Two-color QCD as a laboratory to explore cold and dense medium: Numerical experiments and effective models

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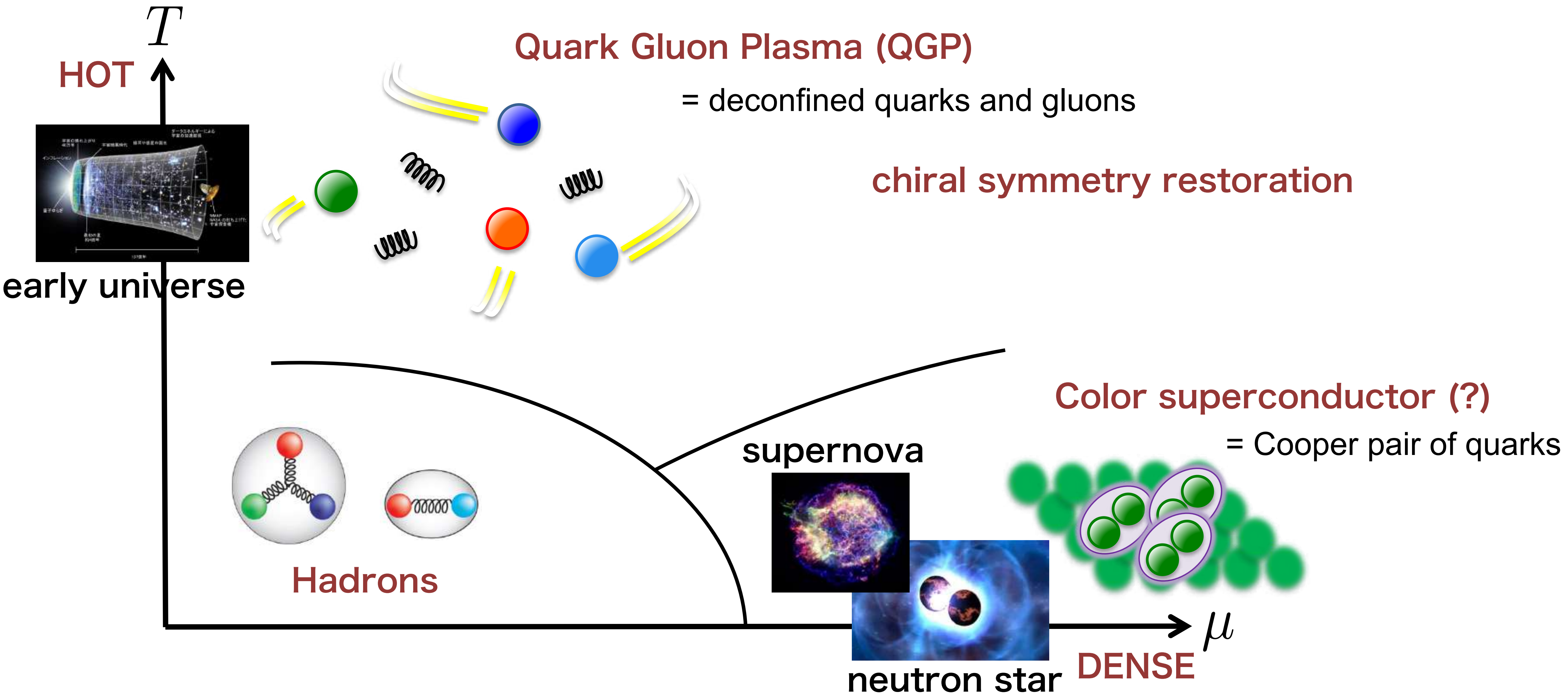
Introduction

- QCD phase diagram



Introduction

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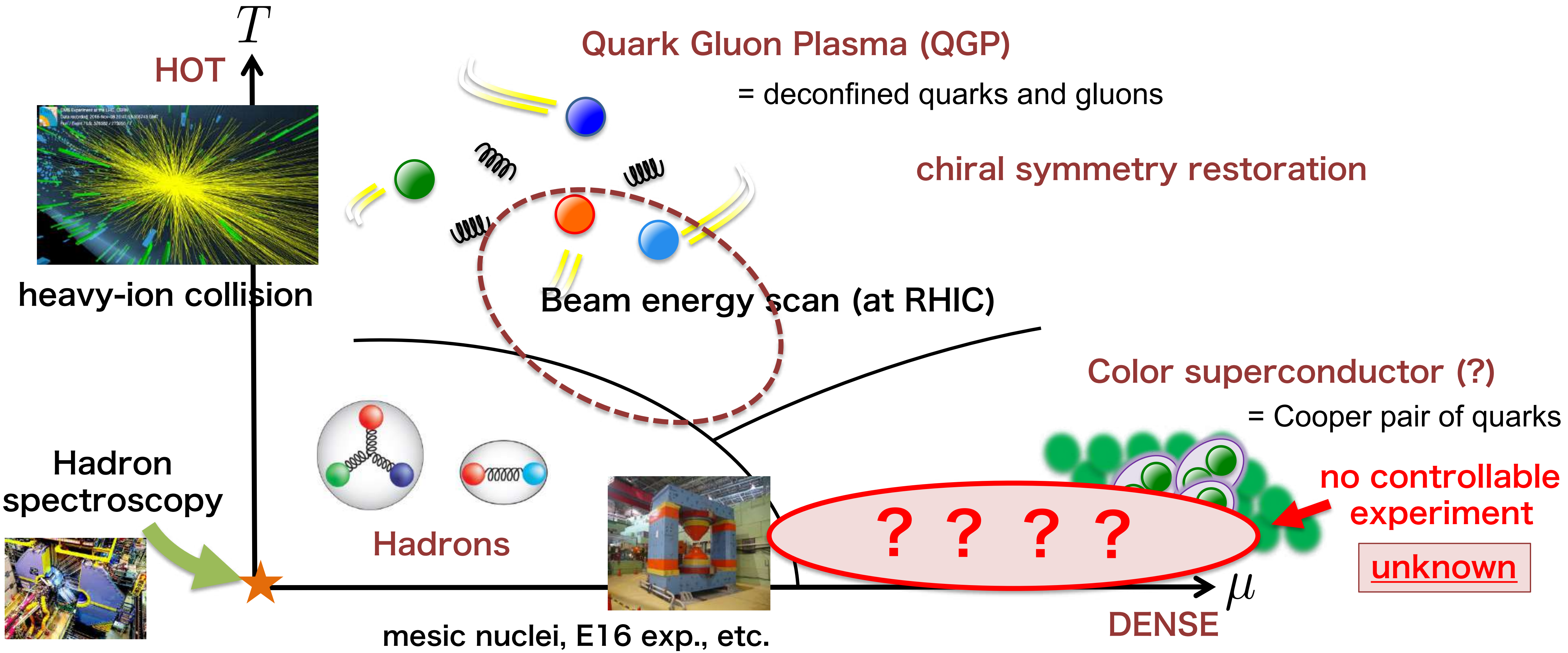

Introduction

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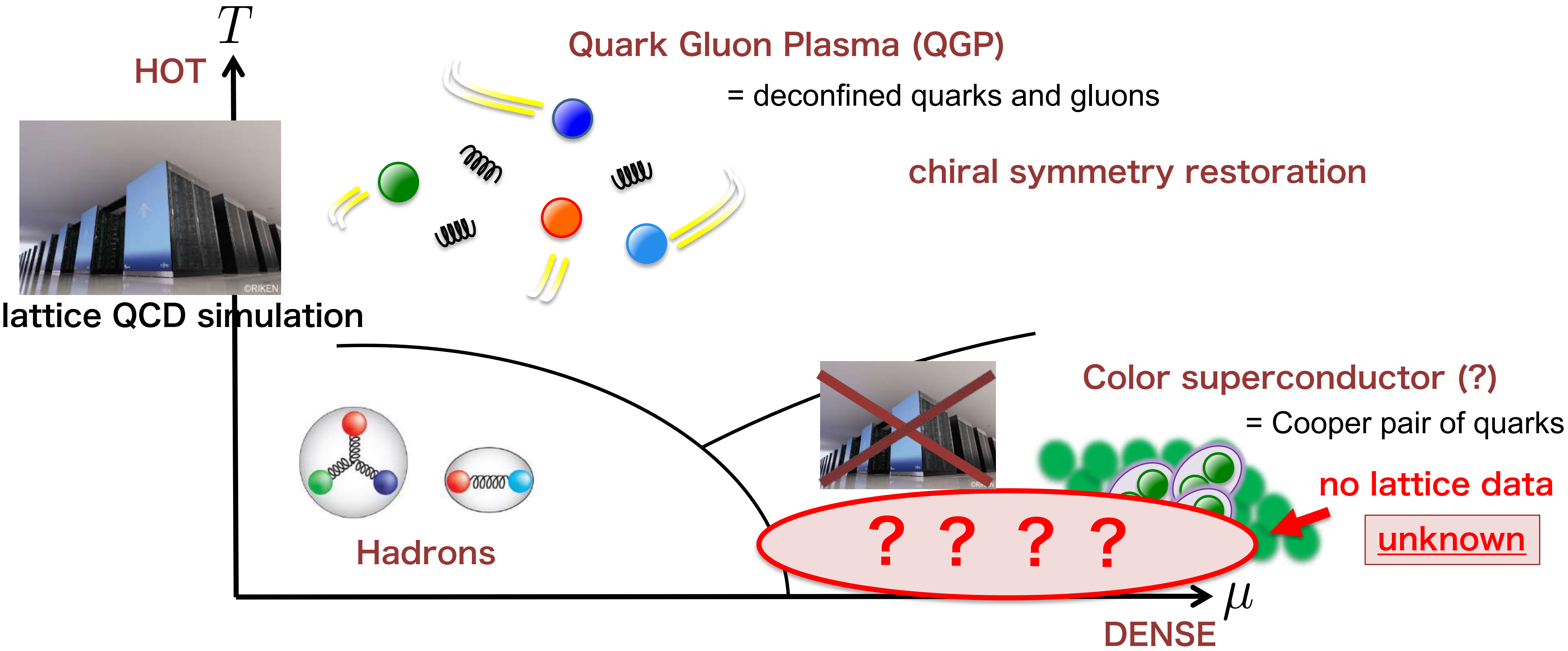
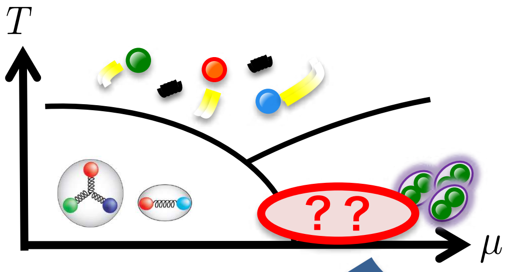


Introduction

- **Lattice study in dense QCD?**

with $N_f = (\mathrm{even})$



<u>three-color QCD (our world)</u>

- Lattice sim. at density is not easy

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m) \in \mathbb{C} \times$$

sign problem

<u>two-color QCD (imaginary world)</u>

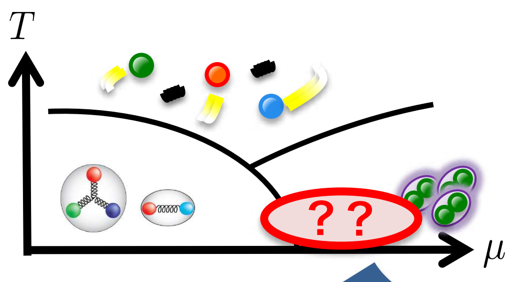
- Lattice sim. at density is possible!

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m)^{N_f} \geq 0 \circledcirc$$

sign problem disappears
thanks to $SU(2)_c$ pseudoreality

Introduction

- **Lattice study in dense QCD?**



with $N_f = \text{(even)}$

<u>three-color QCD (our world)</u>

- Lattice sim. at density is not easy

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m) \in \mathbb{C} \times$$

sign problem

- Baryon is made of three quarks

nucleon

some differences

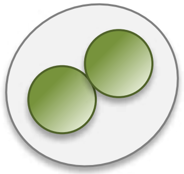
<u>two-color QCD (imaginary world)</u>

- Lattice sim. at density is possible!

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m)^{N_f} \geq 0 \text{ ◎}$$

sign problem disappears
thanks to $SU(2)_c$ pseudoreality

- Baryon is made of two quarks



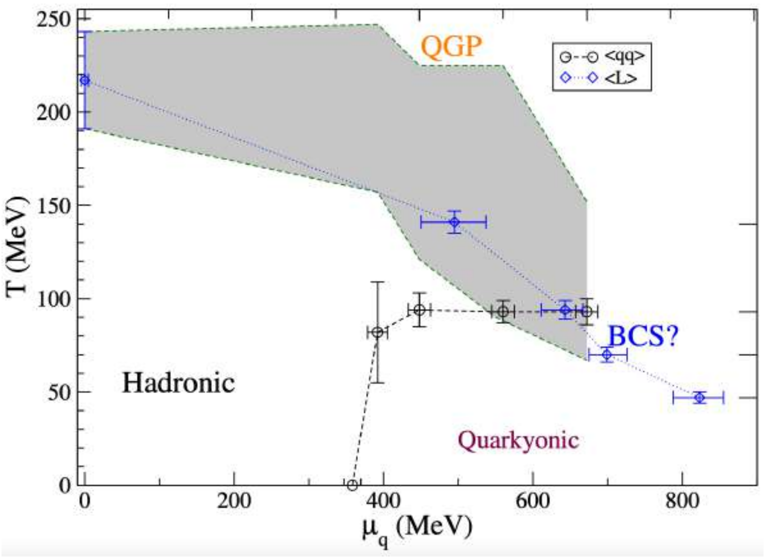
diquark baryon

- **Two-color QCD** is a useful laboratory to explore cold-dense medium with lattice simulations

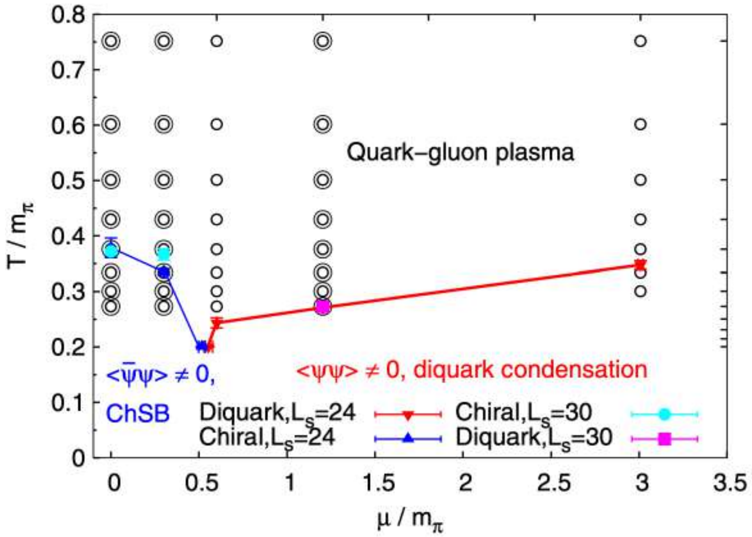
Introduction

- **Phase diagram in two-color QCD (QC_2D)**

  - Examples of simulation results of phase diagram in QC_2D

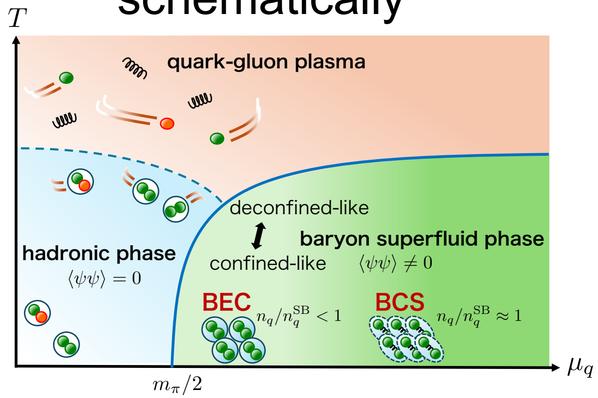


Boz-Cotter-Fister-Mehta-Skullerud (2013)



Buividovich-Smith-Smekal (2020)

schematically



…

- Currently at least four lattice simulation groups are active
  - Ireland/UK group (Hands, Skullerud, …)
  - Russian group (Bornyakov, ...)
  - UK group (Buividovich, ...)
  - Japanese group (Iida-san, Itou-san, …)

Introduction

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  - Examples of simulation results of phase diagram in QC_2D

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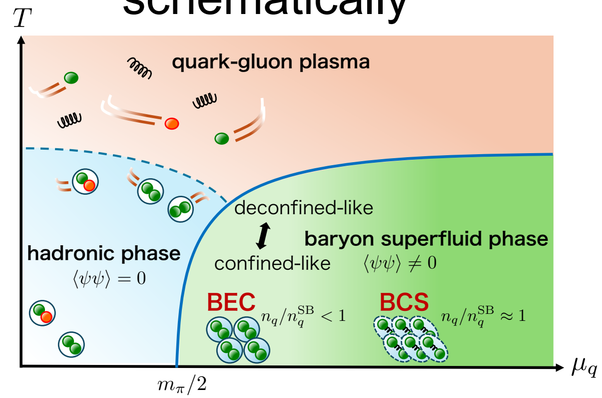

