

The spectral reconstruction problem for thermal dilepton and photon production rates from lattice QCD

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Workshop on recent developments from QCD to nuclear matter

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partly based on: [2403.11647]

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The inverse problem challenge - a mismatch in information

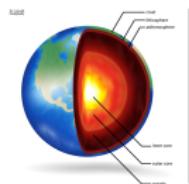
At this conference many talks addressing different physics, but facing a similar core issue: An inverse problem.

In fact, inverse problems are common across science, in general:

- A mismatch between available and desired information
- Need of a robust map / transformation from one into the other
- Inverse problem: This transformation is (numerically) ill-posed or ill-conditioned

Examples outside and inside of (thermal) particle physics:

available information	desired information
seismic waves	geological tomogram
electromagnetic waves	medical images
Thermal meson correlation functions	dilepton/photon production rate electrical conductivity



Electromagnetic source imaging: Backus–Gilbert resolution spread function-constrained and functional MRI-guided spatial filtering

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Particle physics observables - inverse problems and lattice data

The following can be extracted from a **Euclidean expectation value** by solving an **inverse problem** at $T = 0$:

- $N \rightarrow N'$ scattering amplitude at any $s = E_{\text{cm}}^2$ ($\pi\pi \rightarrow \pi\pi$, $N\pi \rightarrow N\pi\pi, \dots$)
- $N + j \rightarrow N'$ transitions at any s ($K \rightarrow \pi\pi$, $D \rightarrow \pi\pi, K\bar{K}$, $\gamma \rightarrow \pi\pi, \dots$)
- Non-local matrix elements (R -ratio, hadronic tensor, $D\bar{D}$ mixing)
 - ~~~ see e.g. [Liu '94], [Hansen, Meyer, Robaina '17], [Bulava, Hansen '19], [ETMC '23]
- Inclusive processes (V_{us} , deep inelastic scattering, heavy semileptonic decays)
 - ~~~ see e.g. [QCDSF '17], [Gambino et al '20], [Alexandrou et al '24]
- Parton distribution functions (PDFs, GPDs, TMDs)
 - ~~~ see e.g. [Ji '13], [Ji et al '23], [HadStruc '24]

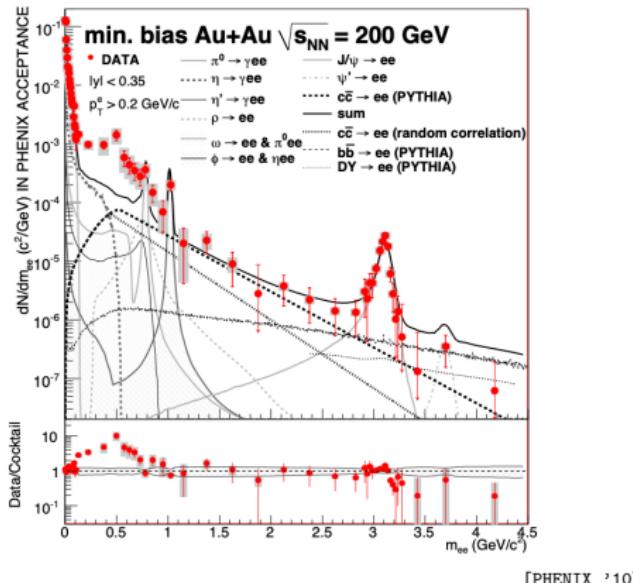
At finite temperature solving an inverse problem of this type gives access to transport coefficients, production rates and thermal broadening:

- Bulk and shear viscosities
 - ~~~ see e.g. [Meyer '07], [Altenkort, AF et al '23]
- Electrical conductivity and heavy quark diffusion
 - ~~~ see e.g. [Kaczmarek, Shu '22], [Brandt, AF et al '16]
- **Dilepton and photon production rates**
 - ~~~ see e.g. [Cè et al '20], [Ali, AF et al '24]

