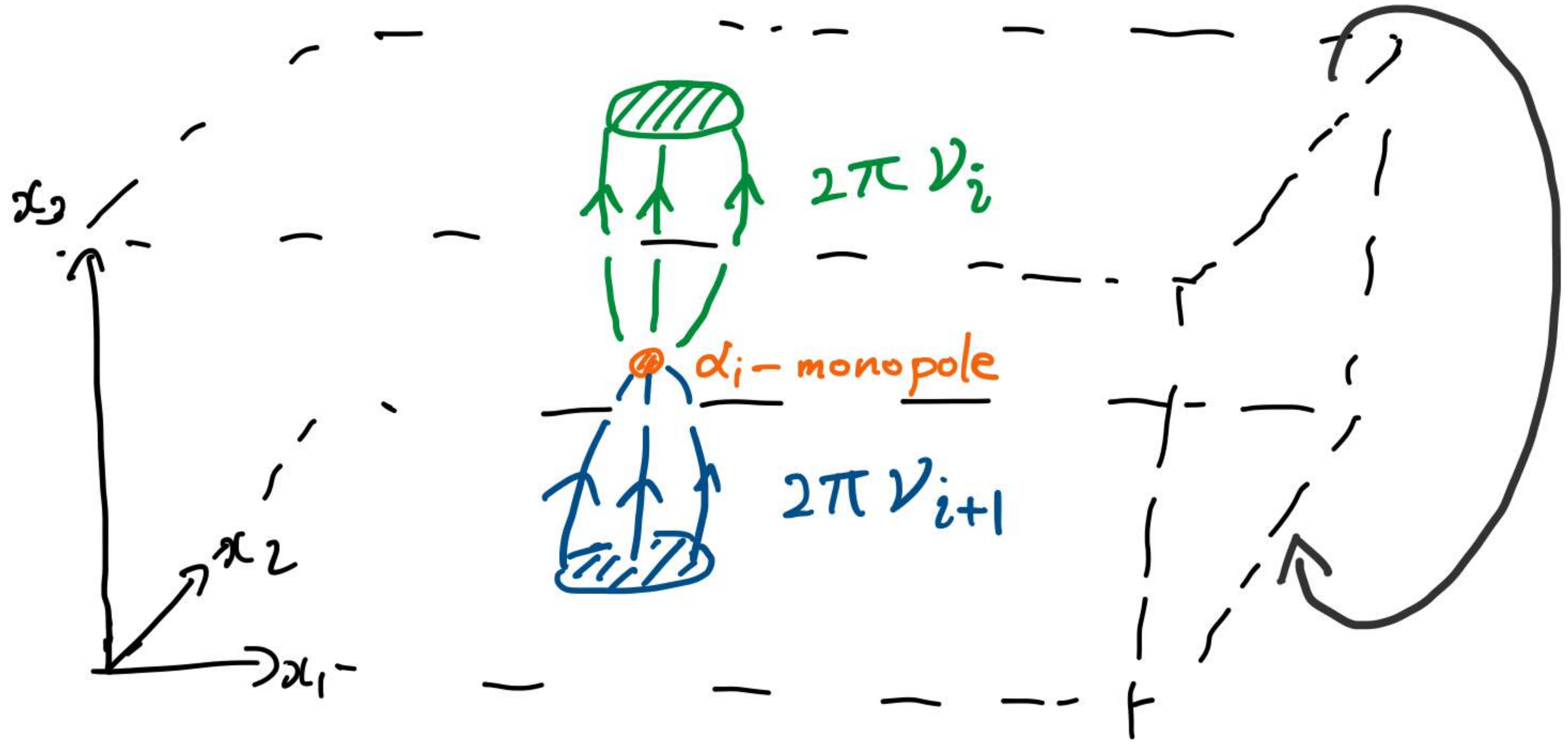
$$\begin{cases} Q_{top} = \pm \frac{1}{N} \\ W(C) \mapsto e^{\pm \frac{2\pi i}{N}} W(C) \end{cases}$$