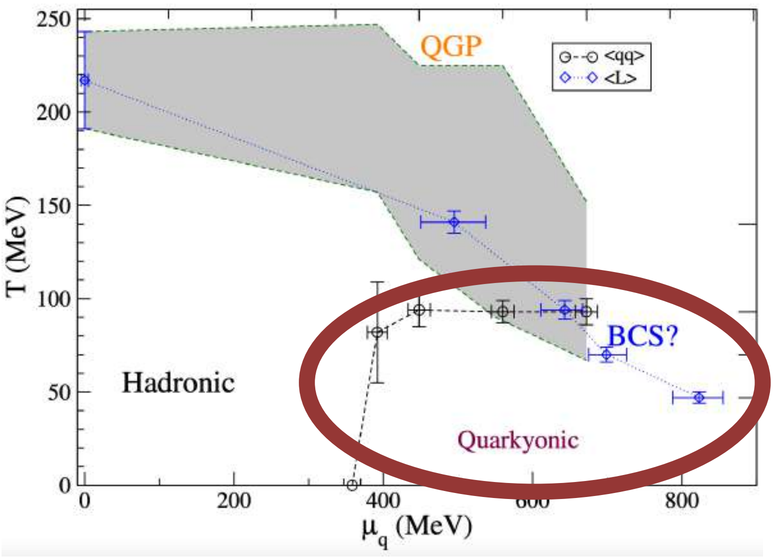

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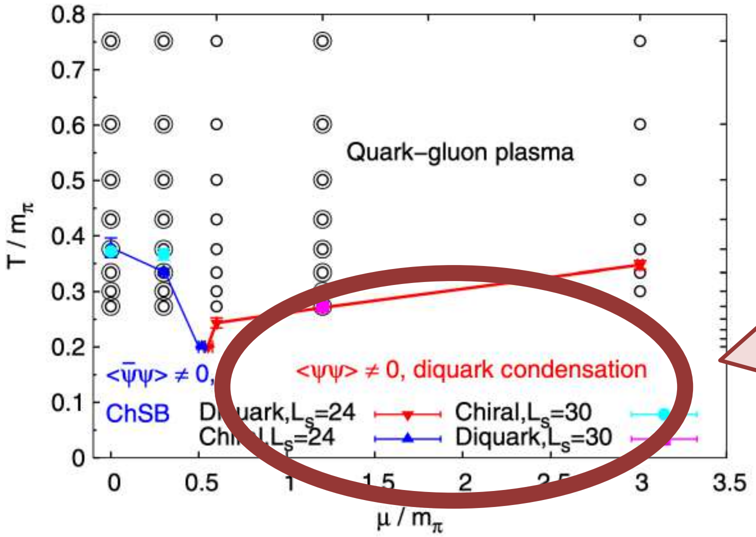

Buividovich-Smith-Smekal (2020)

…

diquark condensed phase (baryon superfluid phase)

$$\langle\psi\psi\rangle \neq 0$$

- Currently at least four lattice simulation groups are active
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  - Russian group (Bornyakov, ...)
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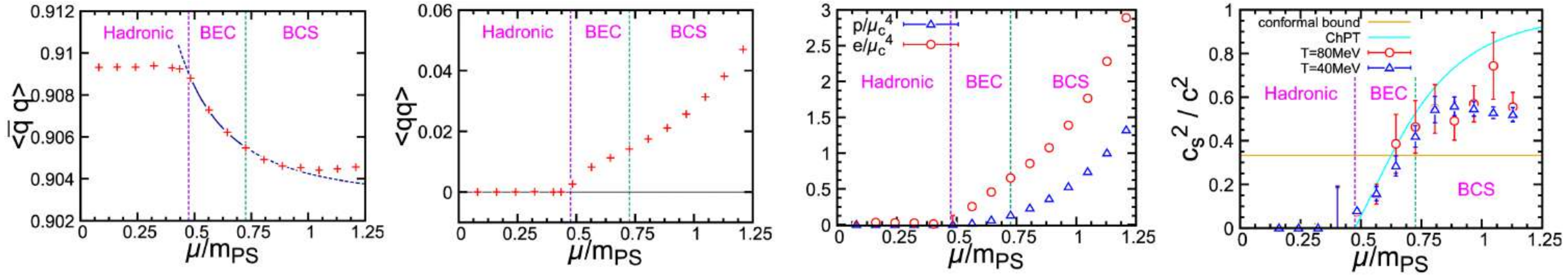
Introduction

- **Lattice results**
  - In addition to phase diagram, hadron mass spectrum, gluon propagator, transport coefficient, EoS, sound velocity, $\langle\bar{\psi}\psi\rangle, \langle\psi\psi\rangle, \langle L\rangle$, etc. have been simulated

Japanese group



. . .

My approach

– (i) Regard QC_2D lattice simulations as useful "numerical experiments" of cold and dense QCD, and (ii) give interpretation from symmetry viewpoints based on effective models

Suenaga (2025) (Review paper)

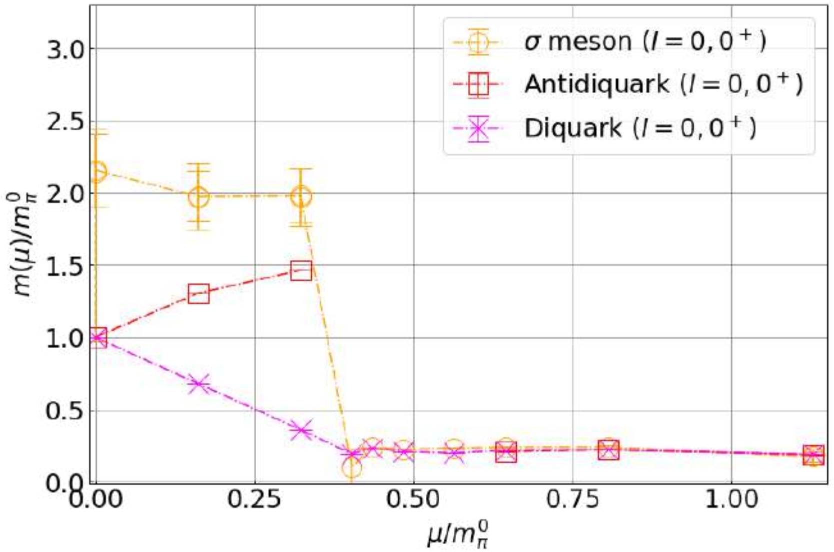
<u>My publications on QC_2D</u>

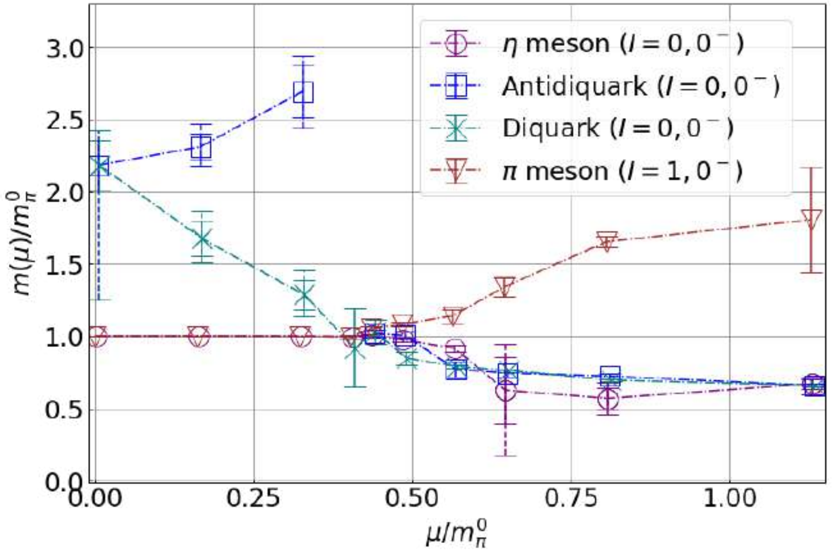
Gluon propagator: **Suenaga**-Kojo(2019), Kojo-**Suenaga**(2021), CSE effect: **Suenaga**-Kojo(2021), Sound velocity: Kojo-**Suenaga**(2022), Kawaguchi-**Suenaga**(2024), Topological susceptibility: Kawaguchi-**Suenaga**(2023), Hadron mass: **Suenaga**-Murakami-Itou-Iida (2023, 2024), FRG analysis: Fejos-**Suenaga**(2025, 2025), LSM with Nf=2+2: Sakai-**Suenaga** (2025), in preparations

Introduction

Lattice results on hadron masses

Murakami et al (2023)





← pion $(I=1,0^-)$

$I=0,0^-$ hadron
lighter than pion!

Introduction

Lattice results on hadron masses

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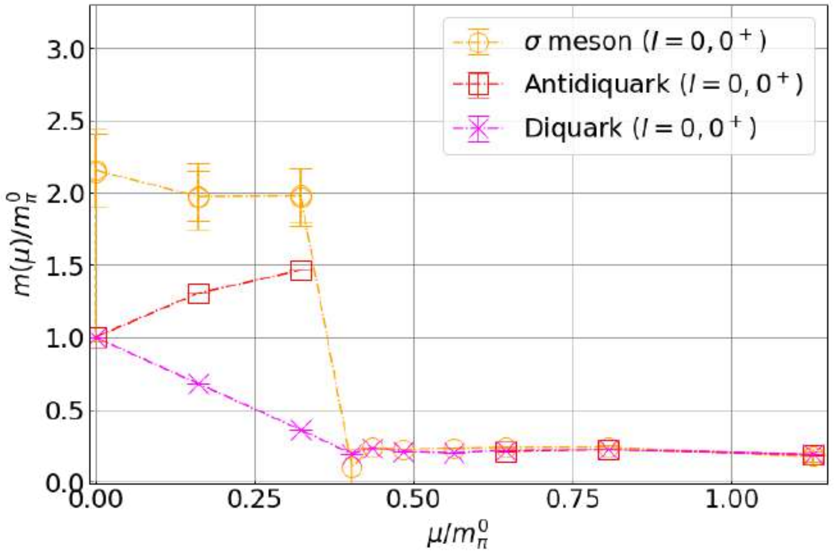
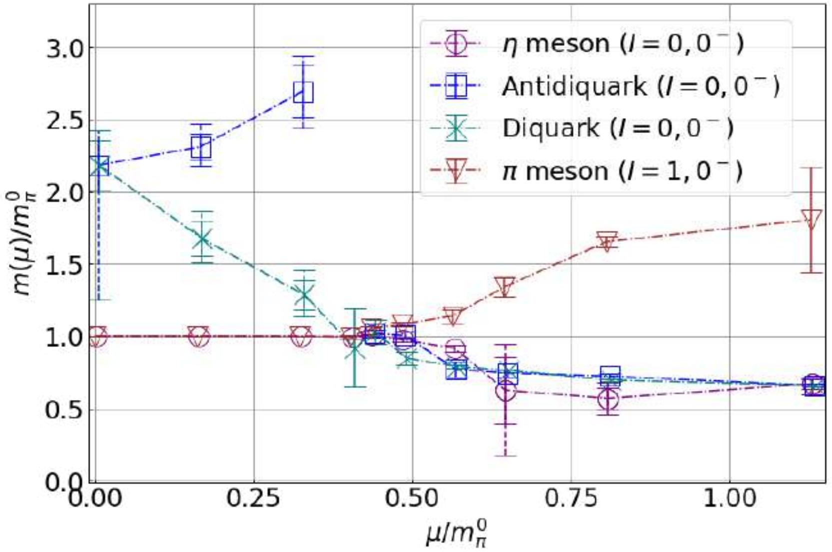




← pion $(I=1,0^-)$

$I=0,0^-$ hadron lighter than pion!

- Pion is no longer light in superfluid phase (for $m_\pi^0/2 \lesssim \mu$) !

- ChPT is powerful but cannot treat $I=0,0^-$ hadron ⇨ ChPT is not the useful "low-energy EFT" in superfluid phase of QC_2D!

Kogut et al (2000)

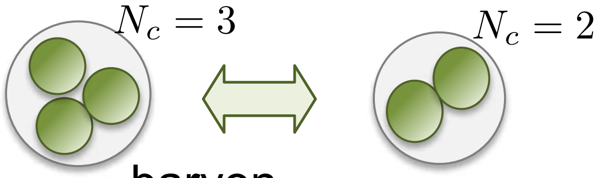
⇨ I constructed another model (linear sigma model) as a reasonable EFT in (dense) QC_2D

Q: What can we learn from QC_2D?

- It is too naïve to bring all obtained information to Nc=3 world

  → detailed dynamics is different

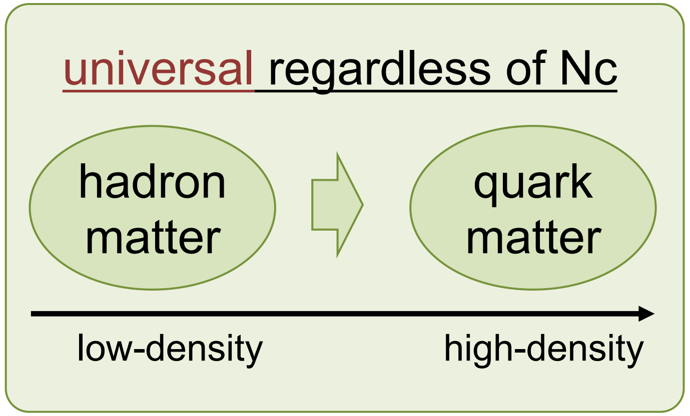
eg



baryon

- But we can get information on
  - to what extent (simple) hadron model description can apply
  - Which representation of hadrons is useful in medium
  - how to incorporate quark dof into hadron model → “unified model”
  - deep understanding of quark matter in high-dense regime

  ⋮



Model

- **Pauli-Gursey SU(4) symmetry**

  - Pseudo reality of $SU(2)_c$ allows us to rewrite QC_2D Lagrangian with massless quarks as

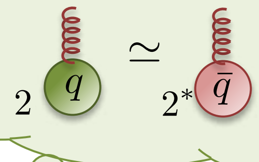
$$\mathcal{L}_{\mathrm{QC_2D}} = \bar{\psi} i \partial\!\!\!/ \psi - g_s \bar{\psi} A\!\!\!/^a T_c^a \psi = \Psi^\dagger i \partial_\mu \sigma^\mu \Psi - g_s \Psi^\dagger A_\mu^a T_c^a \sigma^\mu \Psi$$

pseudoreality: $\sigma^2 \sigma^a \sigma^2 = -(\sigma^a)^*$

In two-flavor: $\Psi = (\psi_R, \tilde{\psi}_L)^T = (u_R, d_R, \tilde{u}_L, \tilde{d}_L)^T$ with $\tilde{\psi}_L = \sigma^2 \tau_c^2 \psi_L^*$

Four-dimensional Pauli matrix: $\sigma^\mu = (1, \sigma^i)$

  - $\mathcal{L}_{\mathrm{QC_2D}}$ is obviously invariant under $\Psi \to g\Psi$ $[g \in SU(4)]$



gluons are blind

$SU(2)_L \times SU(2)_R$ chiral symmetry

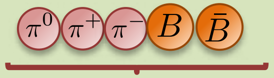
[breaking: $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$]

enlarged

<u>Pauli-Gursey $SU(4)$ symmetry</u>

[breaking: $SU(4) \to Sp(4)$]

Pauli (1957), Gursey (1958)



5 NG bosons

Model

- **Linear sigma model (LSM)**

$$\Psi = (\psi_R, \tilde{\psi}_L)^T = (u_R, d_R, \tilde{u}_L, \tilde{d}_L)^T$$

- LSM is based on linear rep. of a quark bilinear: $\Psi_j \sigma^2 \tau_c^2 \Psi_i$

→ would be reasonable in medium where
  - quark d.o.f. will get seen
  - broken strength of $SU(4) \to Sp(4)$ is changed

- The main building block is

$$\Sigma_{ij} = \frac{1}{2}\begin{pmatrix} 0 & -\frac{B'-iB}{2\sqrt{2}} & \frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & \frac{a_0^+-i\pi^+}{2\sqrt{2}} \\ \frac{B'-iB}{2\sqrt{2}} & 0 & \frac{a_0^--i\pi^-}{2\sqrt{2}} & \frac{\sigma-i\eta-a_0^0+i\pi^0}{4} \\ -\frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & -\frac{a_0^--i\pi^-}{2\sqrt{2}} & 0 & -\frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} \\ -\frac{a_0^+-i\pi^+}{2\sqrt{2}} & -\frac{\sigma-i\eta-a_0^0+i\pi^0}{4} & \frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} & 0 \end{pmatrix}_{ij} \sim \Psi_j \sigma^2 \tau_c^2 \Psi_i$$

cf, $\Sigma = \sigma + i\pi^a\tau^a$ for $N_c = 3$

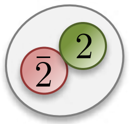
- Hadron assignments are

$$B \sim -\frac{i}{\sqrt{2}}\psi^T C\gamma_5 \tau_c^2 \tau_f^2 \psi \quad B' \sim -\frac{1}{\sqrt{2}}\psi^T C \tau_c^2 \tau_f^2 \psi \quad \sigma \sim \bar{\psi}\psi$$

$$a_0^a \sim \bar{\psi}\tau_f^a\psi \quad \eta \sim \bar{\psi} i\gamma_5 \psi \quad \pi^a \sim \bar{\psi} i\gamma_5 \tau_f^a \psi$$