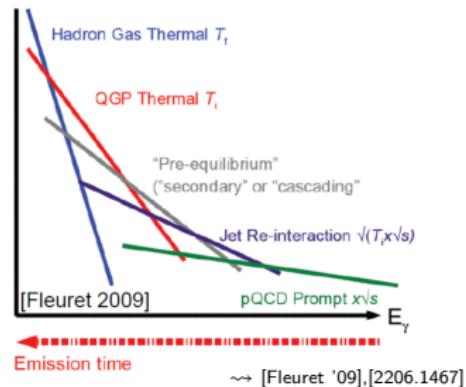
Dilepton and photon production rates - identifying the problem

Probes of the quark gluon plasma

- Dileptons and photons are produced at all stages of Heavy-Ion Collisions.
- Weak coupling to QGP constituents
 - they decouple after production
 - probes of the full thermal medium evolution



Sketch of different photon sources



Thermal QCD goals

- thermal dileptons: understand contribution for $m_{ee} \sim 0.5$ GeV
- thermal photons: understand dominant contribution for $p_T \in [1 : 2]$ GeV

Dilepton and photon production rates - identifying the problem

The photon emissivity is related to thermal vector-vector current spectral functions

$$\rho^{\mu\nu}(\mathcal{K}) = \int d^4x e^{i\mathcal{K}\cdot x} \frac{1}{Z} \sum_n e^{-E_n/T} \langle n | [j^\mu(x), j^\nu(0)] | n \rangle$$

- Rate of dilepton production per unit volume plasma: \rightsquigarrow [McLerran, Toimela '85]

$$d\Gamma_{\ell^+\ell^-}(\mathcal{K}) = \alpha^2 \frac{d^4\mathcal{K}}{6\pi^3\mathcal{K}^2} \frac{-\rho^{\mu\mu}(\mathcal{K})}{e^{\beta\mathcal{K}^0} - 1} \quad (\mathcal{K}^2 \equiv \omega^2 - \mathbf{k}^2)$$

- Rate of photon production per unit volume plasma:

$$d\Gamma_\gamma(\mathbf{k}) = \alpha \frac{d^3k}{4\pi^2k} \frac{-\rho^{\mu\mu}(\mathbf{k}, \mathbf{k})}{e^{\beta k} - 1}.$$

- Electrical conductivity of the quark gluon plasma: \rightsquigarrow see e.g. [1104.3708]

$$\sigma_{el} = e^2 \sum_{f=1}^{N_f} Q_f^2 \lim_{k \rightarrow 0^+} \frac{\rho^i_i(k, 0)}{k}$$



\rightsquigarrow from Meyer, Lattice@CERN '24

Dilepton and photon production rates - identifying the problem

We are interested in three properties of the vector spectral function:

- conductivity: $\sigma_{el} \sim \rho^i_i(0, 0)$ note: $\rho^0_0(0, 0) = \text{const.} =: \chi_q$
- dilepton rate: $d\Gamma_{\ell^+\ell^-}(\mathcal{K}) \sim -\rho^\mu_\mu(\mathcal{K}) \sim \rho^\mu_\mu(\omega, \mathbf{k} = 0)$
- photon rate: $d\Gamma_\gamma(\mathbf{k}) \sim -\rho^\mu_\mu(k, \vec{k}) \sim \rho^\mu_\mu(\omega = k, \mathbf{k} \neq 0)$

The most interesting are in the low-energy, non-perturbative regime of QCD.

~~ Where can we trust perturbative calculations?

~~ Access through lattice QCD?