Center vortex on $\mathbb{R}^2 \times \underbrace{T^2}_{\text{flux}}$ = KvBLLY monopole instanton

$SU(N)$ gauge field on $\mathbb{R}^3 \times S^1$ w/ nontrivial holonomy : N fundamental monopoles (Lee, Yi '97 Lee, Lu '98 Kraan, van Baal '98)

[$\Rightarrow$ 3d semiclassics by Ünsal, ... since 2007]

α_i - monopole emits the magnetic flux $2\pi\alpha_i = 2\pi(\nu_i - \nu_{i+1})$.



$\mathbb{Z}_N$-twisted b.c. (= 't Hooft flux on T^2)

$$\begin{pmatrix} \nu \\ \nu \\ \nu \end{pmatrix} \mapsto \begin{pmatrix} \nu \\ \nu \\ \nu \end{pmatrix}$$

[Hayashi, YT 2405.12402]

$\mathbb{Z}_N$-twisted b.c. gives the perturbative gap $\frac{2\pi}{NL_3}$ $\Rightarrow$ Magnetic flux localizes.

$\boxed{\text{Monopole = Junction of the center vortex}}$

(cf. Ambjorn, Giedt, Greensite '99, de Forcrand, Pepe '00)

Partition funcion on $\underset{\to \mathbb{R}^2}{M_2} \times T^2$ & θ-dependence

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

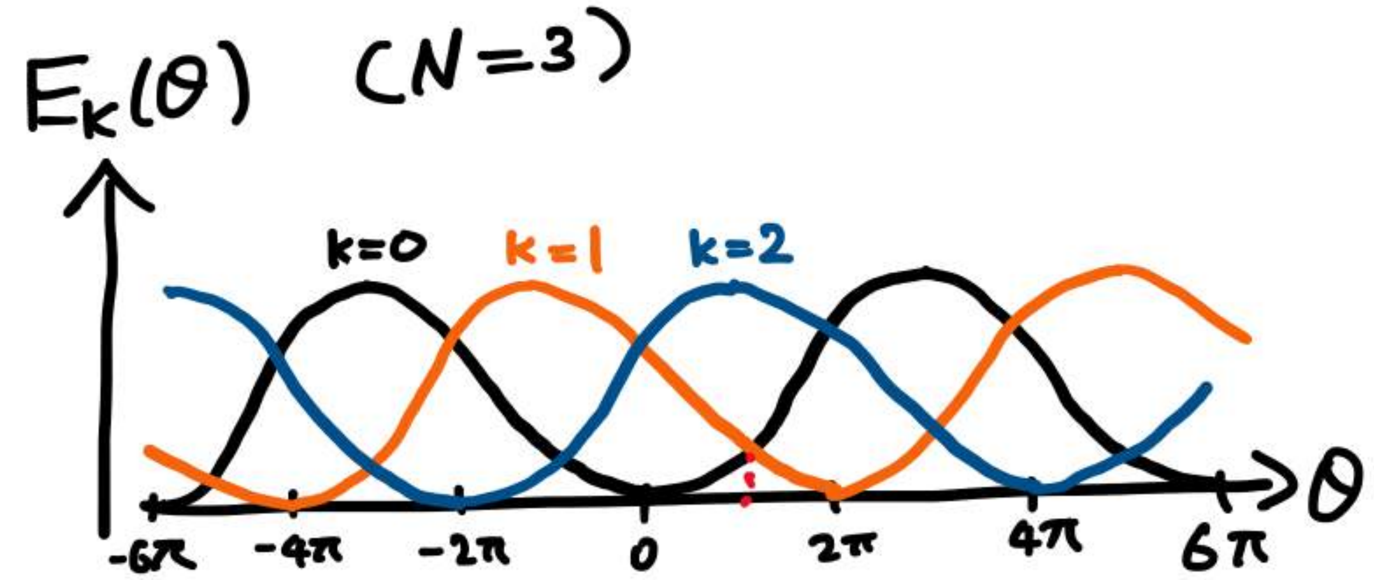
$$\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} k (n-\bar{n})} = \delta_{n-\bar{n}\in N\mathbb{Z}}$$