Hadron	J^P	Quark number	Isospin
σ	0^+	0	0
a_0	0^+	0	1
η	0^-	0	0
π	0^-	0	1
B ($\bar{B}$)	0^+	$+2(-2)$	0
B' ($\bar{B}'$)	0^-	$+2(-2)$	0

mesons



baryons

Model

- **LSM Lagrangian**

  - (approximately) $SU(4)$-invariant LSM Lagrangian is given by

$$\mathcal{L} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - m_0^2 \mathrm{tr}[\Sigma^\dagger \Sigma] - \lambda_1 \left(\mathrm{tr}[\Sigma^\dagger \Sigma]\right)^2 - \lambda_2 \mathrm{tr}[(\Sigma^\dagger \Sigma)^2] + \mathrm{tr}[H^\dagger \Sigma + \Sigma^\dagger H] + c(\det\Sigma + \det\Sigma^\dagger)$$

$D_\mu \Sigma = \partial_\mu \Sigma - i\mu_q \delta_{\mu 0}\{J, \Sigma\}$ with $J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

<u>chemical potential effect</u>

$H = h_q E$ with $E = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$

<u>current-quark mass effect</u>

$U(1)_A$ anomaly

where $\Sigma = \frac{1}{2}\begin{pmatrix} 0 & -\frac{B'-iB}{2\sqrt{2}} & \frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & \frac{a_0^+-i\pi^+}{2\sqrt{2}} \\ \frac{B'-iB}{2\sqrt{2}} & 0 & \frac{a_0^--i\pi^-}{2\sqrt{2}} & \frac{\sigma-i\eta-a_0^0+i\pi^0}{4} \\ -\frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & -\frac{a_0^--i\pi^-}{2\sqrt{2}} & 0 & -\frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} \\ -\frac{a_0^+-i\pi^+}{2\sqrt{2}} & -\frac{\sigma-i\eta-a_0^0+i\pi^0}{4} & \frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} & 0 \end{pmatrix}$

$$\Sigma \to g\Sigma g^T \quad [g \in SU(4)]$$

Hadron	J^P	Quark number	Isospin
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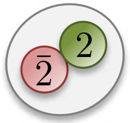
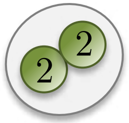
mesons



baryons



- Advantage of LSM
  - treatment of $I = 0, 0^-$ hadrons (mandatory from lattice result!)
  - we can see relations between parity partners

<u>parity partner</u>

$\eta, \pi \leftrightarrow \sigma, a_0$

$B(\bar{B}) \leftrightarrow B'(\bar{B}')$

Mean fields

- **Mean field**

- The mean fields are $\sigma_0 \equiv \langle \sigma \rangle$ and $\Delta \equiv \left\langle \frac{B+\bar{B}}{\sqrt{2}} \right\rangle$

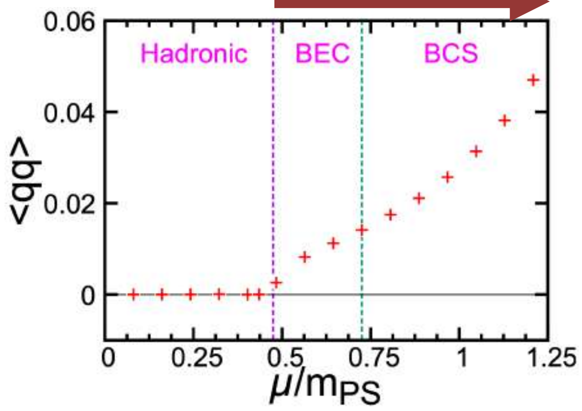
$\sigma_0 \sim \langle \bar{\psi}\psi \rangle$: chiral condensate

$\Delta \sim -\frac{i}{2}\langle \psi^T C\gamma_5 \tau_c^2 \tau_f^2 \psi \rangle + \mathrm{h.c.}$: diquark condensate

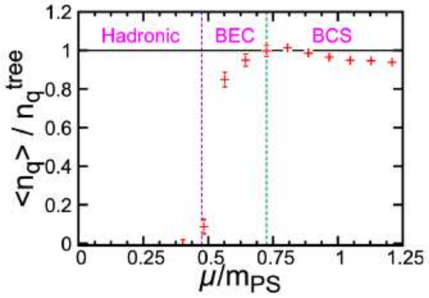
diquark condensate by lattice

Iida et al (2024)

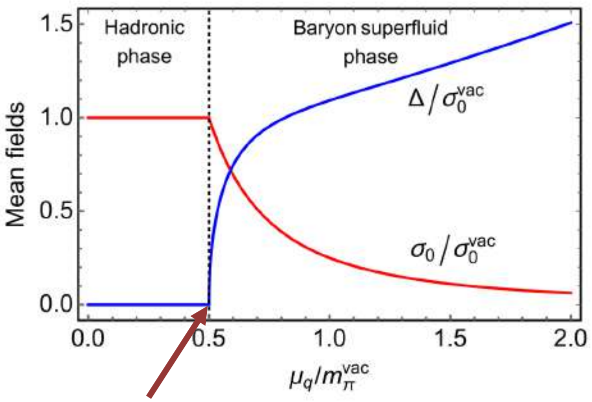
baryon superfluidity



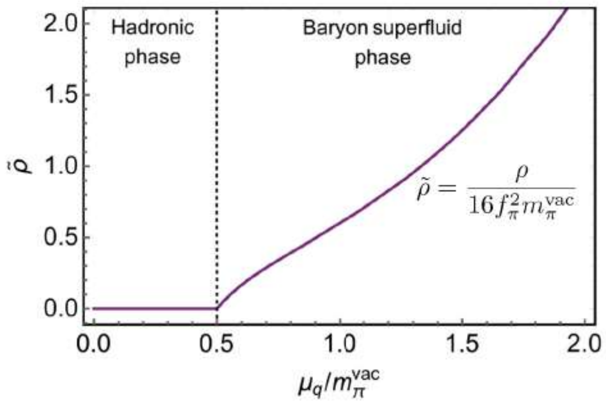
$\begin{cases} \mathrm{BEC}: \langle n_q \rangle / n_q^{\mathrm{tree}} \lesssim 1 \\ \mathrm{BCS}: \langle n_q \rangle / n_q^{\mathrm{tree}} \approx 1 \end{cases}$



σ_0 and Δ vs μ_q



density ρ vs μ_q



2nd order phase transition at $\mu_q = m_\pi^{\mathrm{vac}}/2$

<u>Input here</u>

$\sigma_0^{\mathrm{vac}} = 250\,\mathrm{MeV}$ (put by hand)

$\lambda_1 = c = 0$ (large Nc)

$m_\pi^{\mathrm{vac}} = 738\,\mathrm{MeV}$

$m_{a_0}^{\mathrm{vac}}/m_\pi^{\mathrm{vac}} = 2.18$

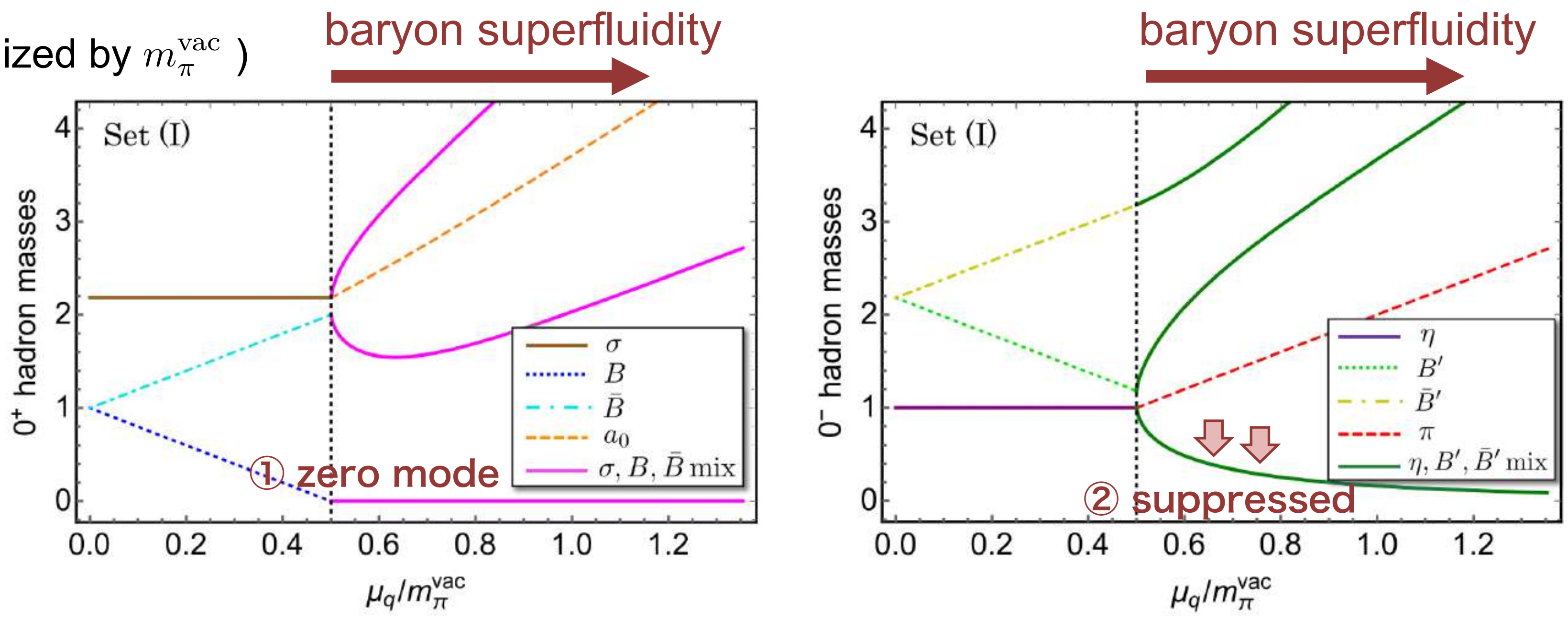
lattice
Murakami et al

Hadron masses

- **Results on hadron mass at finite μ_q (T=0)**

(normalized by m_π^{vac})

baryon superfluidity



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lattice
Murakami et al

- Baryon number violation in superfluid phase

$\sigma \leftrightarrow B \leftrightarrow \bar{B}$ mixing (0+ sector)

$\eta \leftrightarrow B' \leftrightarrow \bar{B}'$ mixing (0- sector)

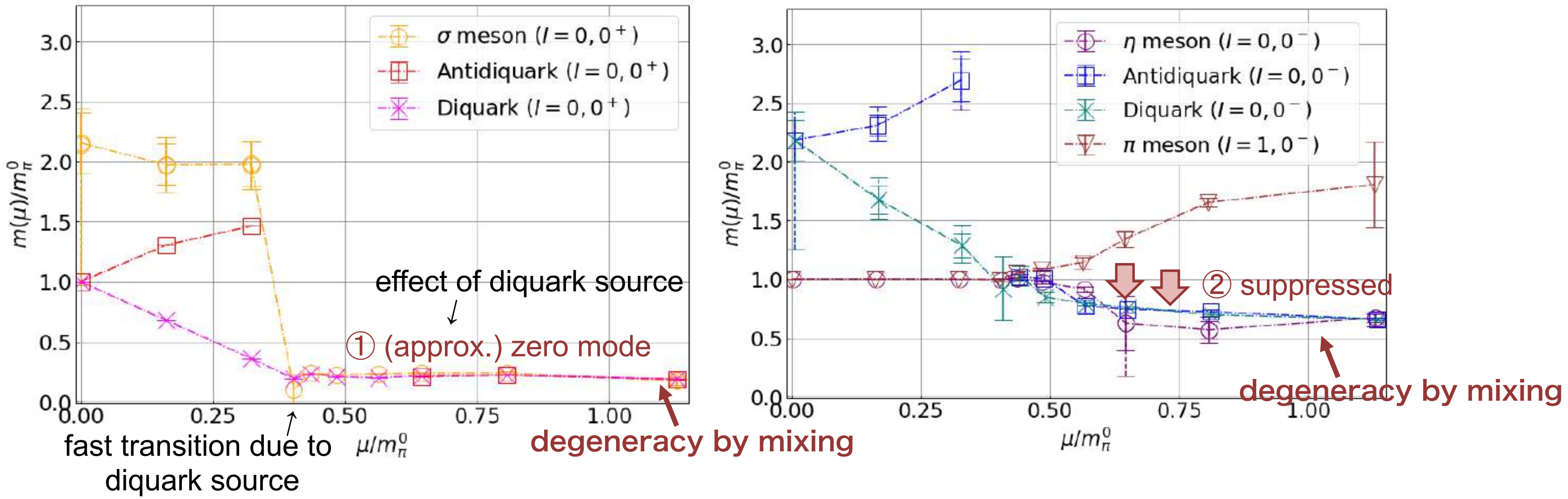
① zero mode (NG mode of $U(1)$ baryon-number breaking)

② nonlinear suppression of “ η “ mass due to the mixing

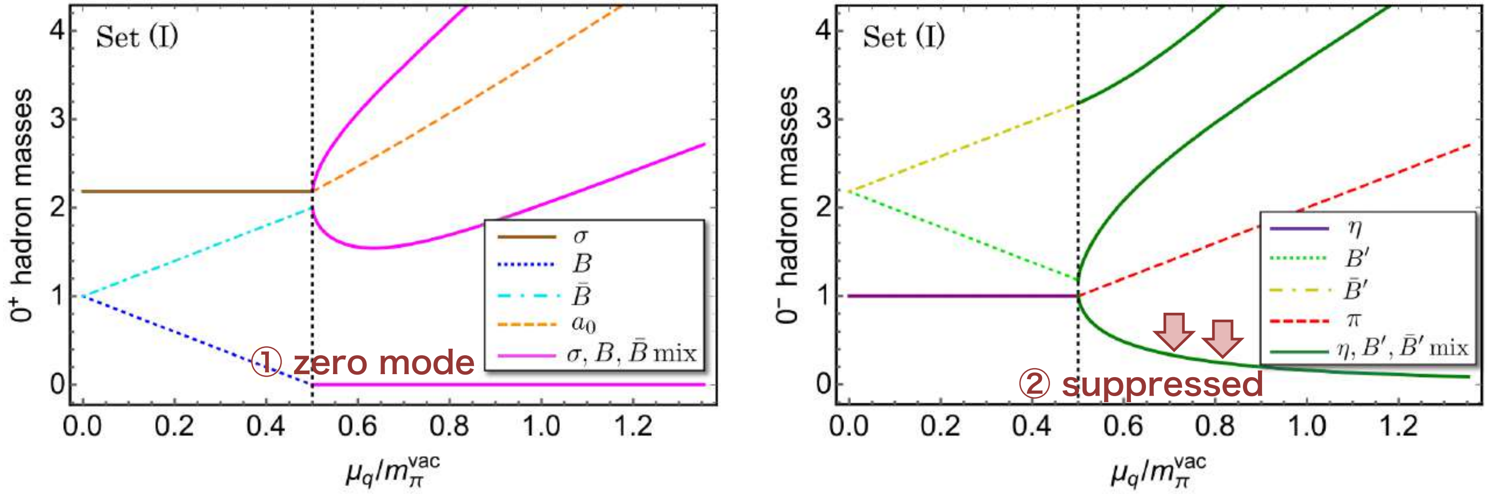
Hadron masses

- **Comparison with lattice**

Lattice (Murakami et al)