On the lattice and at $T > 0$:

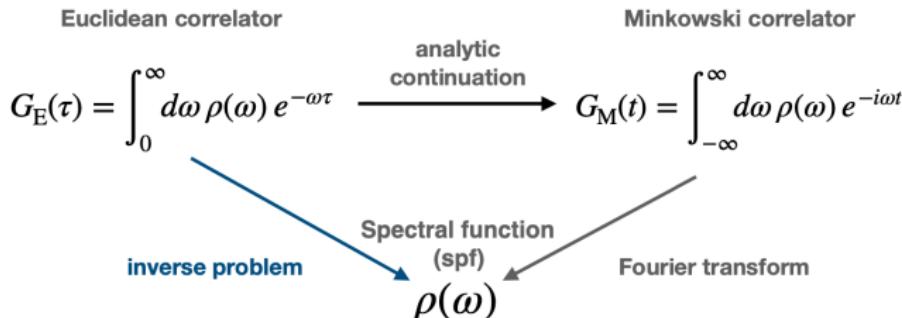
- Based on imaginary-time path-integral rep. of QFT (Matsubara formalism).
- Only **imaginary-time** vector correlators are accessible ($j^\mu = \sum_f Q_f \bar{\psi}_f \gamma^\mu \psi_f$):

$$G^{\mu\nu}(x_0, \mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle j^\mu(x) j^\nu(0) \rangle_T = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \text{Tr} \left\{ \frac{e^{-\beta H}}{Z(\beta)} j^\mu(x) j^\nu(0) \right\},$$

- Their spectral representation is: ~~ inverse problem(!)

$$G^{\mu\nu}(x_0, \mathbf{k}) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\rho^{\mu\nu}(\omega, \mathbf{k})}_{\text{}} \frac{\cosh [\omega(\beta/2 - x_0)]}{\sinh(\beta\omega/2)} \quad , \quad \beta = 1/T.$$

Inverse problem vs. analytic continuation - a net of connections



From the Euclidean correlator calculated in lattice QCD we want to extract the spf
⇒ Problem can be seen as a **simultaneous Wick rotation and Fourier transform**.

Also, formally:

$$\rho(\omega) = \mathcal{L}^{-1} \{ G_E(\tau) \} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\omega\tau} G_E(\tau) d\tau \quad \text{and} \quad \rho(\omega) = \frac{1}{\pi} \text{Im} (G_M(-\omega))$$

⇒ Problem is related to having only real data where complex information is required.

Nature of the inverse problem - setting expectations

Jacques Hadamard established three conditions for a well-posed problem

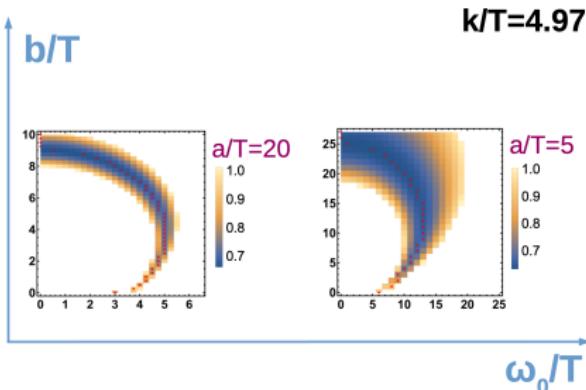
1. Existence
2. Uniqueness
3. Stability (solution changes continuously with the initial conditions)

The problems we consider fail in the sense of 3. This is due to **discrete sampling plus finite precision** and a method-independent statement.



An illustrative example: Extract $\rho(\omega)$ by motivated photon production rate fit-Ansatz with 3 parameters a, b, ω_0 , and minimize:

$$\chi_{fit} = \left[G(\tau, p) - \int d\omega \rho_{fit}(\omega, a, b, \omega_0) K(\omega, \tau, \beta) \right]$$



- o no clear global minimum visible
- o each point represents one "acceptable" spf solution describing the Euclidean data.
- o brute force accuracy increase to arrive at a more constrained result realistic?

