I-Love-Q Relations:
A Nuclear Physicist's Perspective

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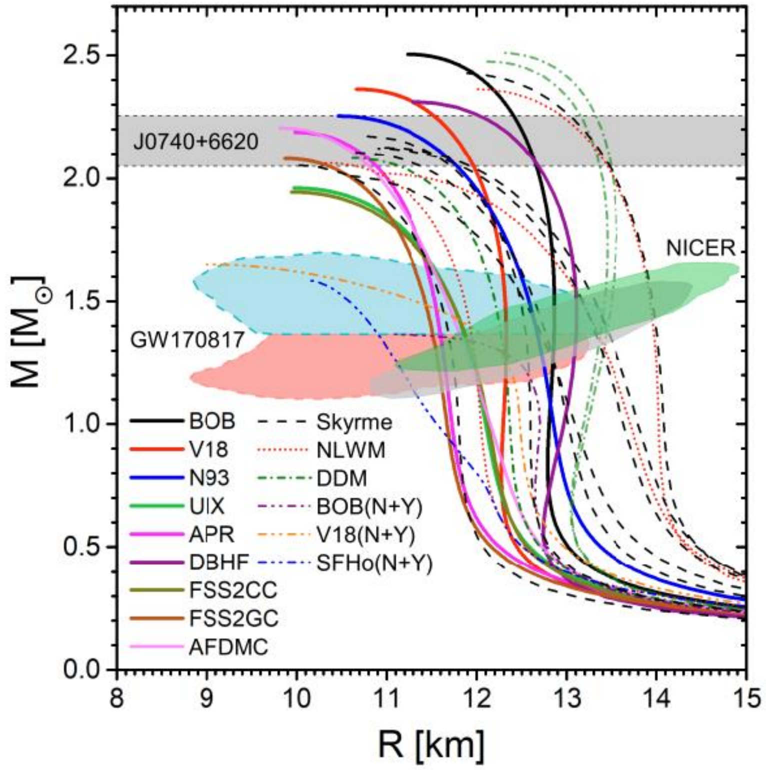
(Based on: K.Fukushima, J.Minamiguchi, S.Kamata, SM in preparation)

Taipei 2025

What is the I-Love-Q relation?

Many observables of neutron stars strongly depend on the EoS



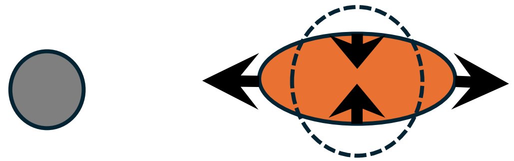
G. F. Burgio et.al., Prog. Part. Nucl. Phys. 120, 103879 (2021)

What is the I-Love-Q relation?

Surprisingly, some combinations of observables of NS (and QS) do not depend on the EoS:

Universal relation

= EoS-insensitive



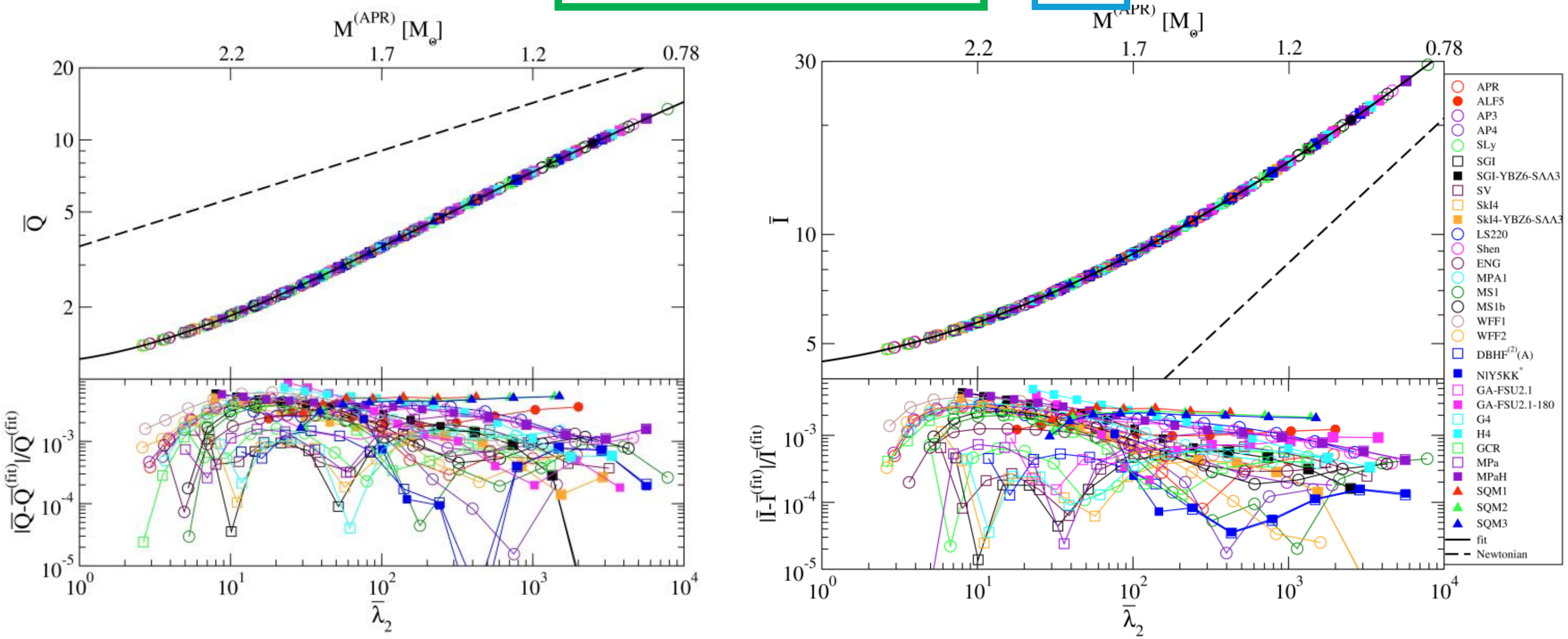
e.g., I-Love-Q relation (Yagi-Yunes 2013)

Characterize the spinning

$$\bar{I} = \frac{I}{M^3}, \bar{Q} = -\frac{Q}{M^3 S^2},$$

k_2^{tid} Characterize the tidal deformation

Highly EoS insensitive within only few percent error



Origin of I-Love-Q?

Since the I-Love-Q relations were discovered, several studies have been carried out to understand their origin:

Many numerical studies exist (e.g., Yagi+2014, Stein+2014, Chatziioannou+2014, …)

But analytical understanding remains limited

- Assuming phenomenological density profile (Sham+2015)

$$\epsilon(r) = \epsilon_0 \left(1 - \delta \frac{r^2}{R^2}\right)$$

- Simplified EoS (Yagi+2014, Chan+2015)

$$\epsilon(p) = c_0 + c_1 p + c_2 p^2 + \cdots$$

Can we analytically explain the universality using a more realistic EoS?