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\ \bar{n}!} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N} + i\frac{\theta}{N}}}_{\text{vortex}}\Big)^{n} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N} - i\frac{\theta}{N}}}_{\text{anti-vortex}}\Big)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\Big[-V\Big(\underbrace{-2K e^{-\frac{8\pi^2}{g^2N}} \cos\Big(\frac{\theta-2\pi k}{N}\Big)}_{E_k(\theta)}\Big)\Big]$$

$E_k(\theta)$: Ground-state energy densities

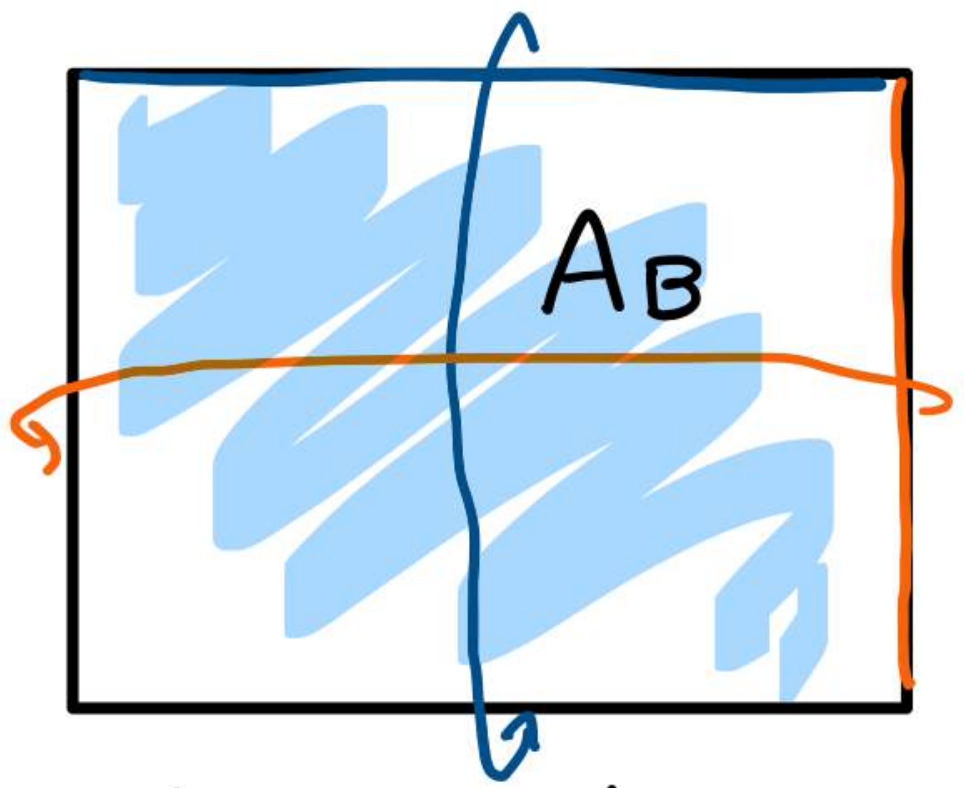
$\Rightarrow$
- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



$U(1)_B$ monopole flux & 't Hooft flux on T^2

Fundamental quarks explicitly violates $\mathbb{Z}_N^{(1)}$ $\Rightarrow$ No 't Hooft B.C., naively.