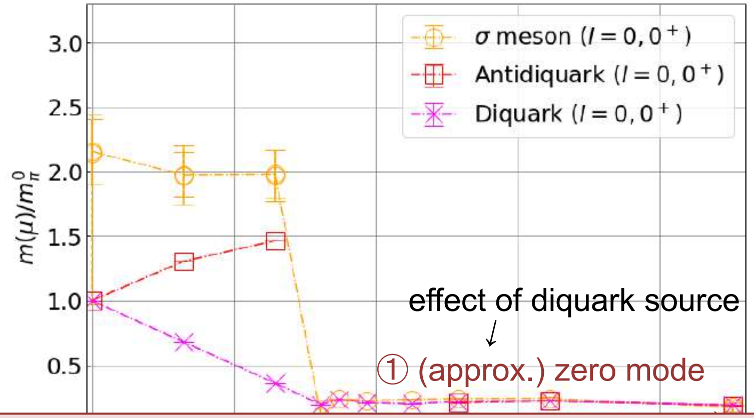
My LSM

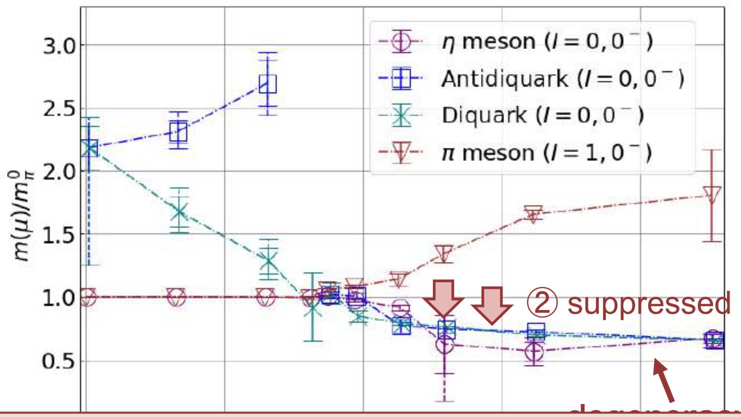


Hadron masses

- Comparison with lattice

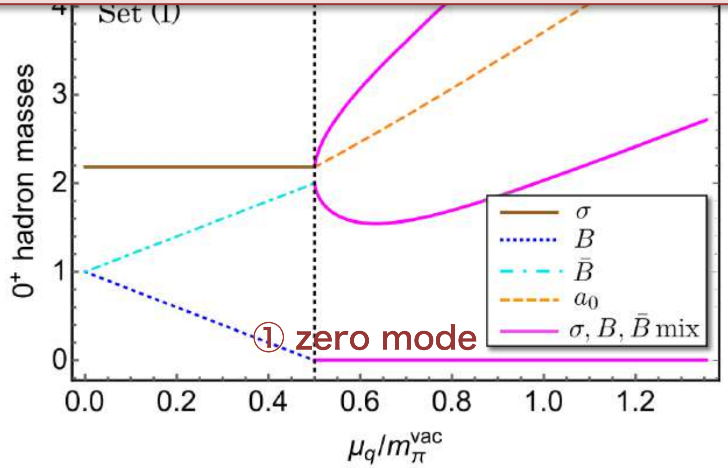
attice (Murakami et al)

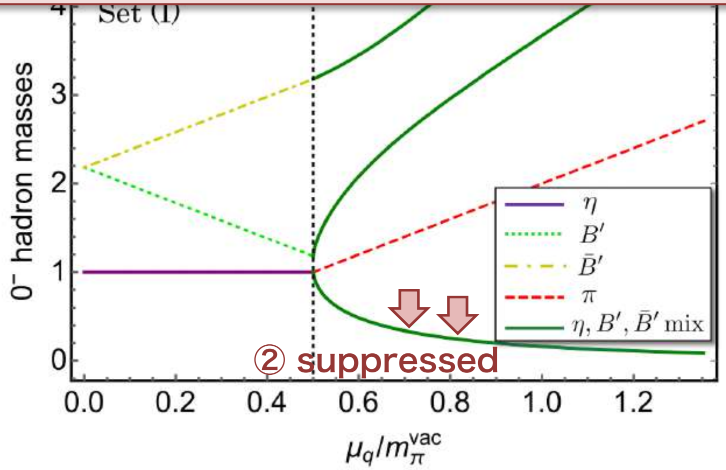




The lowest excitations are qualitatively described by my LSM well!

My LSM





NOTE

ChPT cannot treat $\eta, B', \bar{B}'$

Kogut et al (2000)

linear rep. is essential!

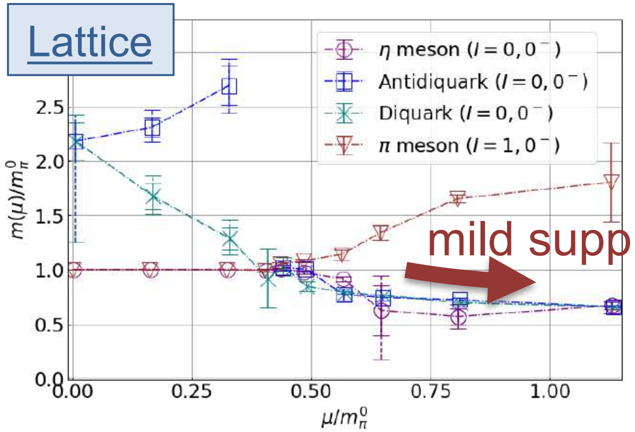
Discussion

- **Anomaly enhancement in superfluid phase?**

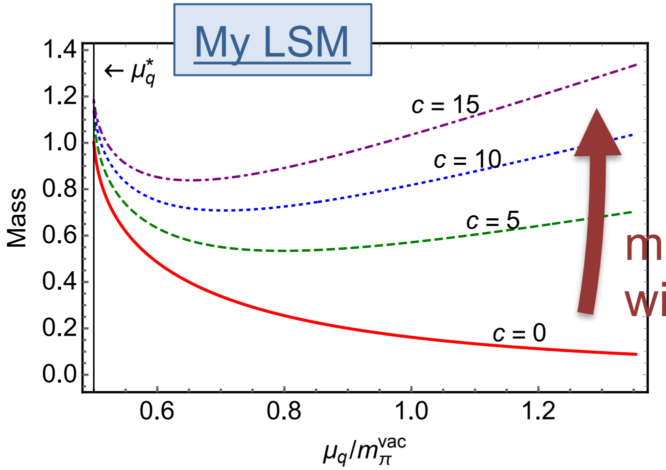
  - In hadronic phase $(\mu < m_\pi^{\rm vac}/2)$, the $U(1)_A$ anomaly effect seems to be small

$m_\eta^{\rm vac} \sim m_\pi^{\rm vac}$ on the lattice even with disconnected diagrams [in preparation]

- In superfluid phase $(m_\pi^{\rm vac}/2 \leq \mu)$, the $U(1)_A$ anomaly effect could be **enhanced**



mild suppression



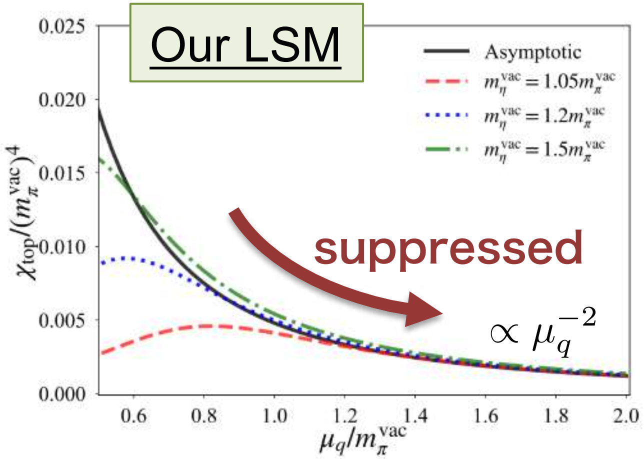
c : anomaly strength

milder suppression
with larger anomaly effects

- Work in progress with FRG analysis

Application 1: Topological susceptibility

- **Top. susceptibility in superfluid phase**



- Suppression of topological susceptibility for large μ_q

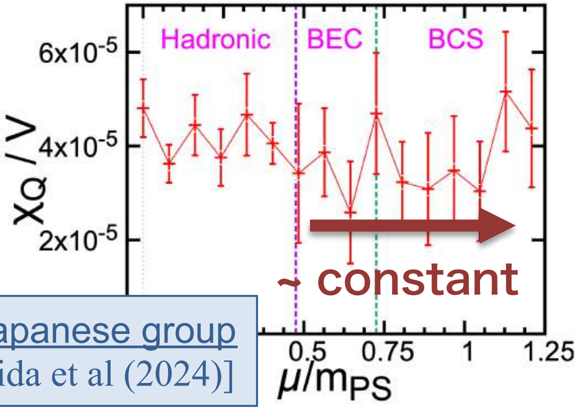
Asymptotic: $\chi_{\rm top} \to \dfrac{(f_\pi^{\rm vac})^2 (m_\pi^{\rm vac})^4}{12} \underline{\mu_q^{-2}}$

Kawaguchi-**Suenaga** (2023)

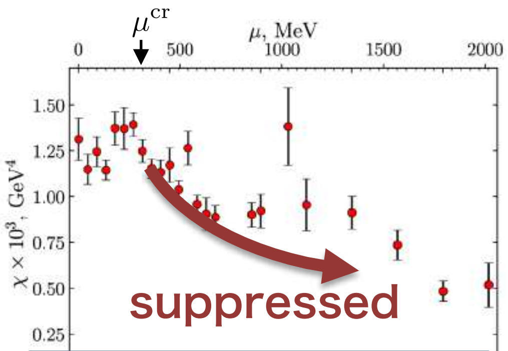
essentially from the chiral restoration $\sigma_0 \propto \mu_q^{-2}$

(axial Ward-Takahashi identity)

comparison with lattice



Japanese group
[Iida et al (2024)]



Russian group
[Astrakhantsev et al (2020)]

- Russian group result shows μ_q^{-2} behavior?
- Japanese group result is constant due to use of Wilson fermion?

No conclusive statement yet!

→ Needs more refined simulations

Application 2: Sound velocity

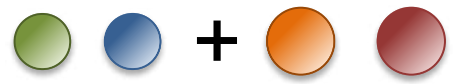
- **Sound velocity at mean-field level within the LSM**

$$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$$
$$\delta\bar{m}^2_{\sigma-\pi} = (m_\sigma^2 - m_\pi^2)/\mu_{\rm cr}^2$$

pressure: $$p = \underbrace{f_\pi^2 m_\pi^2 \left(\bar{\mu}^2 + \frac{1}{\bar{\mu}^2}\right)}_{\text{ChPT result}} + f_\pi^2 m_\pi^2 \left[\frac{4}{\delta\bar{m}^2_{\sigma-\pi}}(\bar{\mu}^2 - 1)^2\right]$$

energy: $$\epsilon = \underbrace{f_\pi^2 m_\pi^2 \left[\frac{(\bar{\mu}^2 + 3)(\bar{\mu}^2 - 1)}{\bar{\mu}^2}\right]}_{\text{ChPT result}} + f_\pi^2 m_\pi^2 \left[\frac{4}{\delta\bar{m}^2_{\sigma-\pi}}(3\bar{\mu}^2 + 1)(\bar{\mu}^2 - 1)\right]$$

sound velocity: $$c_s^2 = \frac{(1 - 1/\bar{\mu}^4) + 8(\bar{\mu}^2 - 1)/\delta\bar{m}^2_{\sigma-\pi}}{\underbrace{(1 + 3/\bar{\mu}^4)}_{\text{ChPT result}} + 8(3\bar{\mu}^2 - 1)/\delta\bar{m}^2_{\sigma-\pi}}$$



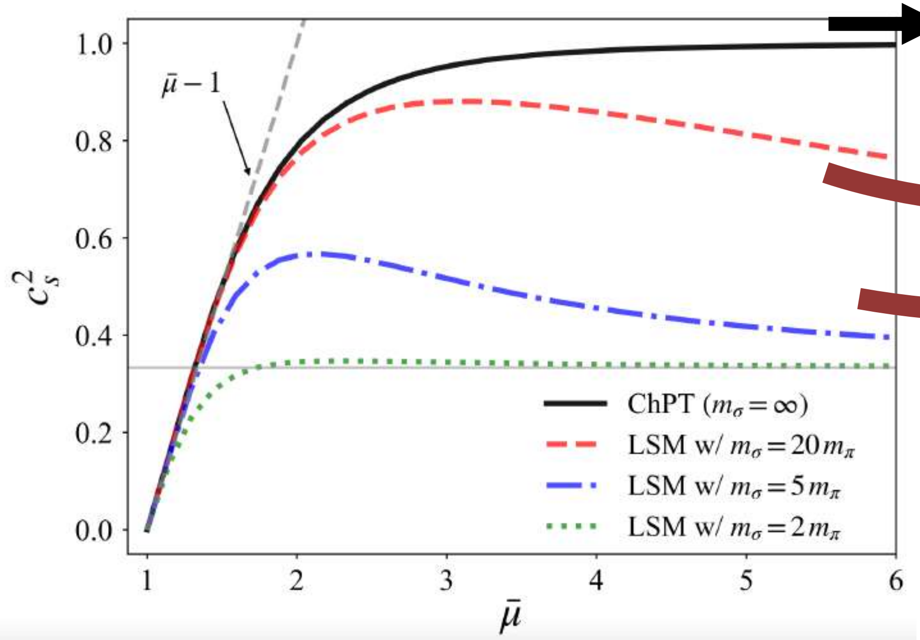
Universal structure: $(\text{LSM result}) = (\text{ChPT result}) + (1/\delta\bar{m}^2_{\sigma-\pi}\ \text{contribution})$

- Integrating out the parity partners $(m_\sigma \to \infty)$ yields the ChPT results $(1/\delta\bar{m}^2_{\sigma-\pi} \to 0)$

Application 2: Sound velocity

- **Sound velocity peak** Kawaguchi-**Suenaga** (2024)

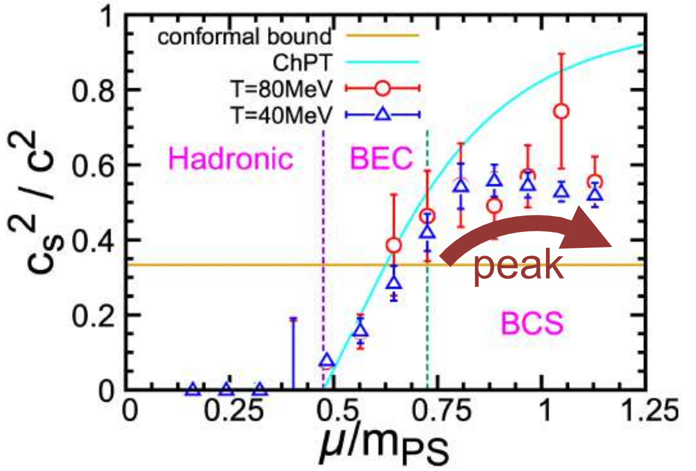
$$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$$

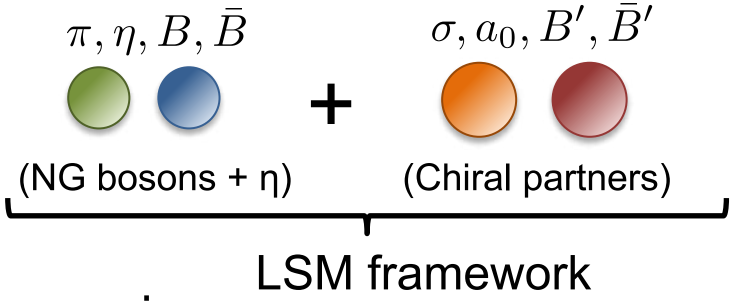


ChPT result $(c_s^2 \to 1)$: no peak

peak when $m_\sigma < \infty$ $(c_s^2 \to 1/3)$

Lattice: Iida et al (2024)





- The peak structure is driven by contributions from parity partners

- Fluctuation and spin-1 hadron effect are needed for more quantitative comparison

- Any connection with crossover to quark matter?