Figure from [1710.07050]

Nature of the inverse problem - general approach

All methods to perform a spectral reconstruction can be understood as a master function

$$\mathcal{F}[\mathbf{G}, \mathbf{C}_G] = (\boldsymbol{\rho}, \mathbf{C}_{\boldsymbol{\rho}})$$

where

- o \mathbf{G} = discrete samples of $G(\tau)$
- o \mathbf{C}_G = covariance of \mathbf{G}
- o $\boldsymbol{\rho}$ = discrete estimator of $\rho(\omega)$
- o $\mathbf{C}_{\boldsymbol{\rho}}$ = covariance of $\boldsymbol{\rho}$

General difficulties:

- o For $\boldsymbol{\rho}_i = \rho(\omega_i)$ we have that $|\mathcal{F}[\mathbf{G} + \delta\mathbf{G}, \mathbf{C}_G + \delta\mathbf{C}_G] - \mathcal{F}[\mathbf{G}, \mathbf{C}_G]|$ and as a consequence $|\mathbf{C}_{\boldsymbol{\rho}}|$ explode.
- o For cases where $|\mathbf{C}_{\boldsymbol{\rho}}|$ is under control, relation between $\rho(\omega) \Leftrightarrow \boldsymbol{\rho}$ may be obscured.

Goal and path:

- We want to understand the properties and limitations of the master function \mathcal{F}
- Crucial to be data focused: The best \mathcal{F} will depend in detail on G , number of slices, properties of C , etc.

Approaching the inverse problem - three basic strategies

Strategy I: Accept the premise and focus on information in the data

Challenges

- Optimal bases for sparse modeling / correlation structures for Gaussian processes?
- Constraints (positivity, sum rules)?

Methods - (non)linear

- Sparse modeling
- Neural Networks
- Gaussian processes

Strategy II: Accept the premise and try to supply as much extra information as possible

Challenges

- Effectively encode more specific information in priors/ Ansätze?
- Control over prior bias and systematics?

Methods - (non)linear

- χ^2 -fits
- Maximum entropy methods
- Stochastic inference / optimization

Strategy III: Reject the premise and focus on smeared spf's or Euclidean quantities

Challenges

- Identify the physics observables that ...
 - can benefit from smeared spf input.
 - simplify / avoid inverse problems altogether.

Methods - linear

- Backus-Gilbert / Hansen-Lupo-Tantalo methods
- Gaussian processes

Each method has its own problem-dependent pros and cons.
Unlikely there is a single best solution.

Results

Roadmap:

1. **dilepton rates / electrical conductivity:** *brief overview of community status*
2. **photon rates:** *new research shown [2403.11647]*

Our lattice setup (HotQCD ensembles)

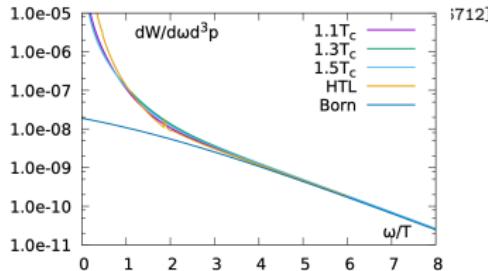
- $n_f = 0$ quenched QCD, $a^{-1} = 9.4, 11.3, 14.1$ GeV, Wilson-Clover
- $n_f = 2 + 1$ full QCD, $a^{-1} = 7.04$ GeV, Wilson-Clover on HISQ sea.
- $m_\pi = 320$ MeV, $m_s = 5 \cdot m_\ell$

Dilepton rate and electrical conductivity

- Dilepton rates available in quenched and full QCD from multiple groups (last update 2019).
- Methods used:
 - χ^2 -fits with additional constraints
 - MEM
 - BG method with Tikhonov regulator
- Fits: Transport (+ BW) + Asymptotic
 - ⇒ Data well described by Ansatz
 - ⇒ Difficult to distinguish Transport and BW
- MEM prior: Positivity + default model