Use $U(1)_B = \frac{U(1)_q}{\mathbb{Z}_N}$ monopole flux.



$$\begin{cases} \psi(L, x_4) = g_3^\dagger(x_4)\, e^{-i\frac{\phi_3(x_4)}{N}}\, \psi(0, x_4) \\ \psi(x_3, L) = \underbrace{g_4^\dagger(x_3)}_{\text{color-transition functions}}\, \underbrace{e^{-i\frac{\phi_4(x_3)}{N}}}_{U(1)_q\text{-transition functions}}\, \psi(x_3, 0) \end{cases}$$

Cocycle condition

$$g_3^\dagger(L)\, g_4^\dagger(0)\, e^{-i\frac{1}{N}(\phi_3(L)+\phi_4(0))} = g_4^\dagger(L)\, g_3^\dagger(0)\, e^{-i\frac{1}{N}(\phi_4(L)+\phi_3(0))}.$$

$U(1)_B = U(1)_q/\mathbb{Z}_N$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_3(L)-\phi_3(0)) - (\phi_4(L)-\phi_4(0))$$

$$\Rightarrow \quad g_3^\dagger(L)\, g_4^\dagger(0) = g_4^\dagger(L)\, g_3^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}}$$

't Hooft flux !!

2d Effective Lagrangian on $\mathbb{R}^2 \times T^2$ w/ baryon-'t Hooft flux

$\mathbb{Z}_N$ – Schwinger model

$$\text{4d QCD} \xRightarrow[\text{Perturbative}]{\text{Classical \& }} \begin{cases} \text{Gauge : Gapped gluons \& Center vortices} \\ \text{Quark : 2d Dirac fermion as 4d Dirac zero modes w/ } \int_{T^2} dA_B = 2\pi. \end{cases}$$

$\Downarrow$ Bosonization & Semiclassical Analysis

$$\begin{cases} U : \text{2d } U(N_f)\text{-valued field } (\pi, K, \eta \ \& \ \eta') \\ k : \mathbb{Z}_N \text{ discrete variable} \end{cases}$$

[YT, Ünsal 2201.06166; Hayashi, YT 2402.02430]

$$\mathcal{L}_{\text{2d effective}} = \frac{1}{8\pi} \operatorname{tr}(\partial_\mu U^\dagger \partial_\mu U) - \operatorname{tr}[M U + \text{c.c.}]$$
$$+ i \frac{1}{12\pi} \operatorname{tr}\left[(U^\dagger dU)^3\right]$$
$$- e^{-\frac{8\pi^2}{g^2 N}} \underbrace{\left(e^{i\frac{\theta - 2\pi k}{N}} (\det U)^{1/N} + \text{c.c.} \right)}_{}$$

$= \cos\left(\frac{\eta' + \theta - 2\pi k}{N}\right)$

Revisiting $U(1)_A$ problem

$U(1)_A$: η' (i.e. $e^{i\eta'} = \det U$) is too massive according to SSB of the chiral symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$.

$\Rightarrow$ $U(1)_A$ is not a symmetry of quantum theory!

η' gets the mass even if $M_{quark} = 0$.

<u>Previous proposals</u>

- Kobayashi-Maskawa-'t Hooft : $-\cos(\eta' + \theta)$
- Witten-Veneziano's large-N : $\frac{1}{2}(\eta'+\theta)^2 + O\left(\frac{(\eta'+\theta)^4}{N^2}\right)$

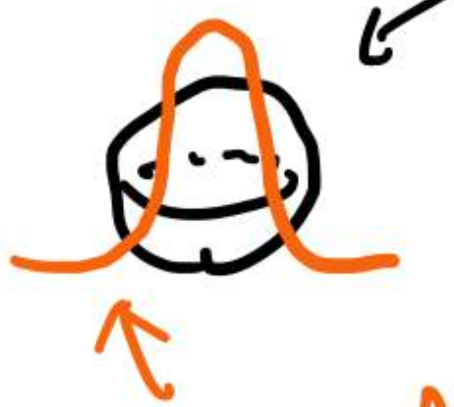
$\frac{1}{N}$ fractionalization

$N \gg 1$

$$\boxed{\text{2d center-vortex theory} \quad : \quad -\cos\left(\frac{\eta' + \theta - 2\pi k}{N}\right)}$$

How does $\frac{1}{N}$ appear? What is its fermionic picture?