FRG for anomaly effect

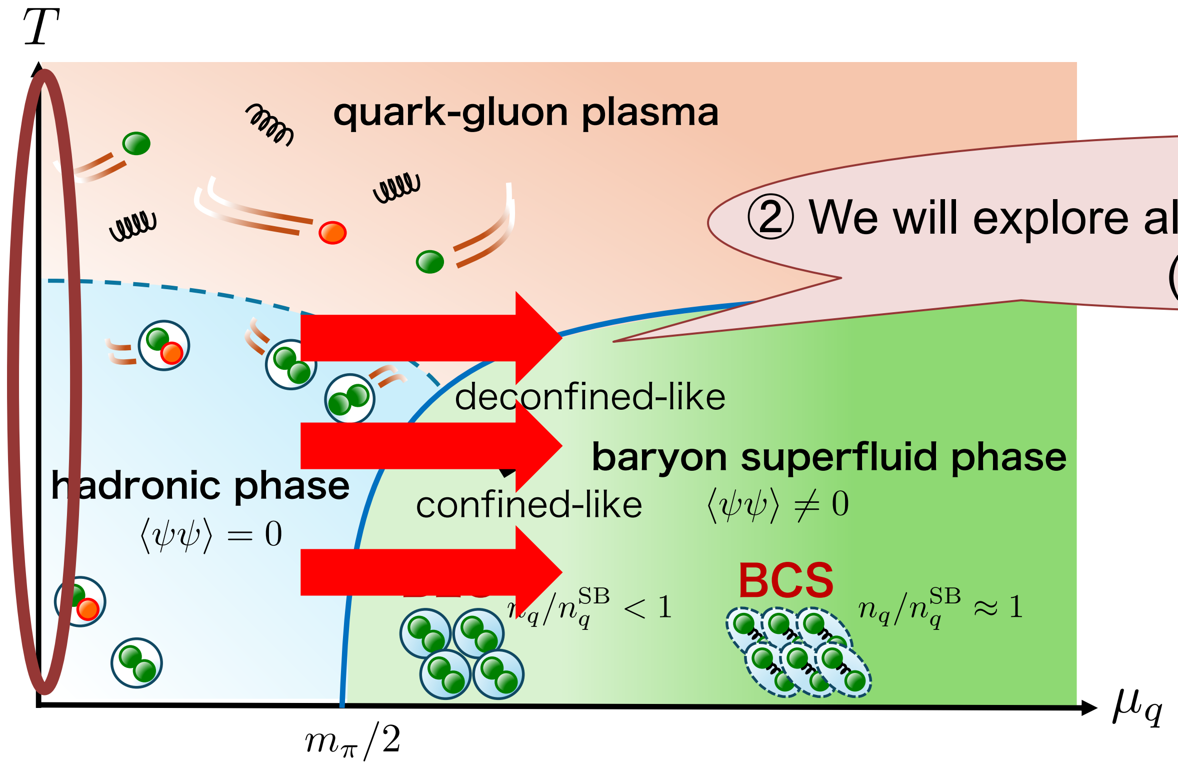
- **FRG analysis in QC$_2$D**

  - to confirm **anomaly enhancement** in superfluid phase

  - FRG in superfluid phase is not easy due to $\begin{cases} C \text{ violation due to finite } \mu \\ U(1)_B \text{ violation due to diquark condensate } \langle\psi\psi\rangle \end{cases}$

① First we studied in hot medium (this talk)

Fejos-**Suenaga** (2025, 2025)



② We will explore all regime including superfluid phase (on-going work)

FRG for anomaly effect

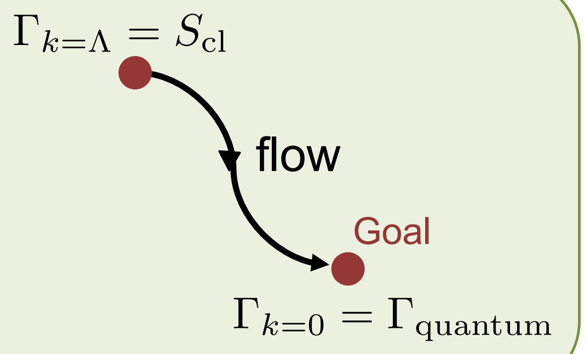
- **FRG analysis in hot $\mathrm{QC_2D}$**

  - The flow equation is $\partial_k \Gamma_k = \frac{1}{2}\tilde{\partial}_k \mathrm{Tr}\,\mathrm{Log}\,[\Gamma_k'' + R_k]$, with $\tilde{\partial}_k \equiv \partial_k R_k \frac{\partial}{\partial R_k}$

    → Regulator R_k allows us to include fluctuations of $p \gtrsim k$

$$\begin{cases} R_k \to \infty \ (k \to \Lambda) \\ R_k \to 0 \ (k \to 0) \end{cases}$$

⇨ 3D Litim regulator
$R_k(\boldsymbol{p}) = (k^2 - \boldsymbol{p}^2)\theta(k^2 - \boldsymbol{p}^2)$



  - k-dependent effective action: $\Gamma_k = \int_x \left(\mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + h\,\sigma \ - V_k[\Sigma] \right)$

    with assumption: $V_k[\Sigma] = m_k^2 I_1 + \tilde{\lambda}_{1,k} I_1^2 + \tilde{\lambda}_{2,k} I_2 + \underbrace{a_k I_A + c_{1,k} I_A^2 + c_{2,k} I_1 I_A}_{\text{anomaly}}$

<u>ansatz</u>
the same structure
as classical one

invariants: $I_1 = \mathrm{Tr}[\Sigma^\dagger \Sigma] \quad I_2 = \mathrm{Tr}[(\Sigma^\dagger \Sigma - \frac{1}{4}\mathrm{Tr}[\Sigma^\dagger \Sigma]\mathbf{1})^2] \quad I_A = \frac{1}{2}\mathrm{Tr}[\tilde{\Sigma}\Sigma + \tilde{\Sigma}^\dagger \Sigma^\dagger] \quad \left(\tilde{\Sigma}_{ij} = \frac{1}{2}\epsilon_{ijkl}\Sigma_{kl}\right)$

I_A: anomalous

⇨ $\Gamma_k'' = \omega_n^2 + \boldsymbol{p}^2 + V_k''$ with $V_k'' = \begin{pmatrix} \frac{\partial^2 V_k}{\partial\sigma\partial\sigma} & \cdots \\ \vdots & \ddots \end{pmatrix}_{12\times 12}$ (12 = # of hadrons)

FRG for anomaly effect

- **Numerical results**

$\mathring{X}$ means vacuum value of X

initial conditions

$\tilde{\lambda}_{2,\Lambda} = 4\tilde{\lambda}_{1,\Lambda}$ (large Nc)

$c_{1,\Lambda} = c_{2,\Lambda} = 0$

physical inputs

$\mathring{M}_\pi = 0.14\,\mathrm{GeV}$

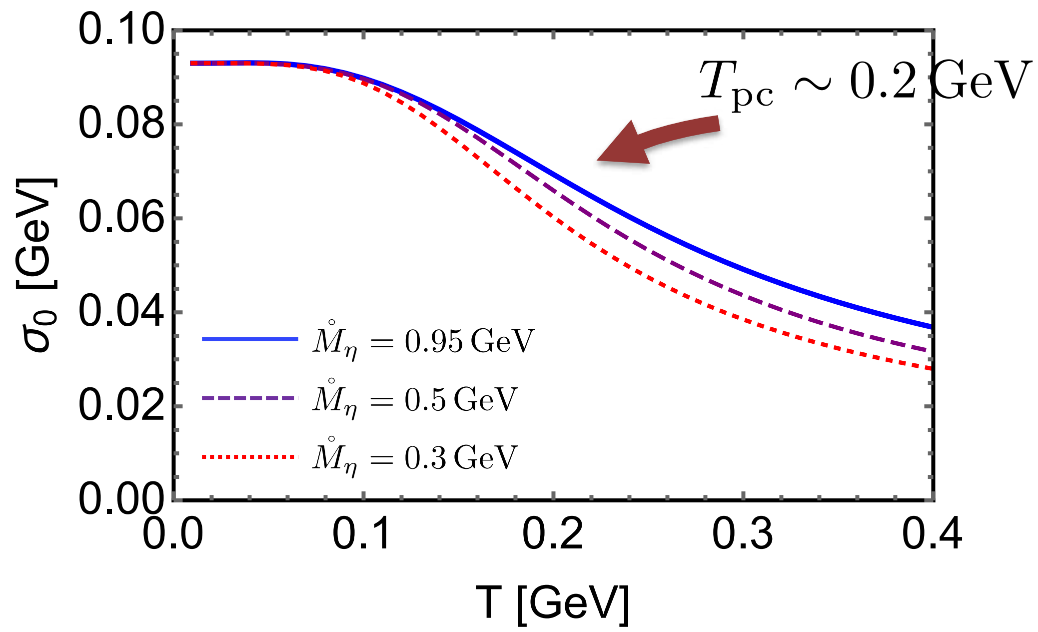
$\mathring{f}_\pi = 0.093\,\mathrm{GeV}$

- $\mathring{M}_\eta$ is treated as a free parameter (anomaly effect is our main interest)

M_η [GeV]	m^2_Λ [GeV^2]	a_Λ [GeV]	$\lambda_{1,\Lambda}$	$\lambda_{2,\Lambda}$	$c_{1,\Lambda}$	$c_{2,\Lambda}$
0.3	-0.0426	-0.032	1.25	5	0	0
0.5	0.0465	-0.108	1.25	5	0	0
0.95	0.385	-0.428	1.25	5	0	0

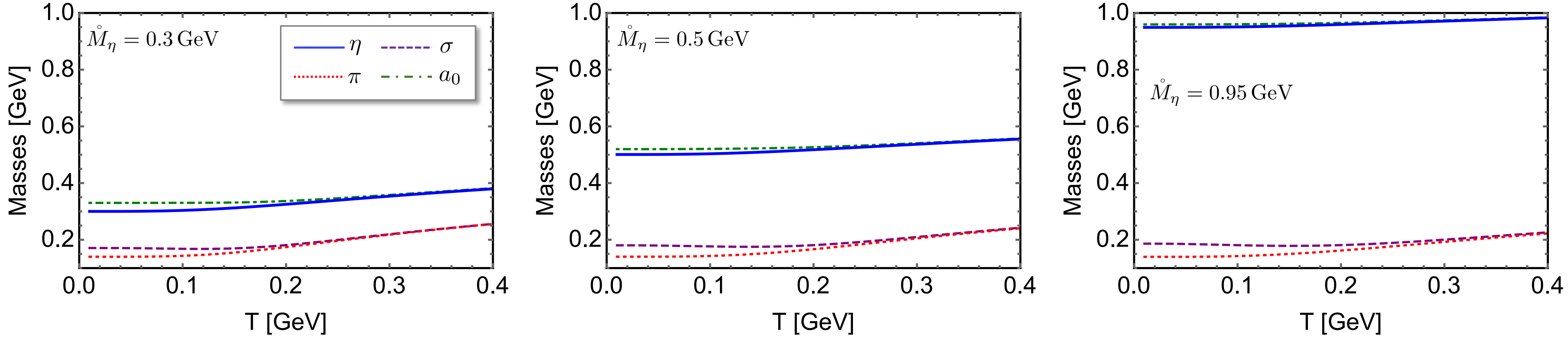
- $\lambda_{2,\Lambda} \lesssim 5$ to yield chiral restoration in hot medium

chiral restoration at finite T



FRG for anomaly effect

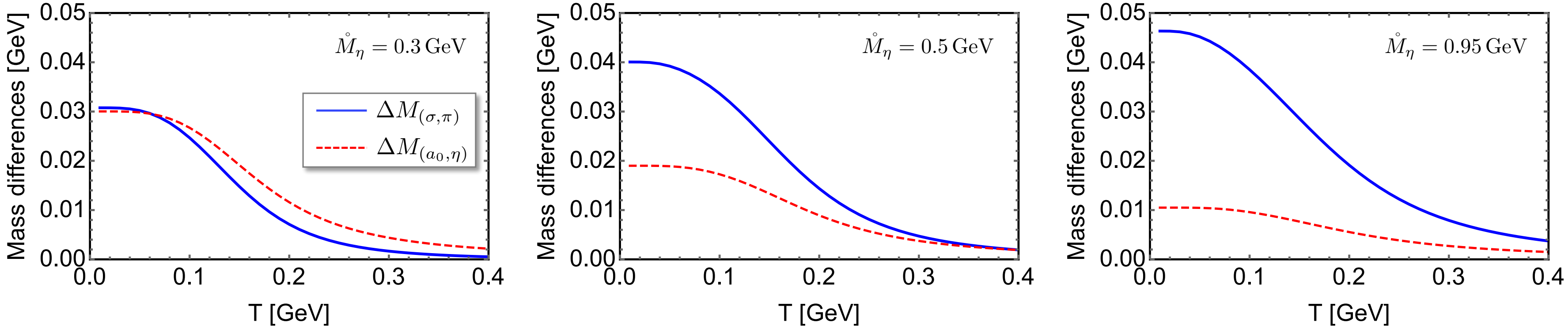
- **Hadron masses at finite T**



$$M_\sigma^2, M_\pi^2 \sim m^2 + a$$
$$M_{a_0}^2, M_\eta^2 \sim m^2 - a$$
$$(a < 0)$$

- Quartic couplings are suppressed at any k

- **Mass differences of parity partners at finite T**



- Mass degeneracies are predicted

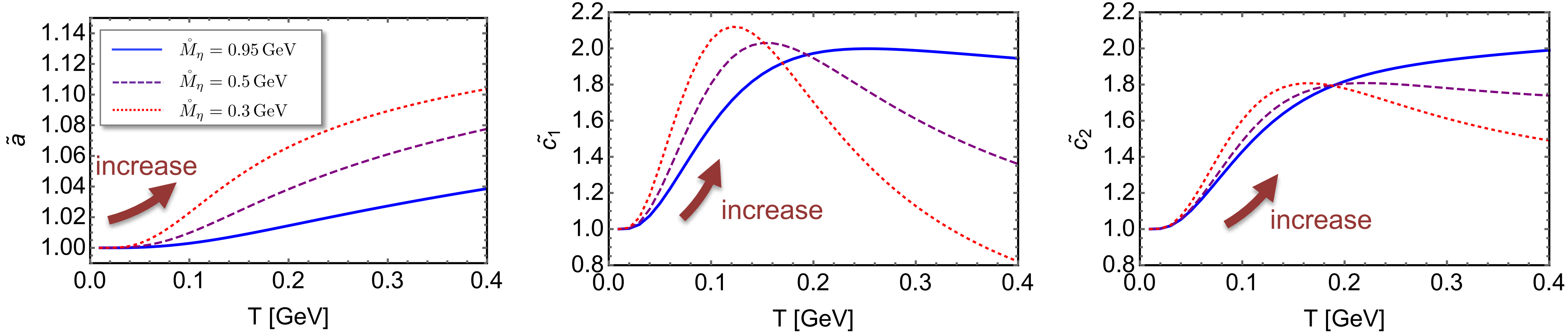
FRG for anomaly effect

- **Anomaly coefficients at finite T**

  - $a < 0,\ c_1 < 0,\ c_2 > 0$ at any k and T

$$\tilde{X} \equiv \frac{X}{X_{T=0}}$$

NOTE: $T_{\rm pc} \sim 0.2\,{\rm GeV}$



- All the anomaly coefficients are **enhanced at finite T**

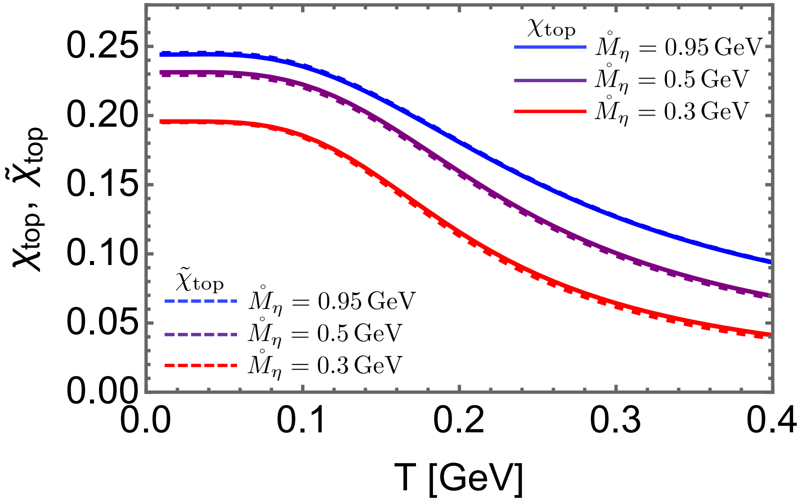
⟺ consistent with $N_f = 2+1,\ N_c = 3$ work by FRG Fejos-Hosaka (2016)