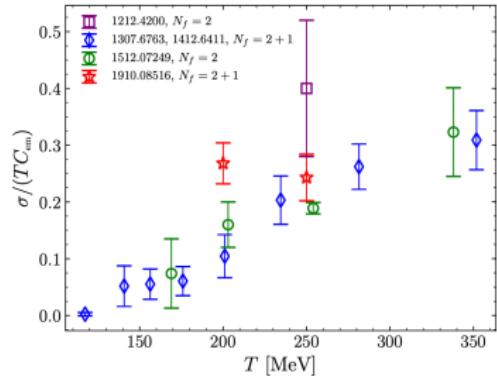
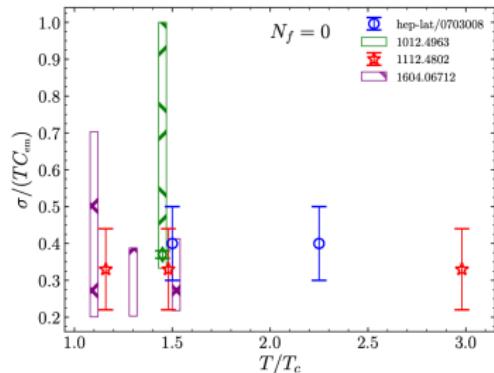
Consistent picture emerging.

Dilepton rate in quenched QCD



Electrical conductivity

[2008.12326]



Photon production rate from the lattice - a better estimator

↔ see [Cè et al '20], [Ali, AF et al '24], [Meyer, Lattice@CERN '24]

Back to:

$$\rho^{\mu\nu}(\mathcal{K}) = \int d^4x e^{i\mathcal{K}\cdot x} \frac{1}{Z} \sum_n e^{-E_n/T} \langle n | [j^\mu(x), j^\nu(0)] | n \rangle$$

at $T > 0$, there are two independent components (longitudinal and transverse):

$$\rho_L(\omega, k) \equiv \left(\hat{k}^i \hat{k}^j \rho^{ij} - \rho^{00} \right), \quad \text{and} \quad \rho_T(\omega, k) \equiv \frac{1}{2} \left(\delta^{ij} - \hat{k}^i \hat{k}^j \right) \rho^{ij},$$

where $(k \equiv |\mathbf{k}|, \hat{k}^i = k^i/k)$

- Due to current conservation: $\omega^2 \rho^{00}(\omega, k) = k^i k^j \rho^{ij}(\omega, k)$
- ρ_L vanishes at light like kinematics, $\mathcal{K}^2 = 0$

→ We can rewrite the photon rate when introducing the estimator:

$$\rho(\omega, k, \lambda) = 2\rho_T + \lambda \rho_L \stackrel{\lambda=1}{=} -\rho^\mu_\mu$$

→ For any value of λ we have

$$d\Gamma_\gamma(\mathbf{k}) = \alpha \frac{d^3k}{4\pi^2 k} \frac{\rho(k, k, \lambda)}{e^{\beta k} - 1}$$

→ Choose λ such that the reconstruction becomes particularly easy.

Lattice photon production rate - estimator and perturbative result

↔ [Ali, AF et al '24]

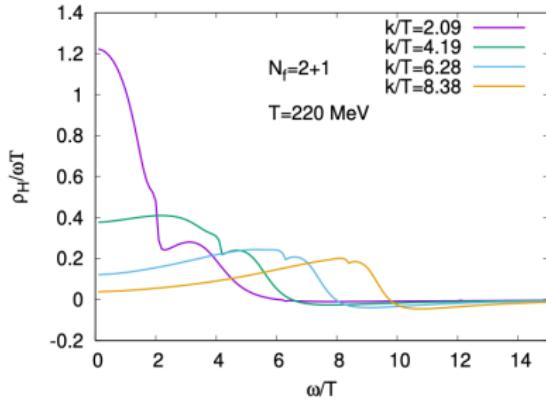
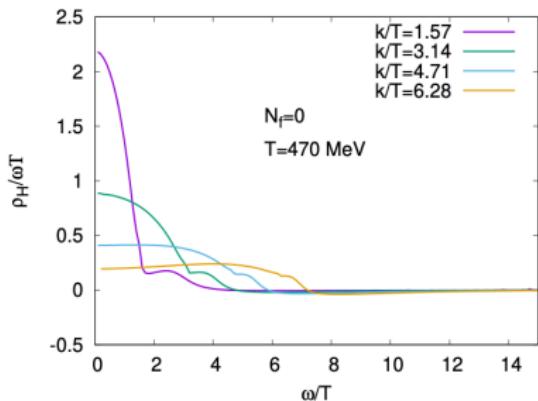
We choose $\lambda = 2$ such that the spf we want to reconstruct becomes:

$$\rho_H(\omega, \vec{k}) = 2 \left\{ \rho_T(\omega, \vec{k}) - \rho_L(\omega, \vec{k}) \right\}$$

- At $T = 0$ this estimator is $\rho_H(\omega, \vec{k}) = 0$. (due to restoration of Lorentz symmetry)
- Purely thermal effects contribute.
- Asymptotically $\rho_H(\omega, \vec{k}) \sim 1/\omega^4$ for $\omega \gg k (\pi T)$.
- Sum rule $\int_0^\infty d\omega \omega \rho_H(\omega) = 0$ is a possible extra constraint.

Can be worked out in perturbation theory at NLO + LPM^{LO}.

↔ [1910.09567]

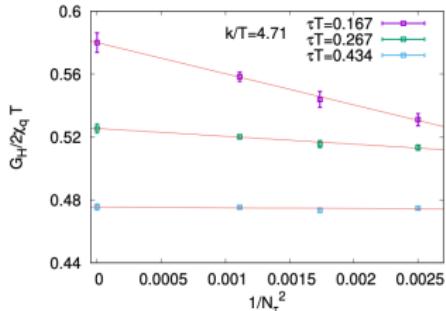


LPM^{LO} means that near the light cone LPM resummation is performed at leading order.

Euclidean correlators - NLO+LPM vs. lattice data

Quenched results

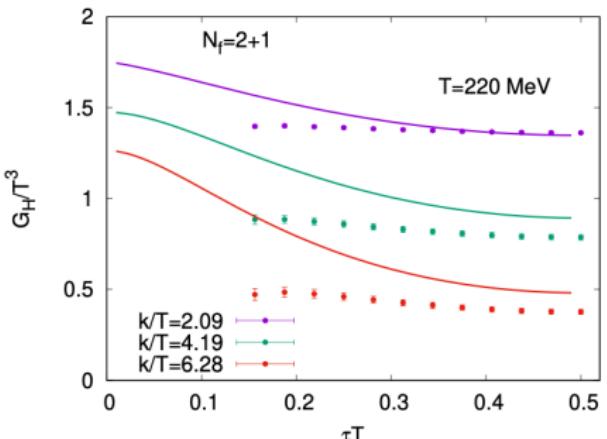
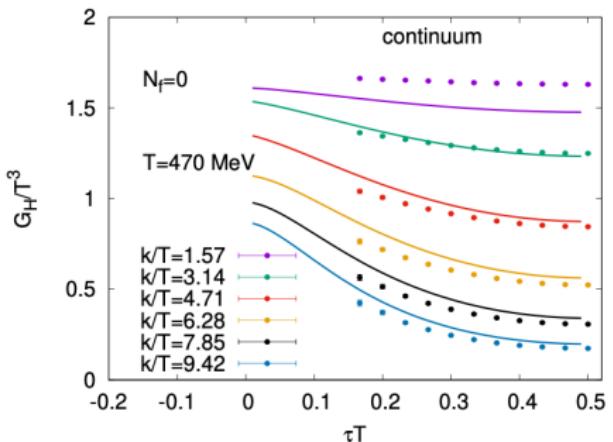
- Continuum limit with $N_\tau = 20, 24, 30$
- non-pert. tuned WCF



- General behavior reproduced, apart from lowest k/T

Full QCD results

- Single lattice spacing
- Differences more visible
- Improvement after continuum limit?