4d instanton



localization of chiral zero mode

$$\xRightarrow[\text{w/ 'tHooft flux}]{\mathbb{R}^2 \times T^2}$$

$\frac{1}{N}$ fraction of the chiral zero mode

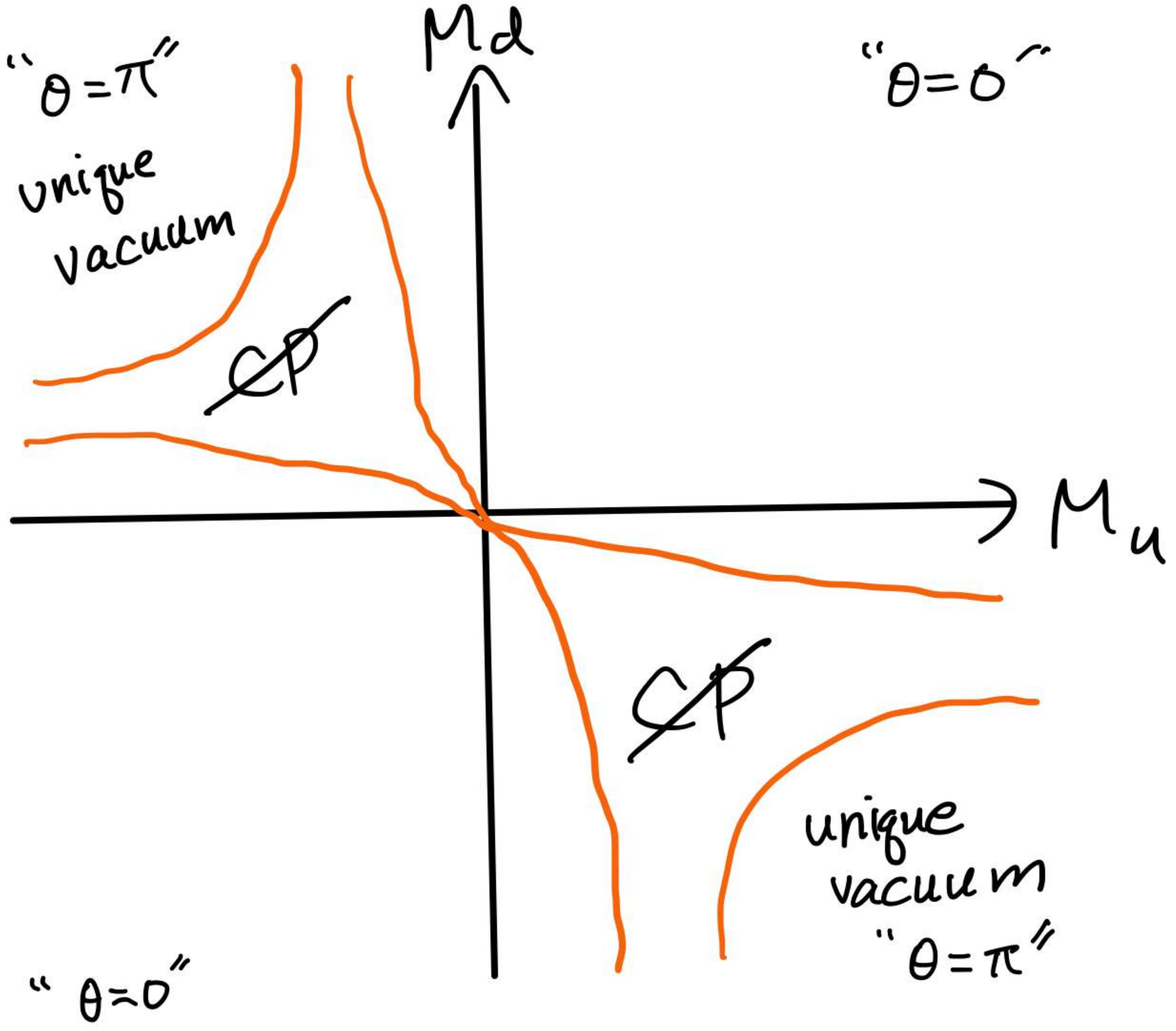
4d instanton $=$ N independant center vortices