FRG for anomaly effect

- **Topological susceptibility at finite T**

Definition: $\bar{\chi}_{\rm top} \equiv -i\int d^4x \langle 0|{\rm T}Q(x)Q(0)|0\rangle = \frac{\mathring{M}_\pi^4 \mathring{\sigma}_0^2}{4}\left(\frac{1}{M_\pi^2} - \frac{1}{M_\eta^2}\right)$ with $Q = (g_s^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$

chiral Ward-Takahashi identity



$$\chi_{\rm top} \equiv \frac{\bar{\chi}_{\rm top}}{\mathring{M}_\pi^2\mathring{\sigma}_0^2} = \frac{\mathring{M}_\pi^2}{4}\left(\frac{1}{M_\pi^2} - \frac{1}{M_\pi^2 + (M_\eta^2 - M_\pi^2)}\right) \quad \text{(← full result : solid)}$$

$$\tilde{\chi}_{\rm top} \equiv \frac{\mathring{M}_\pi^2}{4}\left(\frac{1}{M_\pi^2} - \frac{1}{M_\pi^2 + \underbrace{(\mathring{M}_\eta^2 - \mathring{M}_\pi^2)}}\right) \quad \text{(← reference: dashed)}$$

anomaly effect is constant = no anomaly change at finite T

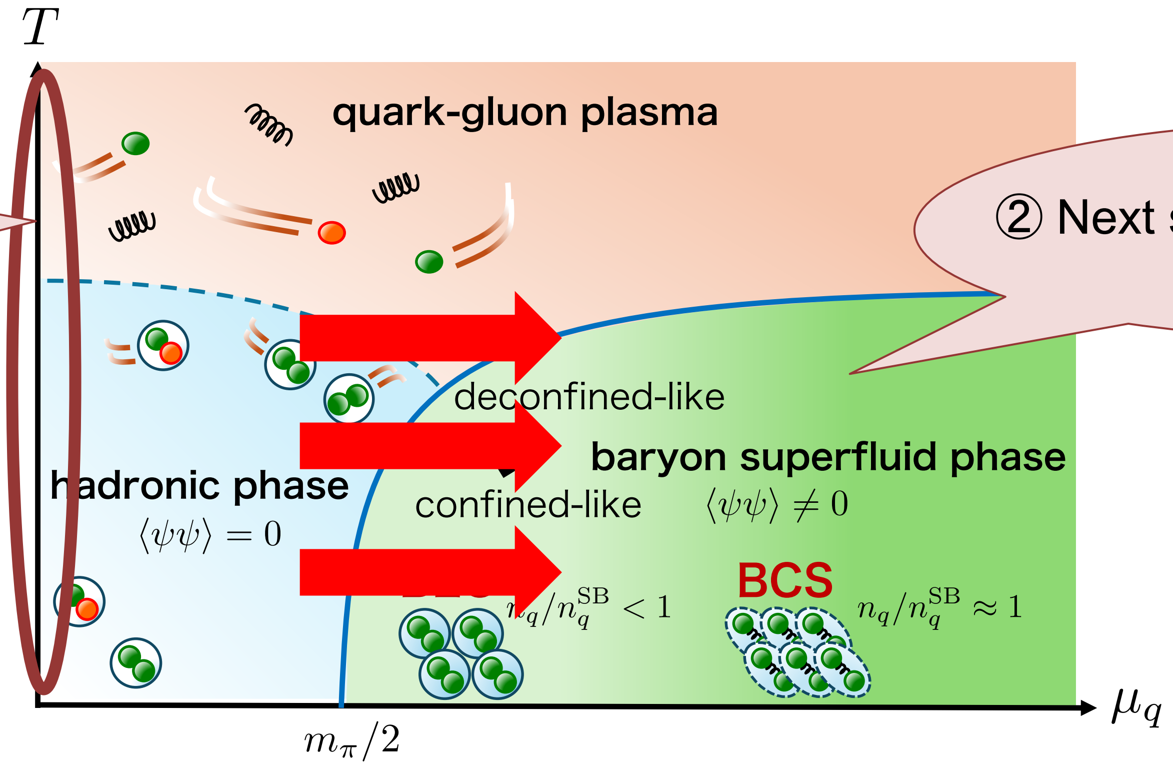
- $\chi_{\rm top}$ is suppressed at higher temperature followed by chiral restoration: $\chi_{\rm top} \sim \frac{\mathring{M}_\pi^2}{4}\frac{1}{M_\pi^2} \propto \sigma_0$
- No sizable difference between $\chi_{\rm top}$ and $\tilde{\chi}_{\rm top}$
  → topological susceptibility is not suitable probe to see corrections of anomaly effects

FRG for anomaly effect

- **FRG analysis in QC_2D**

- FRG in superfluid phase is not easy due to $\begin{cases} C \text{ violation due to finite } \mu \\ U(1)_B \text{ violation due to diquark condensate } \langle\psi\psi\rangle \end{cases}$

① anomaly enhancement is confirmed



② Next step: in superfluid phase (on-going work)

Fejos-**Suenaga** (2025, 2025)

Other directions

- FRG analysis in hot QC_2D medium

Fejos-**Suenaga** (2025), (2025)

- $U(1)_A$ anomaly enhancement in hot medium
- Analysis in dense medium (on-going)

- Mass spectrum in cold-dense QC_2D medium with spin-1 hadrons

Suenaga et al(2024)

- Possibility of (axial)vector condensate

- Analysis of cold-dense QC_2D medium with Nf=2+2 LSM

Sakai-**Suenaga** (2025)

- Enhancement of $\langle \bar{Q}Q \rangle$ in medium
- Possibility of heavy diquark condensate in medium

⋮

Conclusions

- Constructed Linear sigma model (LSM) in QC_2D to study spin-0 hadrons including *parity partners* at μ_q

⇩ Succeeded in explaining lattice mass spectrum qualitatively!

Then
- Applied to topological susceptibility, sound velocity
+ FRG analysis (in hot medium), 2+2 flavor LSM study
+ Extension with spin-1 hadrons

⋮

⇔ Comparison with (future) lattice "numerical experiments"

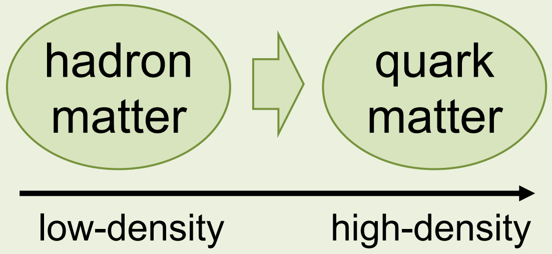
- On-going: Check of anomaly enhancement in superfluid phase with FRG

From QC_2D study we can learn

- to what extent (simple) hadron model description can apply
- Which representation of hadrons is useful in medium
- how to incorporate quark dof into hadron model → "unified model"
- deep understanding of quark matter in high-dense regime

⋮

universal regardless of Nc

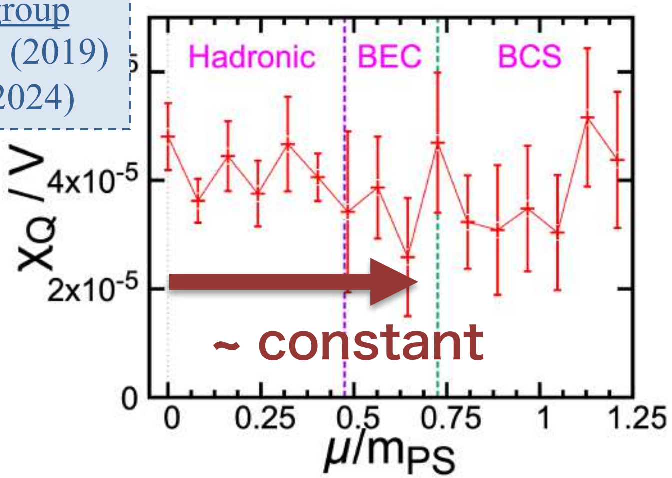


Topological susceptibility

• Topological susceptibility

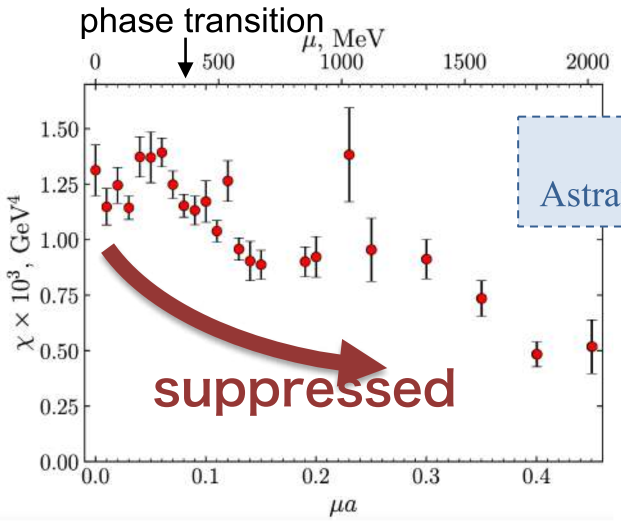
- Lattice results of topological susceptibility by two groups look inconsistent even at qualitative level

Japanese group
Iida-Itou-Lee (2019)
Iida et al (2024)



Why?

inconsistent



Russian group
Astrakhantsev et al (2020)

Definition of topological susceptibility

cf, Instanton, 'tHooft (1986)

$$\chi_{\rm top} = -\int d^4x \frac{\delta^2 \Gamma_{\rm QC_2D}}{\delta\theta(x)\delta\theta(0)}\bigg|_{\theta=0} = -i\int d^4x \langle Q(x)Q(0)\rangle$$ with topological operator $$Q = \frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$$

- We applied LSM to theoretically explore fate of $\chi_{\rm top}$ in dense QC_2D

Topological susceptibility

- **Theoretical background of $\chi_{\rm top}$**

  - QC$_2$D generating functional with a θ-term is

$$Z_{\rm QC_2D}=\int[d\bar{\psi}d\psi][dA]\exp\left[i\int d^4x\left(\bar{\psi}(i\not{D}-m_l)\psi-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}+\theta\frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}\right)\right]$$

U(1)$_A$ axial transformation $\psi\to\exp\left(i\theta/4\gamma_5\right)\psi$

  - θ dependence is absorbed into quark mass term via Fujikawa's method

$$Z_{\rm QC_2D}=\int[d\bar{\psi}d\psi][dA]\exp\left[i\int d^4x\left(\bar{\psi}i\not{D}\psi-m_l\bar{\psi}\exp\left(i\theta/2\gamma_5\right)\psi-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}\right)\right]$$

Ward-Takahashi identity
$\langle\bar{\psi}\psi\rangle=-im_l\chi_\pi$

current quark mass

$$\chi_{\rm top}=-\int d^4x\frac{\delta^2\Gamma_{\rm QC_2D}}{\delta\theta(x)\delta\theta(0)}\bigg|_{\theta=0}=\frac{im_l^2}{4}(\chi_\pi-\chi_\eta)=\frac{f_\pi^2m_\pi^2}{2}\left(1-\frac{\chi_\eta}{\chi_\pi}\right)$$

$$\begin{cases}\chi_\pi\delta^{ab}=\int d^4x\langle(\bar{\psi}i\gamma_5\tau_f^a\psi)(x)(\bar{\psi}i\gamma_5\tau_f^b\psi)(0)\rangle\\ \chi_\eta=\int d^4x\langle(\bar{\psi}i\gamma_5\psi)(x)(\bar{\psi}i\gamma_5\psi)(0)\rangle\end{cases}$$

- Matching $Z_{\rm QC_2D}=Z_{\rm LSM}$ enables us to evaluate χ_π and χ_η within LSM

2+2 flavor LSM

- **LSM for $N_f = 2+2$ in QC$_2$D (no sign problem)**

Sakai-**Suenaga**, submitted to PRD

- Useful to see roles of heavy flavors in dense medium
- Quark bilinear $\Sigma \sim \Psi_j^T \sigma^2 \tau_c^2 \Psi_i$ is now 8×8 matrix

$\Psi = (q_{1R}, q_{2R}, Q_{1R}, Q_{2R}, \tilde{q}_{1L}, \tilde{q}_{2L}, \tilde{Q}_{1L}, \tilde{Q}_{2L})^T$
with $\tilde{\psi}_L = \sigma^2\tau_c^2\psi_L^*$

Building block: $\Sigma = (\mathcal{S}^a - i\mathcal{P}^a)X^a E$ $\begin{cases} X^a : \text{generators of } SU(8)/Sp(8) \\ E : 8\times 8 \text{ symplectic matrix} \end{cases}$

simple extension of $N_f = 2$ one

Hadron	Field parametrization	J^P	N_q	N_Q	I_q	I_Q
η_{qq}	$\frac{1}{\sqrt{2}}\mathcal{P}^0 + \frac{1}{\sqrt{3}}\mathcal{P}^8 + \frac{1}{\sqrt{6}}\mathcal{P}^{15}$	0^-	0	0	0	0
η_{QQ}	$-\frac{1}{\sqrt{3}}\mathcal{P}^8 + \sqrt{\frac{2}{3}}\mathcal{P}^{15}$	0^-	0	0	0	0
$\pi_{0,qq}$	$\mathcal{P}^3$, $\frac{1}{\sqrt{2}}(\mathcal{P}^1 \pm i\mathcal{P}^2)$	0^-	0	0	1	0
$\pi_{0,QQ}$	$\frac{1}{\sqrt{2}}\mathcal{P}^0 - \frac{1}{\sqrt{3}}\mathcal{P}^8 - \frac{1}{\sqrt{6}}\mathcal{P}^{15}$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{13} \pm i\mathcal{P}^{14})$	0^-	0	0	0	1
K_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{P}^4 - i\mathcal{P}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^6 - i\mathcal{P}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^9 - i\mathcal{P}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{11} - i\mathcal{P}^{12})$	0^-	+1	−1	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{K}_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^4 + i\mathcal{P}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^6 + i\mathcal{P}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{P}^9 + i\mathcal{P}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{11} + i\mathcal{P}^{12})$	0^-	−1	+1	$\frac{1}{2}$	$\frac{1}{2}$
B_{qq}	$\frac{1}{\sqrt{2}}(\mathcal{P}^{17} - i\mathcal{P}^{16})$	0^+	+2	0	0	0
$\bar{B}_{qq}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^{17} + i\mathcal{P}^{16})$	0^+	−2	0	0	0
B_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{P}^{19} - i\mathcal{P}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{21} - i\mathcal{P}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{23} - i\mathcal{P}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{25} - i\mathcal{P}^{24})$	0^+	+1	+1	0	0
$\bar{B}_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^{19} + i\mathcal{P}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{21} + i\mathcal{P}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{23} + i\mathcal{P}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{P}^{25} + i\mathcal{P}^{24})$	0^+	−1	−1	0	0
B_{QQ}	$\frac{1}{\sqrt{2}}(\mathcal{P}^{27} - i\mathcal{P}^{26})$	0^+	0	+2	0	0
$\bar{B}_{QQ}$	$\frac{1}{\sqrt{2}}(\mathcal{P}^{27} + i\mathcal{P}^{26})$	0^+	0	−2	0	0

Hadron	Field parametrization	J^P	N_q	N_Q	I_q	I_Q
σ_{qq}	$\frac{1}{\sqrt{2}}\mathcal{S}^0 + \frac{1}{\sqrt{3}}\mathcal{S}^8 + \frac{1}{\sqrt{6}}\mathcal{S}^{15}$	0^+	0	0	0	0
σ_{QQ}	$-\frac{1}{\sqrt{3}}\mathcal{S}^8 + \sqrt{\frac{2}{3}}\mathcal{S}^{15}$	0^+	0	0	0	0
$a_{0,qq}$	$\mathcal{S}^3$, $\frac{1}{\sqrt{2}}(\mathcal{S}^1 \pm i\mathcal{S}^2)$	0^+	0	0	1	0
$a_{0,QQ}$	$\frac{1}{\sqrt{2}}\mathcal{S}^0 - \frac{1}{\sqrt{3}}\mathcal{S}^8 - \frac{1}{\sqrt{6}}\mathcal{S}^{15}$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{13} \pm i\mathcal{S}^{14})$	0^+	0	0	0	1
κ_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{S}^4 - i\mathcal{S}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^6 - i\mathcal{S}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^9 - i\mathcal{S}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{11} - i\mathcal{S}^{12})$	0^+	+1	−1	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{\kappa}_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^4 + i\mathcal{S}^5)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^6 + i\mathcal{S}^7)$, $\frac{1}{\sqrt{2}}(\mathcal{S}^9 + i\mathcal{S}^{10})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{11} + i\mathcal{S}^{12})$	0^+	−1	+1	$\frac{1}{2}$	$\frac{1}{2}$
B'_{qq}	$\frac{1}{\sqrt{2}}(\mathcal{S}^{17} - i\mathcal{S}^{16})$	0^-	+2	0	0	0
$\bar{B}'_{qq}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^{17} + i\mathcal{S}^{16})$	0^-	−2	0	0	0
B'_{Qq}	$\frac{1}{\sqrt{2}}(\mathcal{S}^{19} - i\mathcal{S}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{21} - i\mathcal{S}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{23} - i\mathcal{S}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{25} - i\mathcal{S}^{24})$	0^-	+1	+1	0	0
$\bar{B}'_{Qq}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^{19} + i\mathcal{S}^{18})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{21} + i\mathcal{S}^{20})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{23} + i\mathcal{S}^{22})$, $\frac{1}{\sqrt{2}}(\mathcal{S}^{25} + i\mathcal{S}^{24})$	0^-	−1	−1	0	0
B'_{QQ}	$\frac{1}{\sqrt{2}}(\mathcal{S}^{27} - i\mathcal{S}^{26})$	0^-	0	+2	0	0
$\bar{B}'_{QQ}$	$\frac{1}{\sqrt{2}}(\mathcal{S}^{27} + i\mathcal{S}^{26})$	0^-	0	−2	0	0

2+2 flavor LSM

• Lagrangian and mean fields

- (approximately) $SU(8)$-invariant LSM

$U(1)_A$ anomaly

$$\mathcal{L} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - m_0^2 \mathrm{tr}[\Sigma^\dagger \Sigma] - \lambda_1 \left(\mathrm{tr}[\Sigma^\dagger \Sigma]\right)^2 - \lambda_2 \mathrm{tr}[(\Sigma^\dagger \Sigma)^2] + \bar{c}\,\mathrm{tr}[\zeta^\dagger \Sigma + \Sigma^\dagger \zeta] + c\,\epsilon_{ijklmnop}(\Sigma_{ij}\Sigma_{kl}\Sigma_{mn}\Sigma_{op} + \mathrm{h.c.})$$