Spectral reconstruction - a 3-pronged approach

Strategy

We employ all three basic strategies:

- χ^2 -fits (strategy II)
- Backus-Gilbert method (strategy III)
- Gaussian processes (strategy I)

This makes visible and enables the study of:

- ⇒ Different systematics in all approaches.
- ⇒ Maximal view of possible outcomes.
- ⇒ Aim for robust, conservative final result.

Gaussian process regression

- Gaussian kernel (related to NN)
- Simultaneous reconstruction in (ω, k)
- Continuity only constraint

→ Use mildest possible constraints

χ^2 -fits

Two sets of Ansätze

- (a) Polynomial fit to reproduce IR and UV results (OPE)
- (b) Padé fit with sum rule incorporated (OPE and AdS/CFT)

→ Include as much extra info as possible

BGM

We stabilize/improve the reconstruction by rescaling the spf by asymptotic behavior

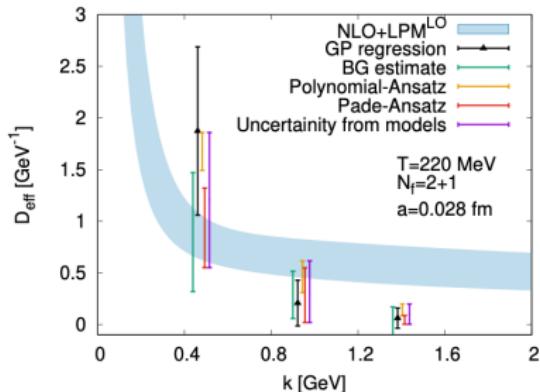
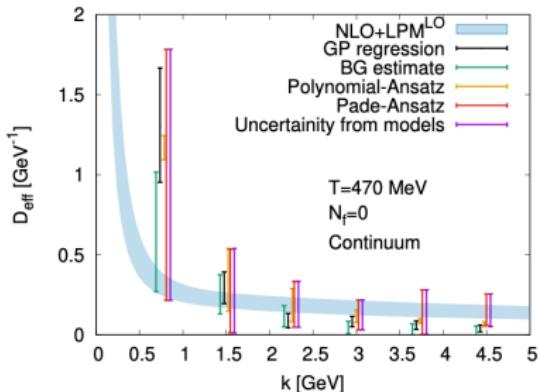
$$\frac{\rho_H^{\text{BG}}(\omega, \vec{k})}{f(\omega, \vec{k})} = \sum_i q_i(\omega, \vec{k}) G_H(\tau_i, \vec{k})$$

Rescaling function:

$$f(\omega, \vec{k}) = \left(\frac{\omega_0}{\omega}\right)^4 \tanh\left(\frac{\omega}{\omega_0}\right)^5$$

→ Work with a smeared spf

Photon rate from the lattice - final results



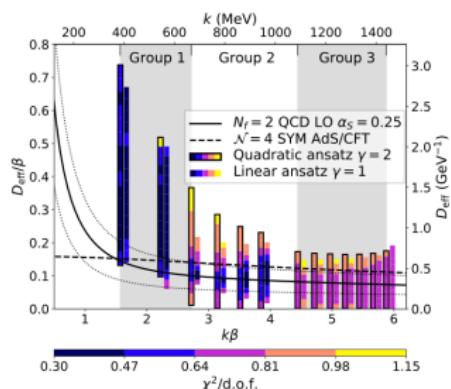
Cé et al [2205.02821]

Plotted:

$$D_{\text{eff}}(k) \equiv \frac{\rho_H(\omega = k, k)}{2\chi_q k}$$

Connection to full photon rate:

$$\begin{aligned} \frac{d\Gamma_\gamma(k)}{d^3k} &= \frac{\alpha}{4\pi^2 k} \frac{\rho(k, k, \lambda = 2)}{e^{\beta k} - 1} \\ &= \frac{\alpha n_b(k) \chi_q}{\pi^2} \left(\sum_{i=1}^{N_f} Q_i^2 \right) D_{\text{eff}}(k) \end{aligned}$$



compatible results, see further material

Summary - inverse problem for thermal spfs from the lattice

I. Inverse problem for thermal spectral functions

- Presented in general terms the inverse problem to obtain (thermal) spectral functions from lattice correlators
- Broadly highlighted applications and connection to dilepton / photon rates
- Gave general approach strategies and their challenges

II. Dilepton rates and electrical conductivity

- Collected a brief overview (last update 2019). Consistent picture emerging.