$$e^{-\frac{8\pi^2}{g^2}} \cdot e^{-i\theta} \det U(x)$$

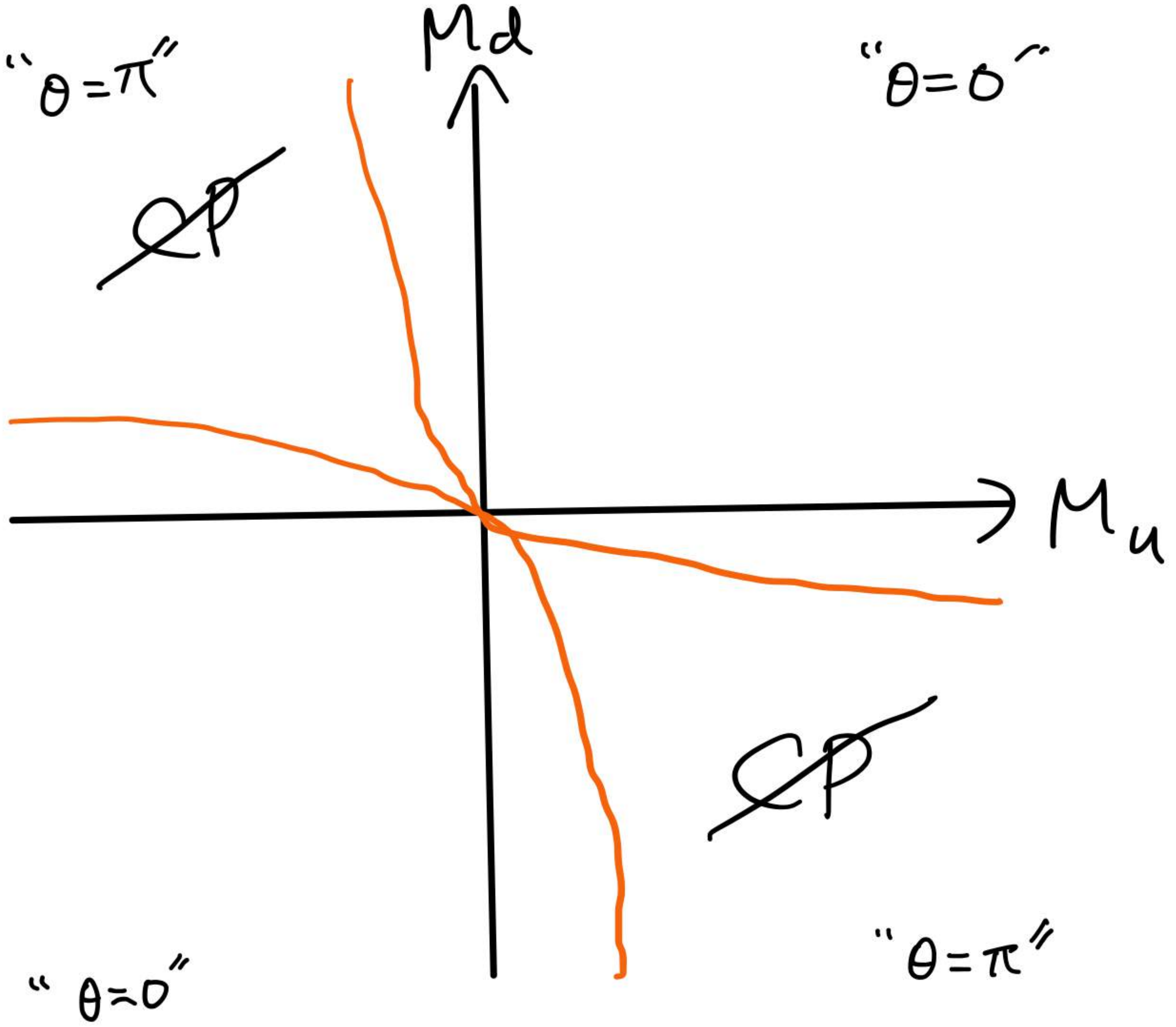
$$\prod_{i=1}^{N} \underbrace{e^{-\frac{8\pi^2}{g^2 N}} \, e^{-i\frac{\theta}{N}} \left(\det U(x_i)\right)^{1/N}}_{\text{each term behaves almost independently}}$$