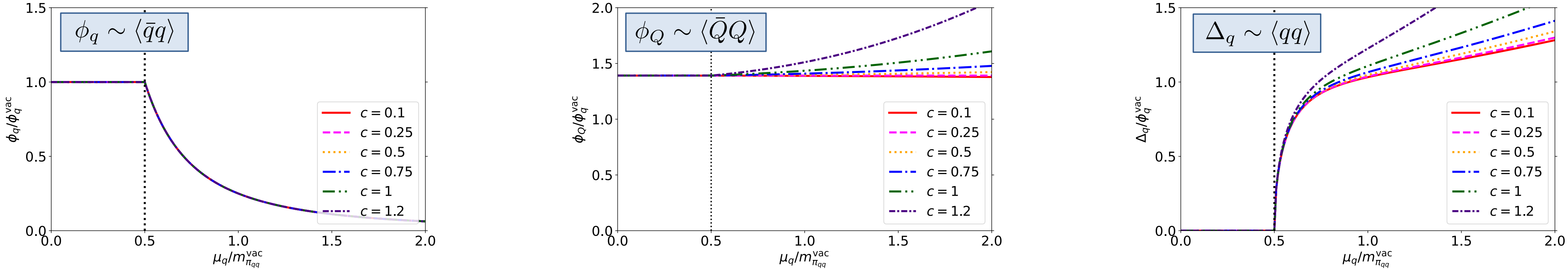
$$D_\mu \Sigma = \partial_\mu \Sigma - i\mu_q \delta_{\mu 0}\{J, \Sigma\} \text{ with } J = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix}_{8\times 8}$$

<u>chemical potential effect</u> $(\mu_q = \mu_Q)$

$$\zeta = \frac{1}{2}(m_q E_q + m_Q E_Q) \text{ with } E_q = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ -\mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{8\times 8} E_Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & -\mathbf{1} & 0 \end{pmatrix}_{8\times 8}$$

<u>current-quark mass effect</u>

- μ_q dependences of the mean fields for several c with $\lambda_1 = 6$ + input $\left\{ \begin{array}{ll} m_{\pi_{qq}}^{\mathrm{vac}} = 138\,\mathrm{MeV} & m_K^{\mathrm{vac}} = 494\,\mathrm{MeV} \\ f_\pi^{\mathrm{vac}} = 92\,\mathrm{MeV} & f_K^{\mathrm{vac}} = 110\,\mathrm{MeV} \end{array} \right.$



2+2 flavor LSM

- **Lagrangian and mean fields**

- (approximately) $SU(8)$-invariant LSM

$U(1)_A$ anomaly

$$\mathcal{L} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - m_0^2 \mathrm{tr}[\Sigma^\dagger \Sigma] - \lambda_1 \left(\mathrm{tr}[\Sigma^\dagger \Sigma]\right)^2 - \lambda_2 \mathrm{tr}[(\Sigma^\dagger \Sigma)^2] + \bar{c}\,\mathrm{tr}[\zeta^\dagger \Sigma + \Sigma^\dagger \zeta] + c\,\epsilon_{ijklmnop}(\Sigma_{ij}\Sigma_{kl}\Sigma_{mn}\Sigma_{op} + \mathrm{h.c.})$$

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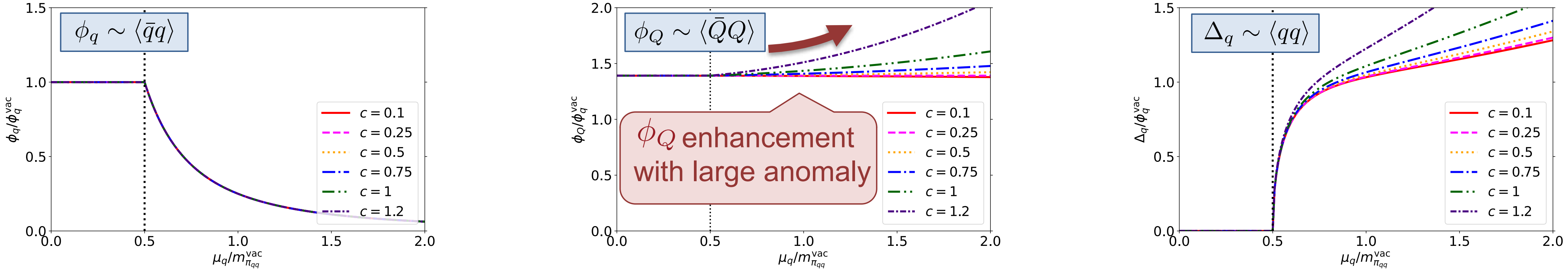
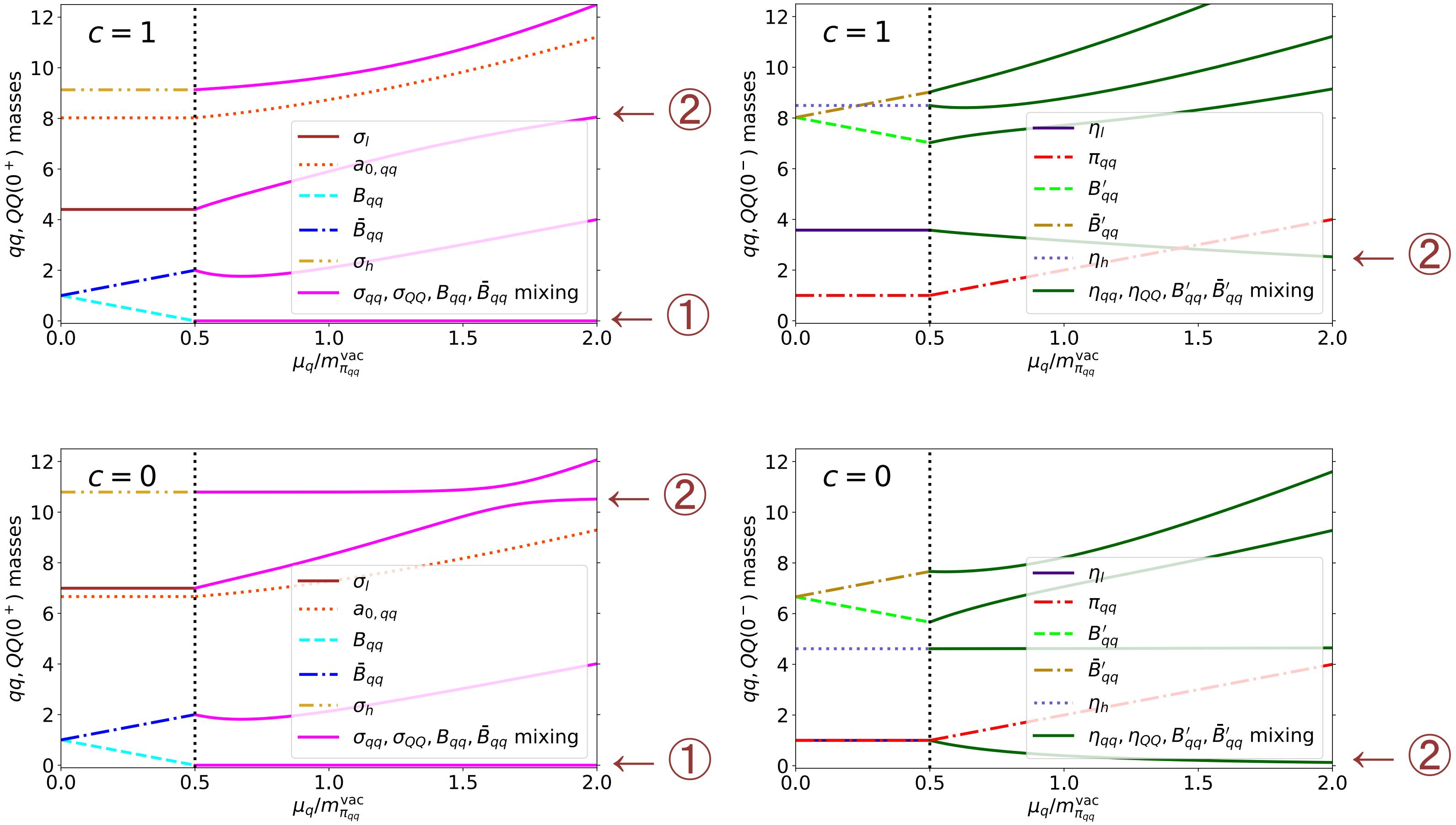
- μ_q dependences of the mean fields for several c with $\lambda_1 = 6$ + input $\begin{cases} m_{\pi_{qq}}^{\mathrm{vac}} = 138\,\mathrm{MeV} & m_K^{\mathrm{vac}} = 494\,\mathrm{MeV} \\ f_\pi^{\mathrm{vac}} = 92\,\mathrm{MeV} & f_K^{\mathrm{vac}} = 110\,\mathrm{MeV} \end{cases}$



Light-light hadron masses

w/ anomaly →

w/o anomaly →



① zero mode (NG mode of $U(1)$ light-quark number breaking)

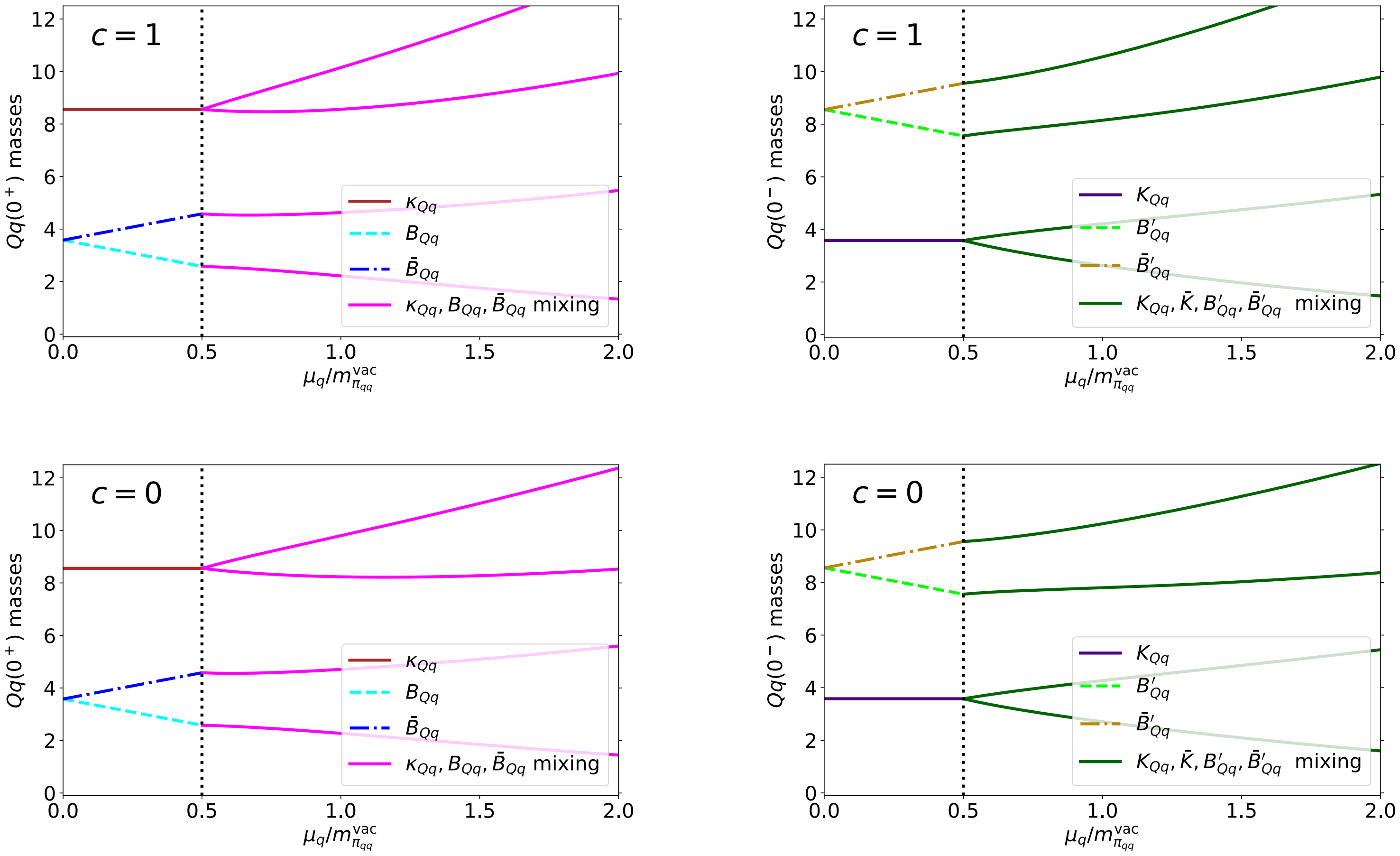
② large anomaly dependences in " σ " and " η " modes

- mixings among $(\sigma_{qq}, \sigma_{QQ}, B_{qq}, \bar{B}_{qq})$ and $(\eta_{qq}, \eta_{QQ}, B'_{qq}, \bar{B}'_{qq})$ occur

Heavy-light hadron masses

w/ anomaly →

w/o anomaly →

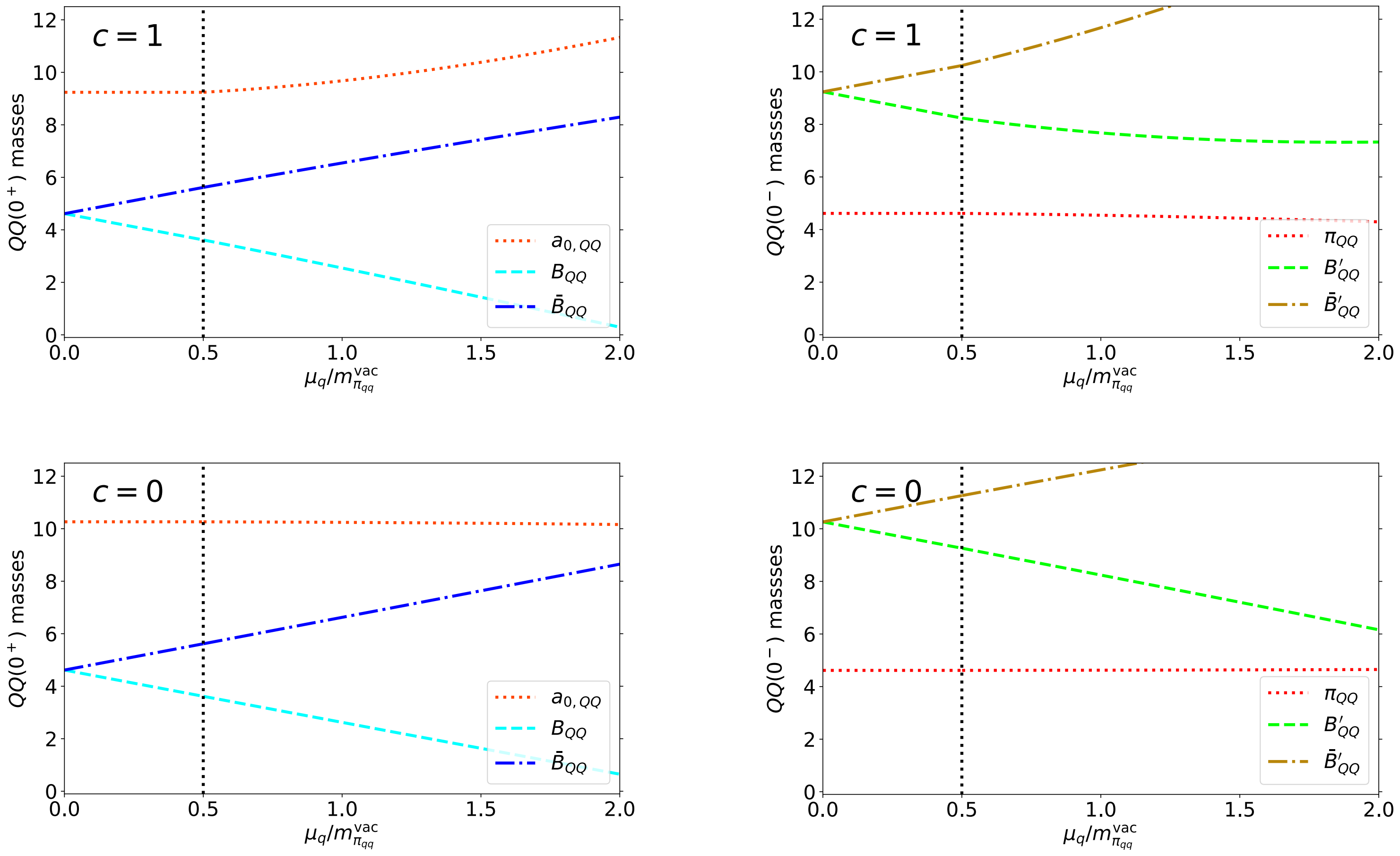


- mixings among $(\kappa_{Qq}, \bar{\kappa}_{Qq}, B_{Qq}, \bar{B}_{Qq})$ and $(K_{Qq}, \bar{K}_{Qq}, B'_{Qq}, \bar{B}'_{Qq})$ occur
- tiny anomaly dependences

Heavy-heavy hadron masses

w/ anomaly →

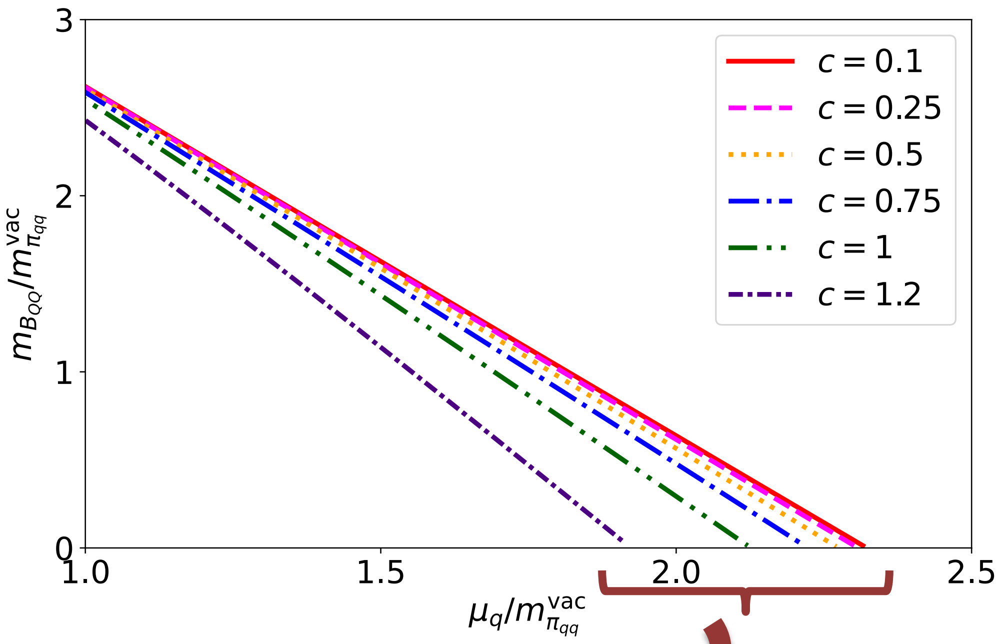
w/o anomaly →



- no mixings occur ($U(1)$ heavy-quark number symmetry is preserved)
- small but non-negligible anomaly dependences

Heavy-heavy hadron masses

- **Possibility of novel diquark condensate $\langle QQ \rangle$ ($B_{QQ} \sim QQ$)**



what happened with quark d.o.f?