Model and Notation

1. Model: slow-rotation & small tidal-deformation (Hartle+1967, Hinderer 2008)

$$\frac{dp}{dc} = -\frac{(\epsilon + p)(c + 4\pi r^2 p)}{(4\pi\epsilon r^2 - c)(1-2c)},$$

$$\frac{dr}{dc} = \frac{r}{4\pi\epsilon r^2 - c},$$

$$\frac{d\bar{N}}{dc} = -2\frac{4\pi r^2 p + c}{(4\pi\epsilon r^2 - c)(1-2c)}\bar{N},$$

$$\frac{dH_0}{dc} = \frac{\beta r}{4\pi\epsilon r^2 - c},$$

$$\frac{d\beta}{dc} = \frac{2H_0 r}{(4\pi\epsilon r^2 - c)(1-2c)}\left[-2\pi\left(5\epsilon + 9p + \frac{\epsilon + p}{c_s^2}\right) + \frac{3}{r^2} + \frac{2(c + 4\pi r^2 p)^2}{r^2(1-2c)}\right] + \frac{2\beta}{(4\pi\epsilon r^2 - c)(1-2c)}\left[-1 + c + 2\pi r^2(\epsilon - p)\right],$$

$$\frac{d\omega_1}{dc} = \frac{\phi r}{4\pi\epsilon r^2 - c},$$

$$\frac{d\phi}{dc} = -\frac{4\{1 - \pi(\epsilon + p)r^2\}\phi - 16\pi(\epsilon + p)\omega_1 r}{(4\pi\epsilon r^2 - c)(1-2c)},$$

$$\frac{dK_2}{dc} = -\frac{dh_2}{dc} + \frac{1}{(4\pi\epsilon r^2 - c)}\left[\frac{1 - 3c - 4\pi r^2 p}{(1-2c)}h_2 + \frac{1 - c + 4\pi r^2 p}{r(1-2c)^2}\mu_2\right],$$

$$\frac{dh_2}{dc} = -\frac{1 - c + 4\pi r^2 p}{(1-2c)}\frac{dK_2}{dc} + \frac{1}{(4\pi\epsilon r^2 - c)}\left[\frac{r^4}{12}\bar{N}\phi^2 + \frac{(3h_2 + 2k_2) - 4\pi r^2(\epsilon + p)(h_2 + \frac{r^2}{3}\bar{N}\omega_1^2)}{(1-2c)} + \frac{1 + 8\pi r^2 p}{r(1-2c)^2}\mu_2\right],$$

with Piecewise polytropic EoS

$$\epsilon(p) = K_N p^{\gamma_N}, \quad p_N \le p < p_{N+1},$$

2. Collective notation

$$\mathcal{O} = \{p, r, \bar{N}, H_0, \beta, \omega_1, \phi, K_2, h_2\}$$

3. Use $c = \dfrac{m(r)}{r}$ as an expansion parameter (instead of r)

Surface of star: $c = C < 0.5$

C: compactness of star

Analytical Strategy

Let $\mathcal{U} = \{\bar{I}, \bar{Q}, k_2^{tid}\}$: the I-Love-Q quantities

1. Specify the general form of the series solution $\mathcal{O}(c)$ satisfying the initial and boundary conditions

$\bar{\Sigma}_p^{(N)}, \bar{\Sigma}_r^{(N)}$: some integration constants not determined by BC and IC

IC, BC

e.g., $$\mathcal{O}(c) \sim \sum_{n,k,\ell} \boxed{\mathcal{O}_{nk\ell}}\, c^n \left(\bar{\Sigma}_p^{(N)}\right)^k \left(\bar{\Sigma}_p^{(N)}\right)^\ell$$

Substitute → $$\frac{d\mathcal{O}}{dc} = F(\mathcal{O}; c)$$

(LHS)$\sim \sum_{n,k,\ell} \mathcal{O}_{nk\ell}\, c^n \left(\bar{\Sigma}_p^{(N)}\right)^k \left(\bar{\Sigma}_r^{(N)}\right)^\ell$ (RHS)$\sim \sum_{n,k,\ell} \mathcal{O}_{nk\ell}\, c^n \left(\bar{\Sigma}_p^{(N)}\right)^k \left(\bar{\Sigma}_r^{(N)}\right)^\ell$

Expanded form must be consistent

→ **2.** Evaluate at the surface $\mathcal{O}(C) = \mathcal{O}(c = C)$

$$\mathcal{O}(C) \sim \sum_{n,k,\ell} \mathcal{O}_{nk\ell}\, C^n \left(\bar{\Sigma}_p^{(N)}\right)^k \left(\bar{\Sigma}_r^{(N)}\right)^\ell$$

e.g., $$I = \frac{r(C)^4 \frac{d\omega_1}{dr}(C)}{2\left[r(C)\frac{d\omega_1}{dr}(C) + 3\omega_1(C)\right]}$$

↓

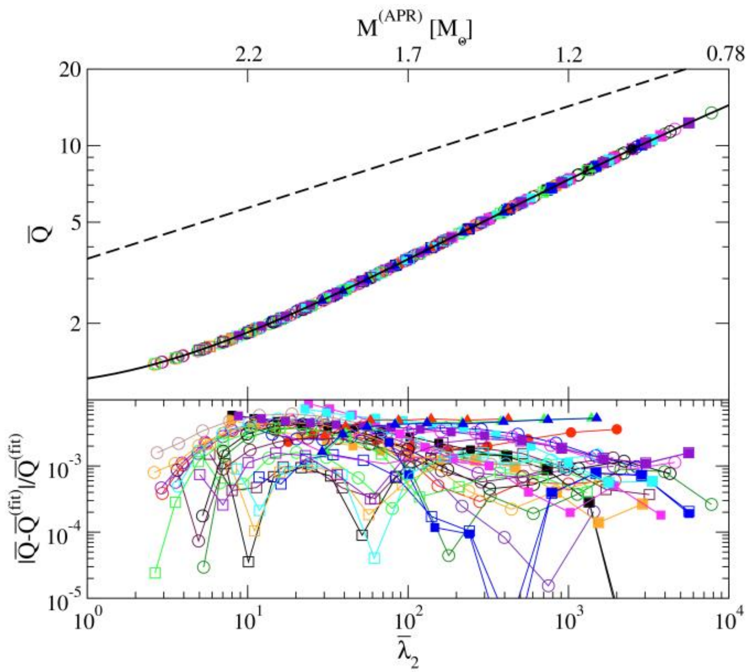
3. From $\mathcal{O}(C)$, one can express $\mathcal{U}$ as the C-expanded form

$$\mathcal{U}(C) \sim \sum_{n,k,\ell} \mathcal{U}_{nk\ell}\, C^n \left(\bar{\Sigma}_p^{(N)}\right)^k \left(\bar{\Sigma}_p^{(N)}\right)^\ell$$

↓

4. From $\mathcal{U}_1(C)$ *and* $\mathcal{U}_2(C)$, eliminate C to obtain the I-Love-Q expansion $\mathcal{U}_1(\mathcal{U}_2)$