Phase diagram in (M_u, M_d) space

KMT vertex & $\frac{1}{N}$-fractionalized vertex can give different predictions.



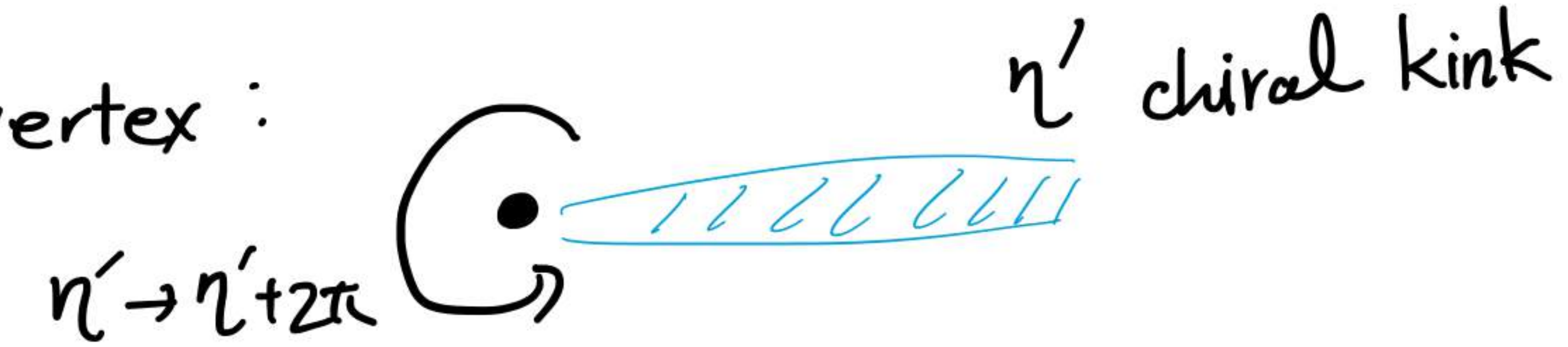
w/ KMT vertex
[S. Aoki, Creutz '14]
(inconsistent w/ anomaly matching)



w/ $\frac{1}{N}$-fractionalized vertex
[Hayashi, YT '24]