- $m_{B_{QQ}} \to 0$ signals the appearance of $\Delta_Q \sim \langle QQ \rangle$
- Larger anomaly effects make the onset μ_q smaller (= anomaly is a “catalyst”)

Conclusions

- Constructed Linear sigma model (LSM) in QC_2D to study spin-0 hadrons including *parity partners* at μ_q

⇩ Succeeded in explaining lattice mass spectrum qualitatively!

Then
- Applied to topological susceptibility, sound velocity
+ FRG analysis (in hot medium), 2+2 flavor LSM study
+ Extension with spin-1 hadrons (similarly to eLSM by Frankfurt group)
⋮

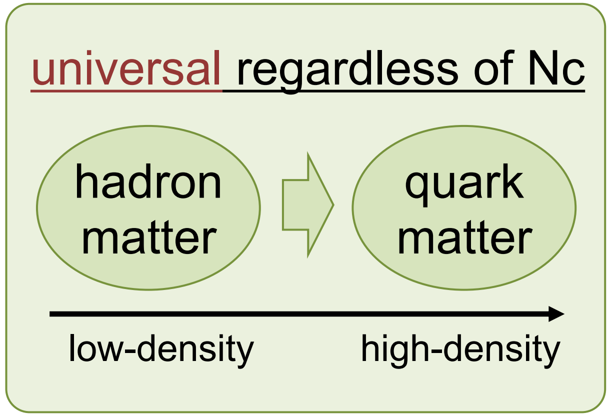
Suenaga et al (2024)

⇔ Comparison with (future) lattice

- On-going: Check of anomaly enhancement in superfluid phase with FRG

From QC_2D study we can learn

- **to what extent simple hadron model description can apply**
- **how to incorporate quark d.o.f. into hadron model (→ unified model)**
- **deep understanding of quark matter in high-dense regime**
⋮



LSM with spin-1 hadrons

Spin-1 hadron matrix

$$\Phi^\mu = \frac{1}{2}\begin{pmatrix} \frac{\omega+\rho^0-(f_1+a_1^0)}{\sqrt{2}} & \rho^+ - a_1^+ & \sqrt{2}B_S^{I_z=+1} & B_S^{I_z=0} - B_{AS} \\ \rho^- - a_1^- & \frac{\omega-\rho^0-(f_1-a_1^0)}{\sqrt{2}} & B_S^{I_z=0} + B_{AS} & \sqrt{2}B_S^{I_z=-1} \\ \sqrt{2}\bar{B}_S^{I_z=-1} & \bar{B}_S^{I_z=0} + \bar{B}_{AS} & -\frac{\omega+\rho^0+f_1+a_1^0}{\sqrt{2}} & -(\rho^- + a_1^-) \\ \bar{B}_S^{I_z=0} - \bar{B}_{AS} & \sqrt{2}\bar{B}_S^{I_z=+1} & -(\rho^+ + a_1^+) & -\frac{\omega-\rho^0+f_1-a_1^0}{\sqrt{2}} \end{pmatrix}^\mu$$

	Hadron	J^P	Quark number	Isospin
mesons	ω	1^-	0	0
	ρ	1^-	0	1
	f_1	1^+	0	0
	a_1	1^+	0	1
diquark baryons	B_S ($\bar{B}_S$)	1^+	+2 (−2)	1
	B_{AS} ($\bar{B}_{AS}$)	1^-	+2 (−2)	0

SU(4) transformation laws: $\Phi^\mu \to U\Phi^\mu U^\dagger \quad [U \in SU(4)]$

explicit breaking $\quad U(1)_A$ anomaly

$$\begin{aligned}\mathcal{L}_{\mathrm{eLSM}} =\ & \mathrm{tr}[D_\mu\Sigma^\dagger D^\mu\Sigma] - m_0^2\mathrm{tr}[\Sigma^\dagger\Sigma] - \lambda_1\left(\mathrm{tr}[\Sigma^\dagger\Sigma]\right)^2 - \lambda_2\mathrm{tr}[(\Sigma^\dagger\Sigma)^2] + \underbrace{\mathrm{tr}[H^\dagger\Sigma + \Sigma^\dagger H]}_{\text{explicit breaking}} + \underbrace{c(\det\Sigma + \det\Sigma^\dagger)}_{U(1)_A\text{ anomaly}} \\ & - \frac{1}{2}\mathrm{tr}[\Phi_{\mu\nu}\Phi^{\mu\nu}] + m_1^2\mathrm{tr}[\Phi_\mu\Phi^\mu] + ig_3\mathrm{tr}\left[\Phi_{\mu\nu}[\Phi^\mu,\Phi^\nu]\right] + h_1\mathrm{tr}[\Sigma^\dagger\Sigma]\mathrm{tr}[\Phi_\mu\Phi^\mu] + h_2\mathrm{tr}[\Sigma\Sigma^\dagger\Phi_\mu\Phi^\mu] \\ & + h_3\mathrm{tr}[\Phi_\mu^T\Sigma^\dagger\Phi^\mu\Sigma] + g_4\mathrm{tr}[\Phi_\mu\Phi_\nu\Phi^\mu\Phi^\nu] + g_5\mathrm{tr}[\Phi_\mu\Phi^\mu\Phi_\nu\Phi^\nu] + g_6\mathrm{tr}[\Phi_\mu\Phi^\mu]\mathrm{tr}[\Phi_\nu\Phi^\nu] + g_7\mathrm{tr}[\Phi_\mu\Phi_\nu]\mathrm{tr}[\Phi^\mu\Phi^\nu]\end{aligned}$$

$$\begin{cases} \Phi_{\mu\nu} \equiv D_\mu\Phi_\nu - D_\nu\Phi_\mu \\ D_\mu\Sigma \equiv \partial_\mu\Sigma - iG_\mu\Sigma - i\Sigma G_\mu^T - ig_1\Phi_\mu\Sigma - ig_2\Sigma\Phi_\mu^T \\ D_\mu\Phi_\nu \equiv \partial_\mu\Phi_\nu - i[G_\mu,\Phi_\nu] \\ H = h_q E \ \text{ with } \ E = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \end{cases}$$

$\pi - a_1$ mixings, etc.

MF in superfluid phase

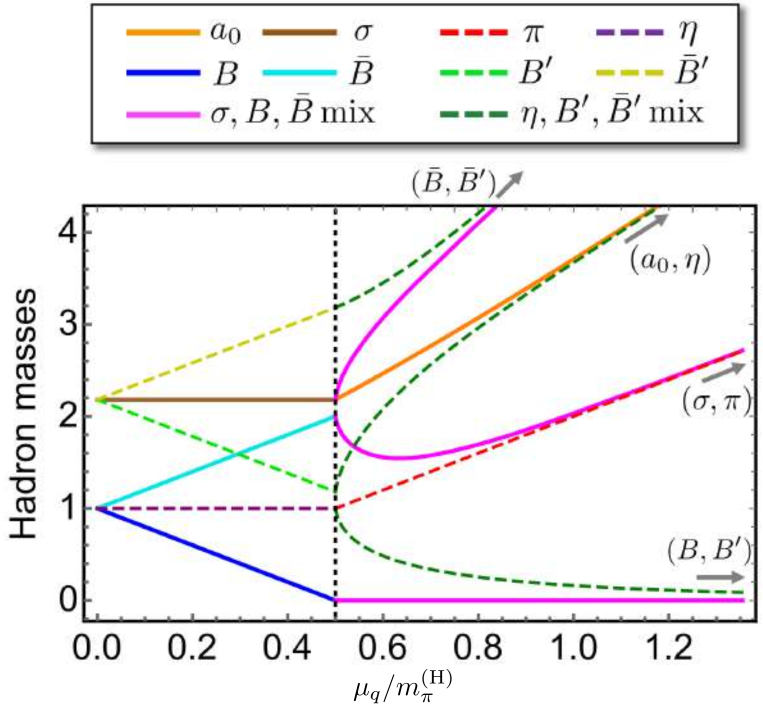
mesons: $\sigma_0 = \langle\sigma\rangle \qquad \bar{\omega} = \langle\omega^{\mu=0}\rangle$

diquarks: $\Delta = \left\langle\frac{B+\bar{B}}{\sqrt{2}}\right\rangle \qquad \bar{V} = \left\langle\frac{\bar{B}_{AS}^{\mu=0} - B_{AS}^{\mu=0}}{\sqrt{2}i}\right\rangle$

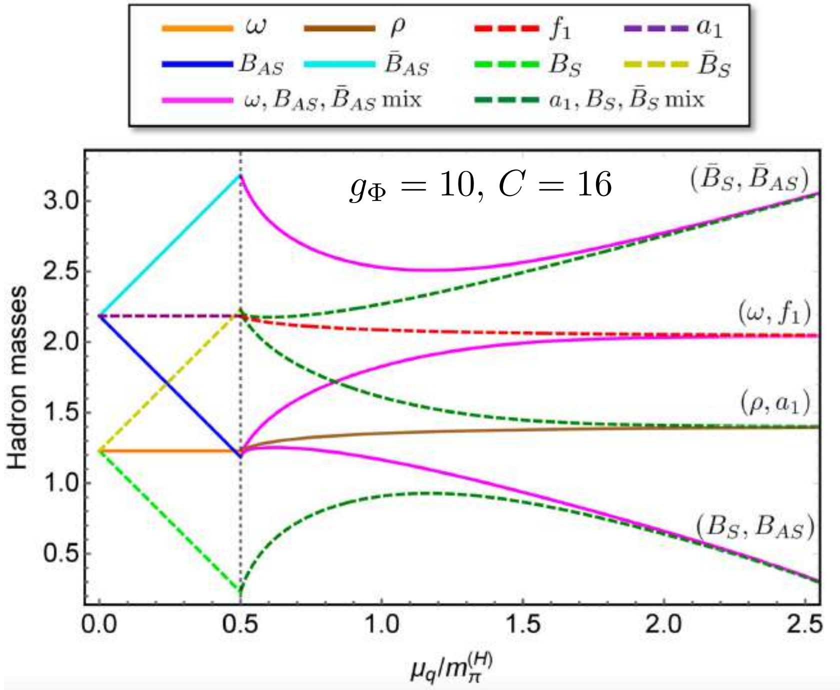
LSM with spin-1 hadrons

Suenaga et al (2024)

Spin-0 hadrons



Spin-1 hadrons

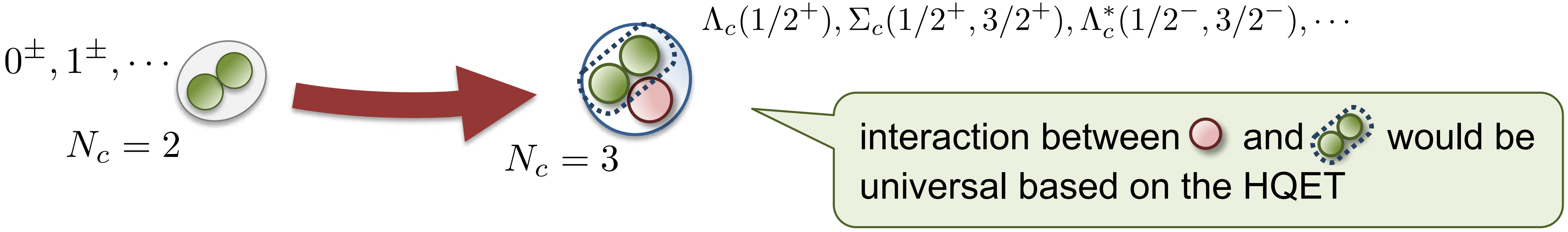


- Mass degeneracy of chiral partners at high μ_q is clearly predicted
  → challenging issue for future lattice simulation

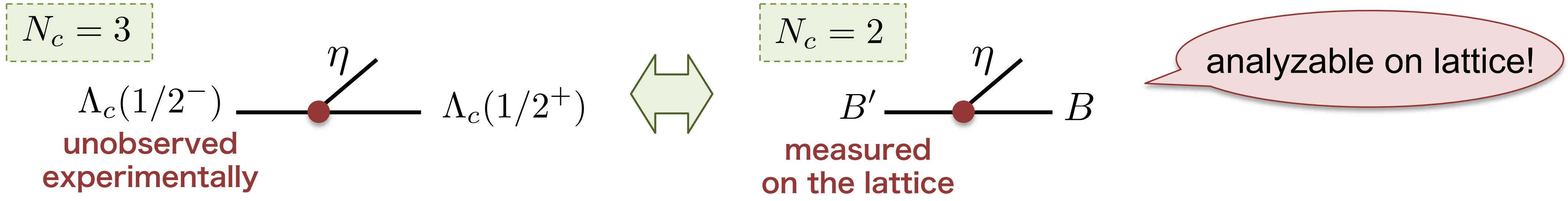
Other directions

- **Analysis with Nf=2+1 at vanishing μ_q**

- QC_2D is useful to extract information of singly heavy baryon (SHB) spectrum



- Hint for unobserved (HQS-singlet) $\Lambda_c(1/2^-)$ from B' study in QC_2D?



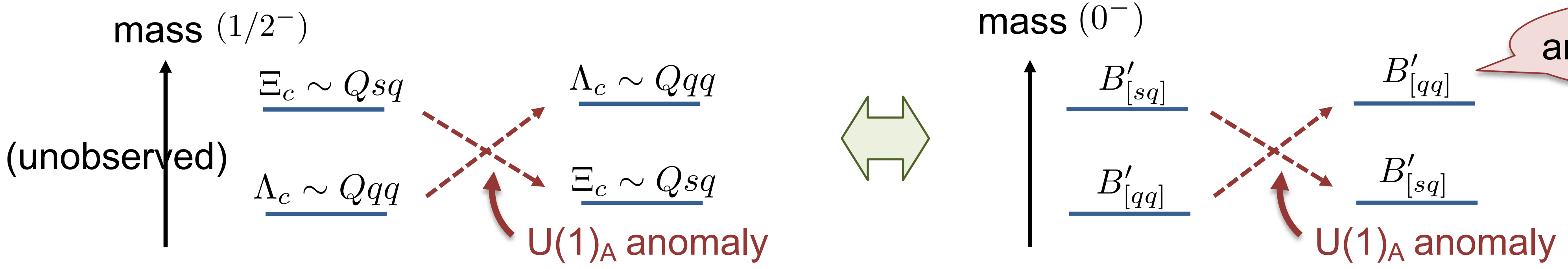
Other directions

- **Analysis with Nf=2+1 at vanishing μ_q**

  - QC$_2$D is useful to extract information of singly heavy baryon (SHB) spectrum

interaction between ○ and ○○ would be universal based on the HQET

  - Insights into **inverse mass hierarchy?**



analyzable on lattice!

theoretical prediction, Harada et al (2020)