I-Love-Q expansion



Yagi-Yunes 2013

Consistent

For small $\bar{\Sigma}_p, \bar{\Sigma}_r$

$$\log \mathcal{U}_1(\mathcal{U}_2; \bar{\Sigma}_p^{(N)}, \bar{\Sigma}_r^{(N)}) \sim \frac{\delta_{\mathcal{U}_1}}{\delta_{\mathcal{U}_2}} \log \mathcal{U}_2 + \sum_{k,n_1,n_2 \in \mathbb{N}_0} \mathcal{M}^{\log}_{(k,n_1,n_2)}(\gamma_N)(\bar{\Sigma}_p^{(N)})^{n_1}(\bar{\Sigma}_r^{(N)})^{n_2}\mathcal{U}_2^{\frac{1}{\delta_{\mathcal{U}_2}}(k-n_1-\frac{3}{2}n_2)}$$

Inconsistent

For Large $\bar{\Sigma}_p, \bar{\Sigma}_r$

$$\log \mathcal{U}_1(\mathcal{U}_2; \bar{\Sigma}_p^{(N)}, \bar{\Sigma}_r^{(N)}) \sim \log \bar{\Sigma}_p^{(N)} + \log \mathcal{M}^{\log}_{(0,0,0)}(\gamma_N) + \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{N}_0} \sum_{\substack{k \in \mathbb{N}_0 \\ |n_1|+k>0}} \mathcal{M}^{\log}_{(k,n_1,n_2)}(\gamma_N)(\bar{\Sigma}_p^{(N)})^{n_1}(\bar{\Sigma}_r^{(N)})^{n_2}\widetilde{\mathcal{U}}_2^{k+|n_1|-\frac{3}{2}n_2}.,$$

$$\log \mathcal{U}_1(\mathcal{U}_2; \bar{\Sigma}_p^{(N)}, \bar{\Sigma}_r^{(N)}) \sim \frac{2}{3} \log \bar{\Sigma}_r^{(N)} + \sum_{n_1 \in \mathbb{N}_0} \sum_{n_2 \in \mathbb{Z}} \sum_{\substack{k \in \mathbb{N}_0 \\ k-n_1-n_2 \geq 0}} \mathcal{M}^{\log}_{(k;n_1,n_2)}(\gamma_N)(\bar{\Sigma}_p^{(N)})^{n_1}(\bar{\Sigma}_r^{(N)})^{\frac{2}{3}n_2}\widehat{\mathcal{U}}_2^{k-n_1-n_2}$$

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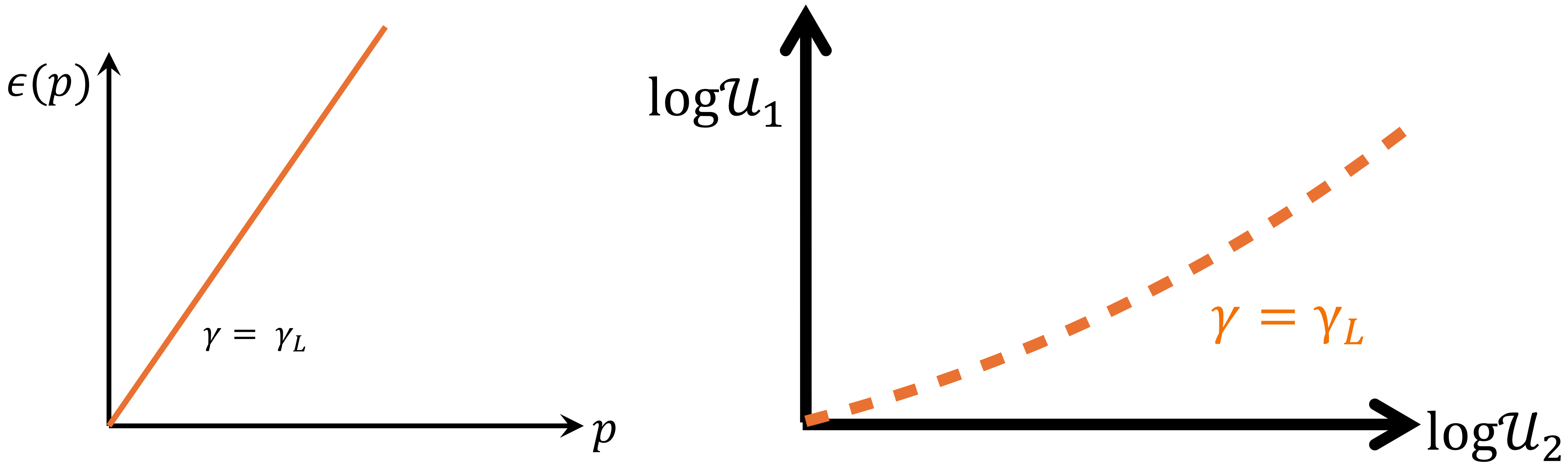
Why universal? 1. Memory Loss

From the expansion for small $\bar{\Sigma}_p, \bar{\Sigma}_r$:



The I-Love-Q relations transition between the results obtained for individual single-polytropic EoS

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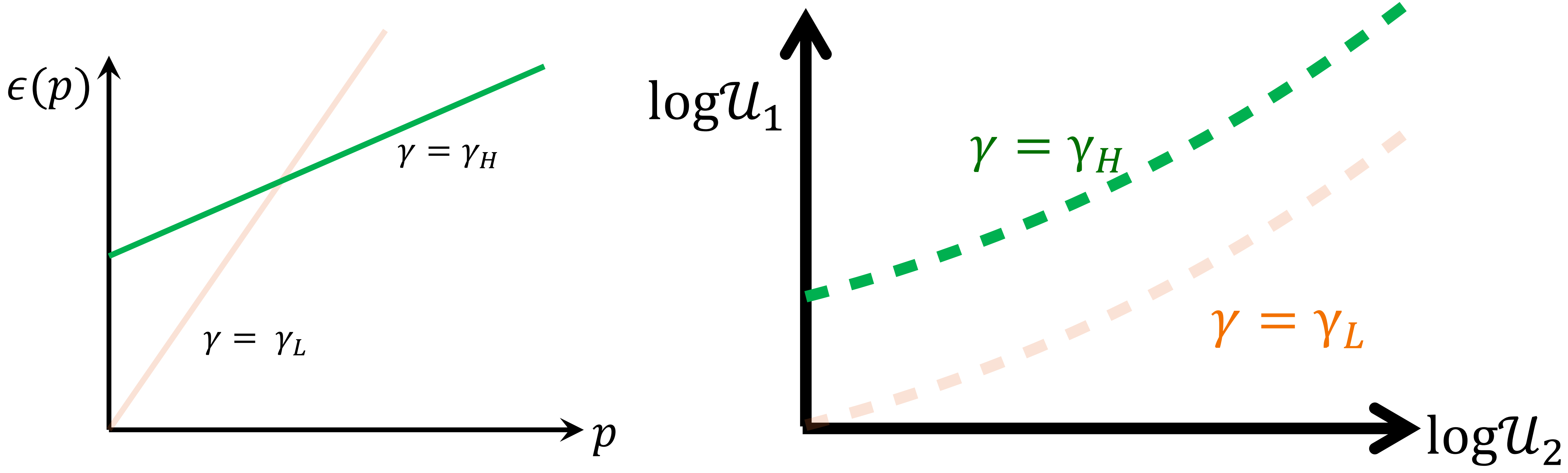