η' – Vortex

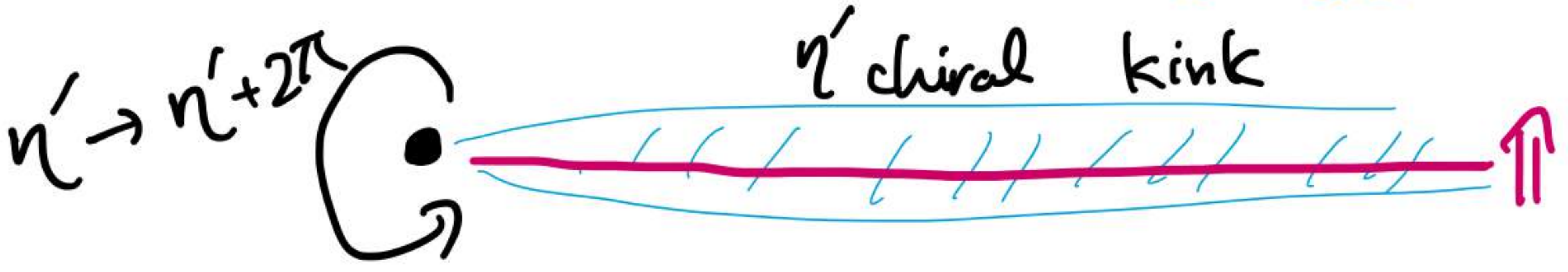
Usual KMT vertex :



$\frac{1}{N}$ fractionalized vertex : $V(\eta', k) = -\cos\left(\frac{\eta' + \theta - 2\pi k}{N}\right)$ does not come back

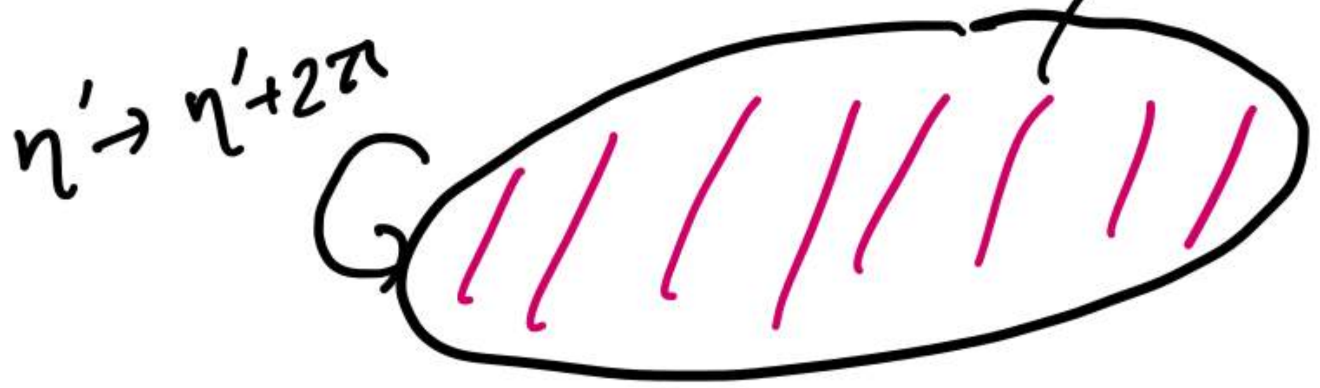
even if we shift $\eta' \to \eta' + 2\pi$.

k should also jump!



$\Rightarrow$ In $\mathbb{R}^2 \times T^2_{\text{twist}}$, $\boxed{\eta' - \text{vortex} \; = \; \text{quark.}}$

This corresponds to the 2d reduction of the "pancake baryon" construction. (Komargodski 1812.09253)



Fractional quantum Hall system

$U(1)_{-N}$ Chern-Simons.

QCD on $\mathbb{R}^2 \times T^2$ w/ Baryon-'t Hooft flux

- Weakly-coupled if $NL\Lambda \ll 1$

$\rightsquigarrow$ We can solve $\underbrace{\text{QCD}}_{\Downarrow}$ analytically!

2d $\mathbb{Z}_N$ - Schwinger model

- η' - potential involves the discrete label $k \in \mathbb{Z}_N$: [YT, Ünsal '22; Hayashi, YT '24]

$$\cancel{-\cos(\eta' + \theta)} \Longrightarrow -\cos\left(\frac{\eta' + \theta + 2\pi k}{N}\right)$$

- η' - vortex = Boundary of $U(1)_N$ Chern-Simons quantum liquid