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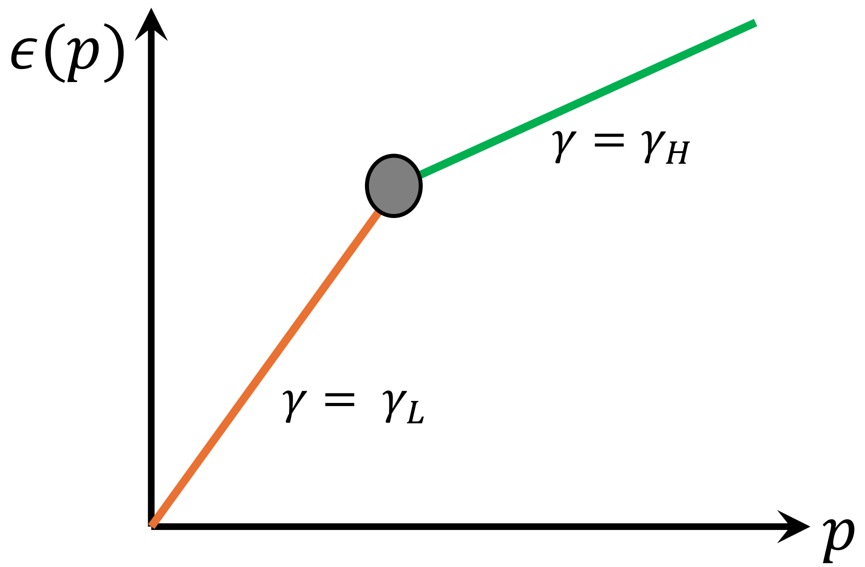
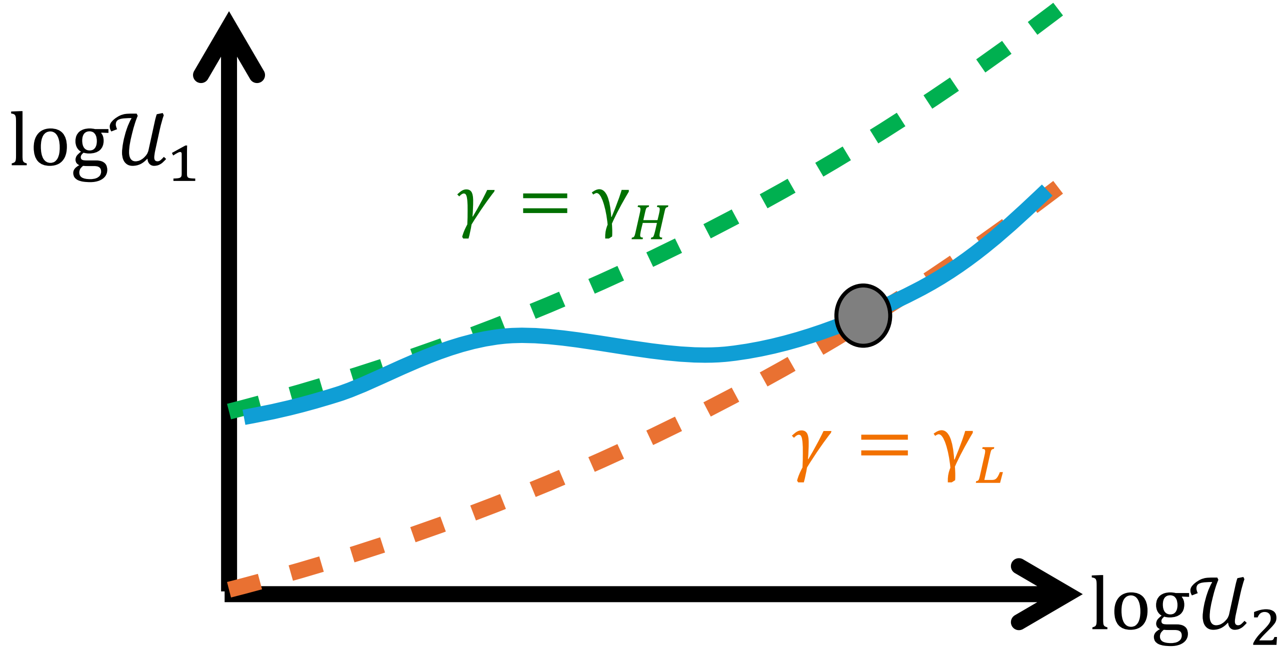


Memory loss



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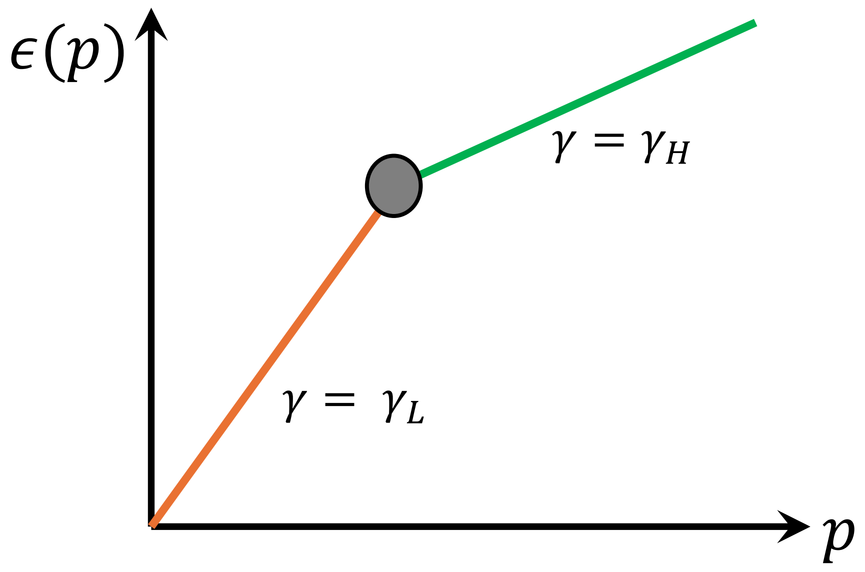

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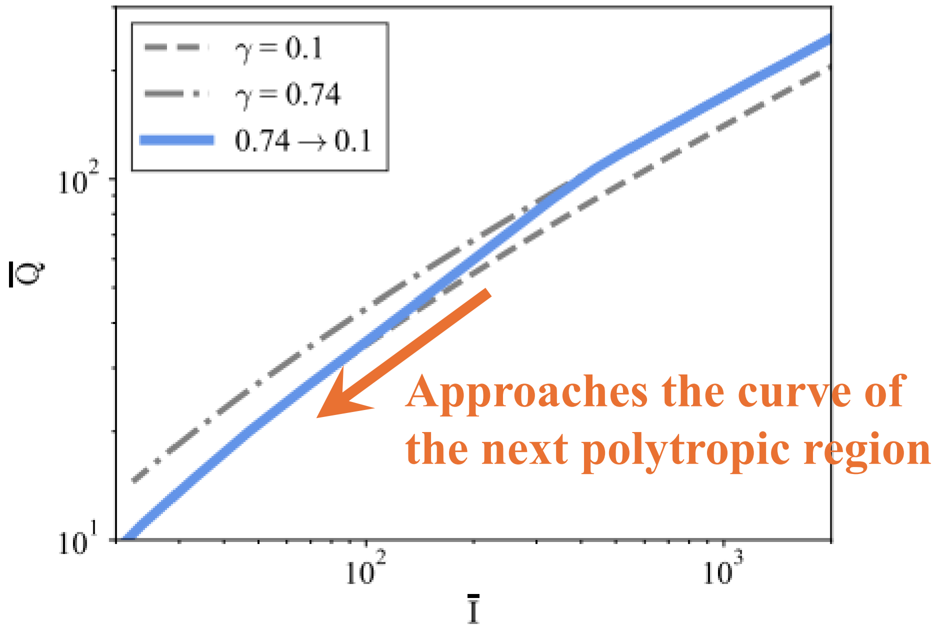


The I-Love-Q relations transition between the results obtained for individual single-polytropic EoS

Why universal? 2. Newtonian analysis

Variations in γ control the residual EoS dependence

$$\log \mathcal{U}_1(\mathcal{U}_2) \sim \underbrace{\frac{\delta_{\mathcal{U}_1}}{\delta_{\mathcal{U}_2}} \log \mathcal{U}_2 + \mathcal{M}^{\log}_{(0)}(\gamma)}_{\text{Newtonian}}$$

Up to this constant term, the expansion is completely fixed by the Newtonian limit.

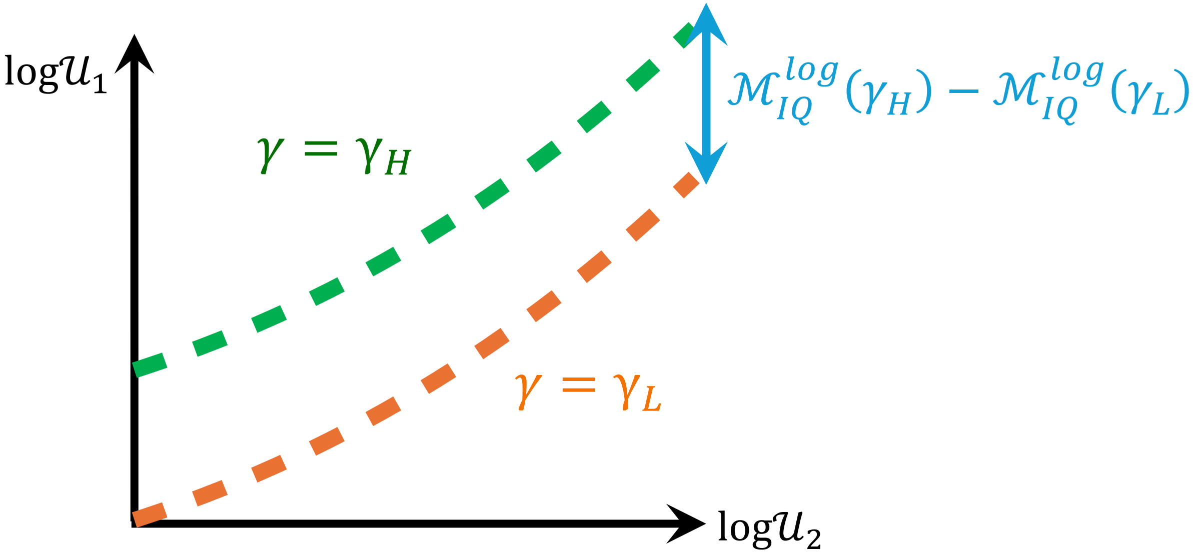
e.g., I-Q relation:

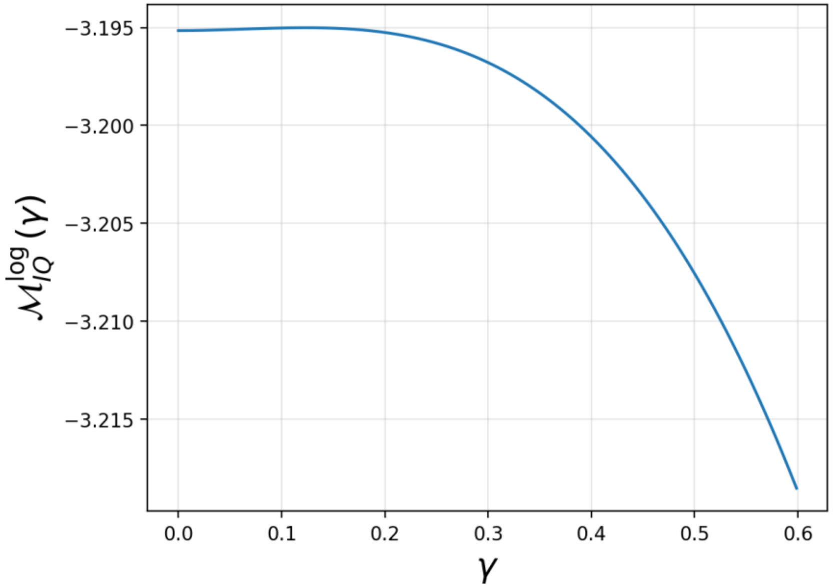
$$\mathcal{M}_{IQ}^{log}(\gamma) = -3.195 \times (1 - 0.0091\gamma^2 + 0.049\gamma^3 + O(\gamma^4))$$

For typical EoS, $\gamma < 0.6$: Error is at most about 0.6%

$$\frac{\mathcal{M}_{IQ}^{log}(\gamma_H) - \mathcal{M}_{IQ}^{log}(\gamma_L)}{\mathcal{M}_{IQ}^{log}(\gamma_L)} < 0.006$$

γ-dependence is very weak, i.e. universality





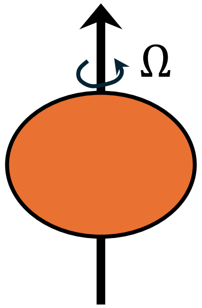
Summary

- Analytical verification of universal relations
- Working assumptions:
  slow rotation, weak tidal deformation, and a piecewise-polytropic EoS
- Origin of universality
  - Memory loss: The I-Love-Q relations transition between the results obtained for individual single-polytropic EoS
  - Small variation induced by changing the single-polytrope EoS

Backup: Physical Quantities

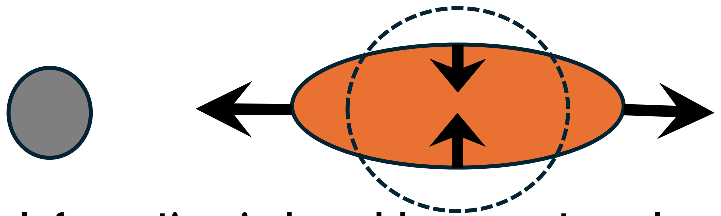
Measure of Spinning:

- Moment of Inertia I
- (Spin) Quadrupole moment Q



Linear Response to an external gravitational field:

- Tidal Deformability



Characterize the deformation induced by an external gravitational field

$$Q_{ij} = -\lambda E_{ij}$$

(Quadrupole moment) = − (Tidal Deformability) × (External field)

- (Electric) Tidal Love number

$$k_2^{tid} \sim \lambda \times C^5 \times \frac{1}{M^5}$$

$$C = M/R$$

Backup: Construction of $\mathcal{O}(c)$

- In each polytropic region, we can find the general solution

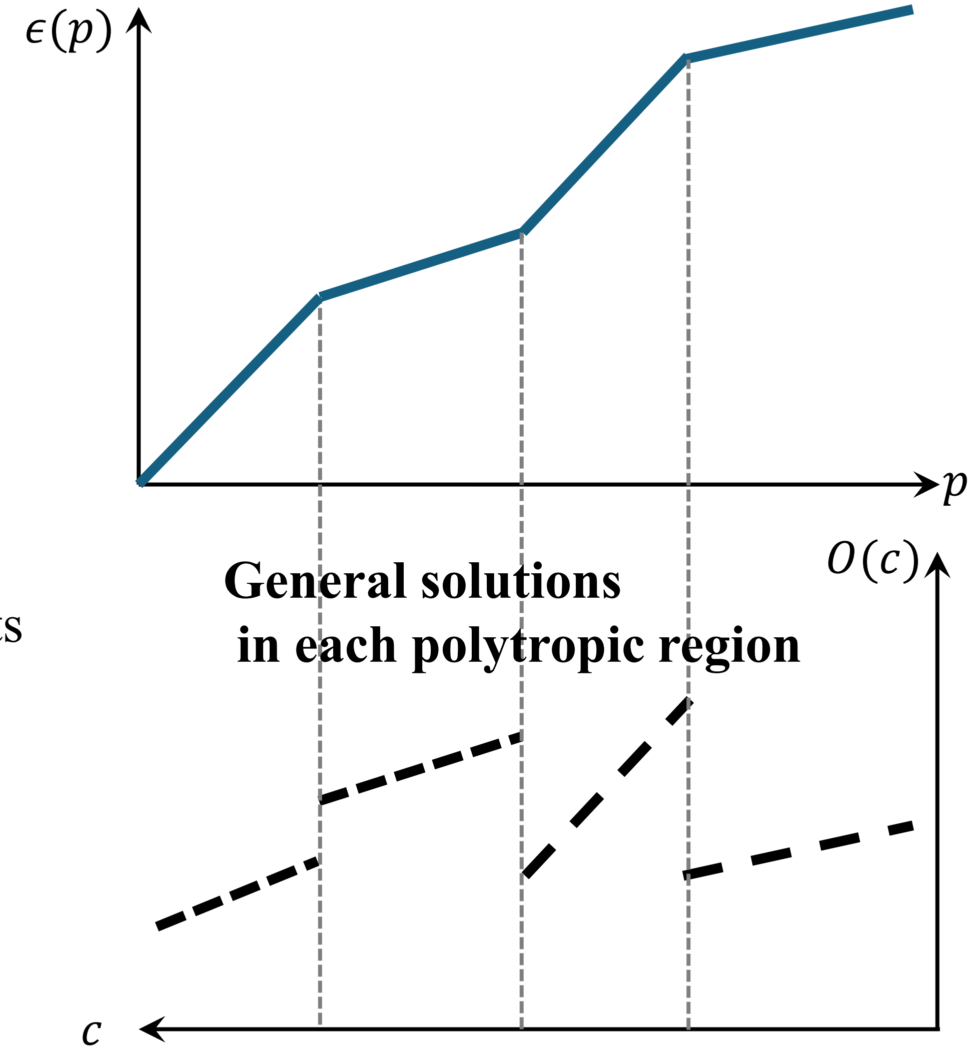
$$\frac{d\mathcal{O}}{dc} = F(\mathcal{O}; c) \qquad \epsilon(p) = K_N p^{\gamma_N}$$

$$\downarrow$$

$$\mathcal{O}(c) \sim c^a \sum_{n=0}^{\infty} \mathcal{O}_n\, c^n \quad (p_N \leq p(c) < p_{N+1})$$

"a"= determined by the initial condition

- General solutions are discontinuous at the transition points



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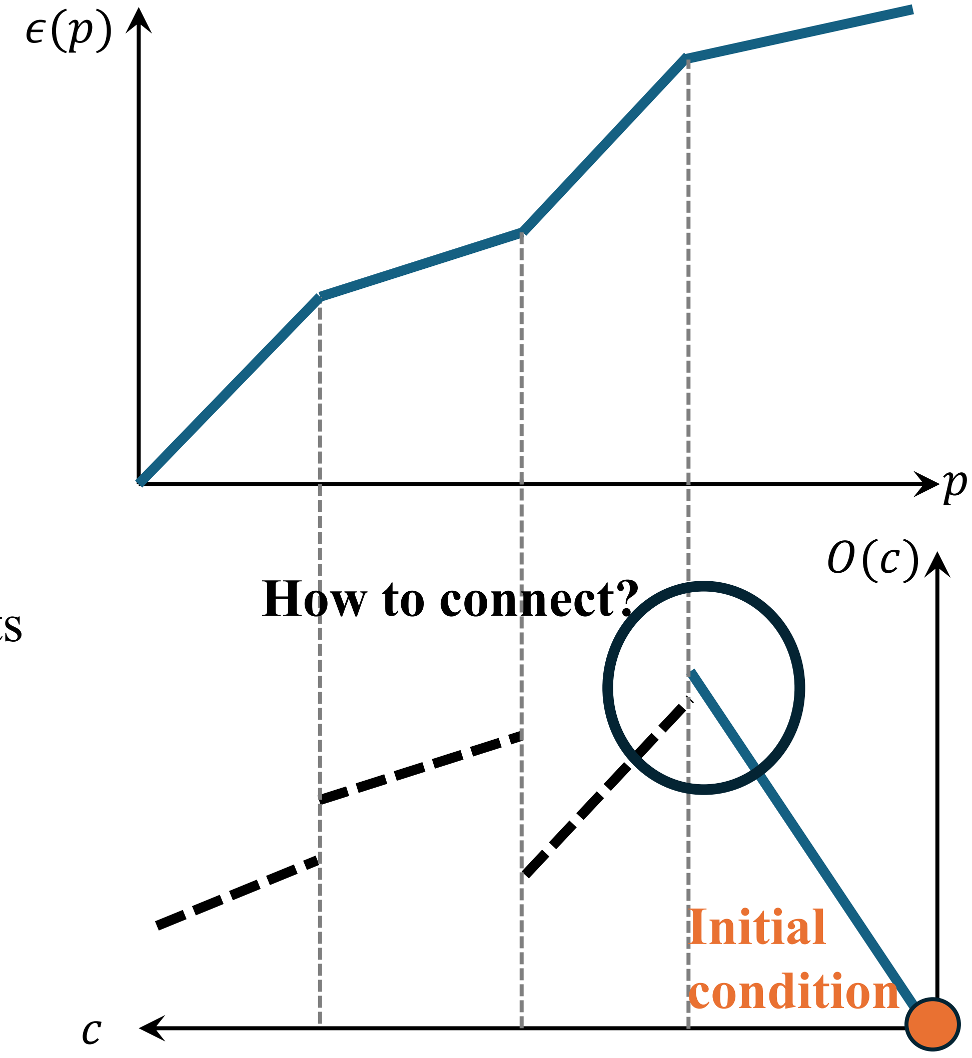

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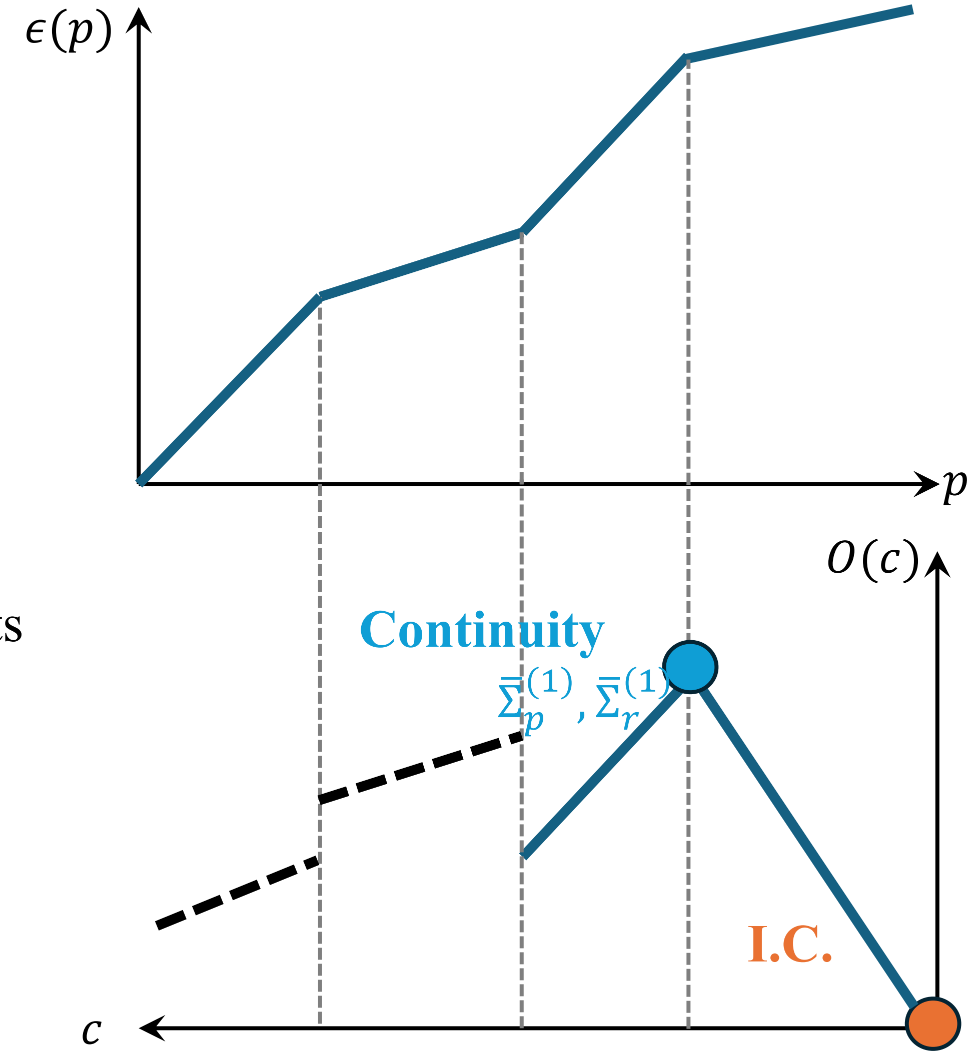
- We find the integration constants for p and r ($\bar{\Sigma}_p, \bar{\Sigma}_r$) can be used to connect these solutions.

$\longleftrightarrow$ Other integration constants are fixed by the boundary conditions



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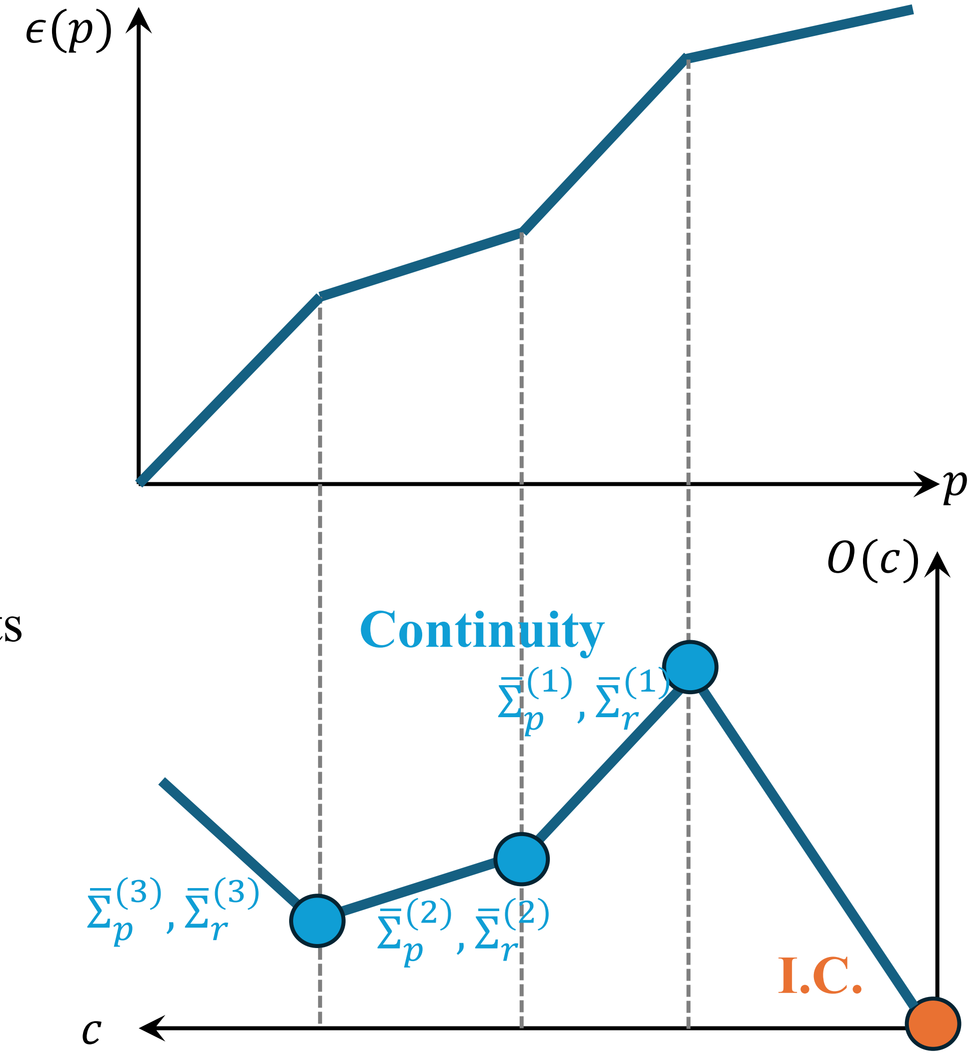

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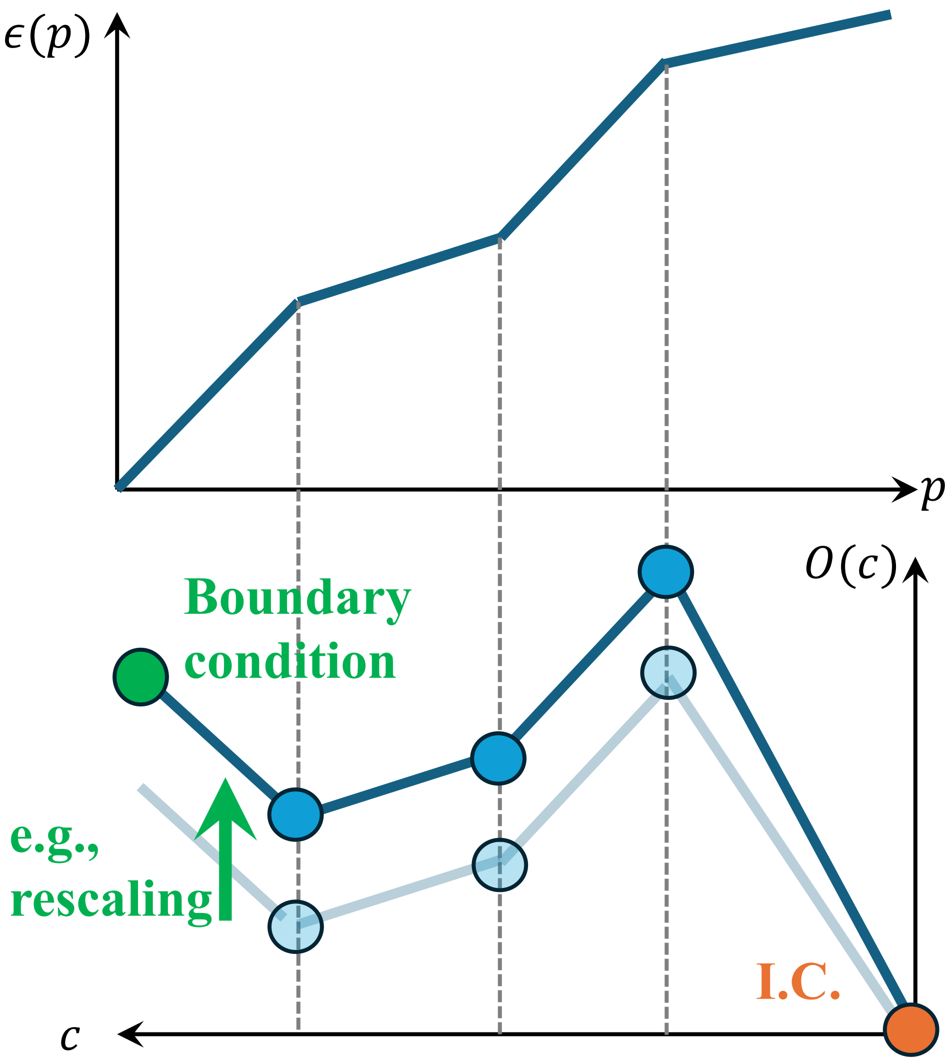

Backup: Role of other expansions

Rapid changes in EoS breaks the universality

$$\log\mathcal{U}_1(\mathcal{U}_2;\bar{\Sigma}_p^{(N)},\bar{\Sigma}_r^{(N)}) \sim \log\bar{\Sigma}_p^{(N)} + \log\mathcal{M}^{\log}_{(0,0,0)}(\gamma_N) + \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{N}_0}\sum_{\substack{k\in\mathbb{N}_0\\|n_1|+k>0}} \mathcal{M}^{\log}_{(k,n_1,n_2)}(\gamma_N)(\bar{\Sigma}_p^{(N)})^{n_1}(\bar{\Sigma}_r^{(N)})^{n_2}\widetilde{\mathcal{U}}_2^{k+|n_1|-\frac{3}{2}n_2}.,$$

$$\log\mathcal{U}_1(\mathcal{U}_2;\bar{\Sigma}_p^{(N)},\bar{\Sigma}_r^{(N)}) \sim \frac{2}{3}\log\bar{\Sigma}_r^{(N)} + \sum_{n_1\in\mathbb{N}_0}\sum_{n_2\in\mathbb{Z}}\sum_{\substack{k\in\mathbb{N}_0\\k-n_1-n_2\geq 0}} \mathcal{M}^{\log}_{(k;n_1,n_2)}(\gamma_N)(\bar{\Sigma}_p^{(N)})^{n_1}(\bar{\Sigma}_r^{(N)})^{\frac{2}{3}n_2}\widehat{\mathcal{U}}_2^{k-n_1-n_2}$$

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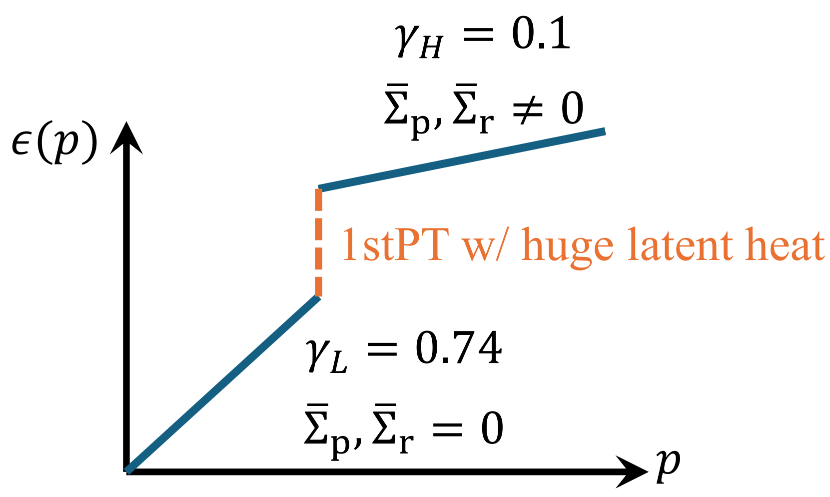
- When $\bar{\Sigma}_\mathrm{p}, \bar{\Sigma}_\mathrm{r}$ is too large, the form of expansion is modified, and log-log structure of expansion disappears.



$$\epsilon(p) = K_N p^{\gamma_N}$$