III. Photon rate

- Showed the improved estimator for photon rate
- Applied all basic strategies to inverse problem to arrive at new result

IV. Future questions

- Finding improved estimator has made the study of the photon rate (and heavy quark diffusion) tractable. *Are there more observables like this?*
- In the $T = 0$ community there is a lot of work on using smeared spf's and clarifying the role of the finite volume. *Possibilities also at $T > 0$?*
- Gaussian processes related to NN. Connections between BG-type methods worked out. *New methods that combine the best of all strategies possible?*

Thank you for your attention.



Further material

The role of the kernel - different physics, different problems

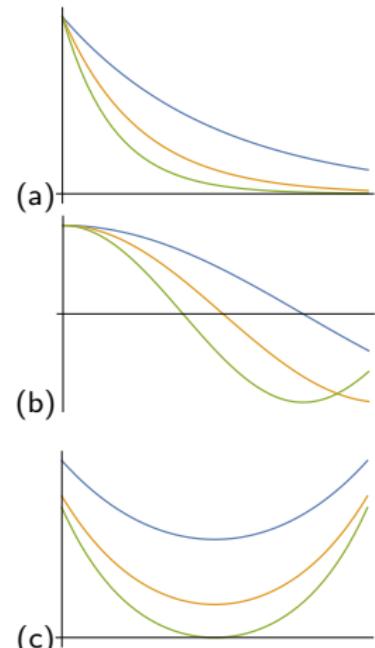
The specific problem sets the kernel entering the inverse problem

$$G_E(\tau) = \int_0^\infty d\omega \rho(\omega) \kappa(\omega, \tau)$$

- (a) Zero-temperature quantities: $\kappa(\omega, \tau) = e^{-\omega\tau}$
~~ Need to perform the inverse Laplace transform.
- (b) qPDFs: $\kappa(\nu, x) = \cos(\nu x) \Theta(1 - x)$
~~ Need to perform Fourier transform.
- (c) Nonzero-temperature: $\kappa(\omega, \tau) = \frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\omega\beta/2)}$

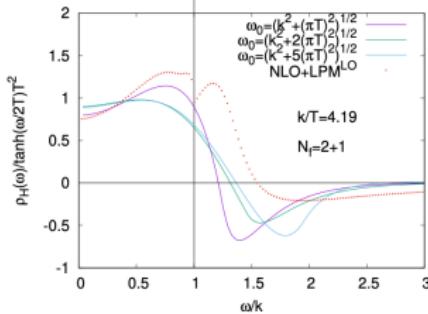
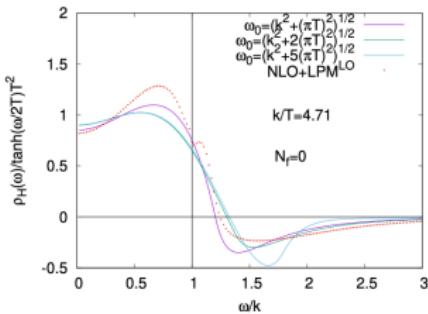
Focusing on the $T > 0$ case and target spectral information at low energies:

- o Laplace transform in limit $\lim_{T \rightarrow 0} \kappa(\omega, \tau) = e^{-\omega\tau}$
- o Often $T = 1/N_\tau$ short in lattice calculations
~~ could indicate a benefit of anisotropic calculations
- o low- ω contributions suppressed at short τ
- o low- ω contributions compete with kernel T -effects at $\tau = N_\tau/2$

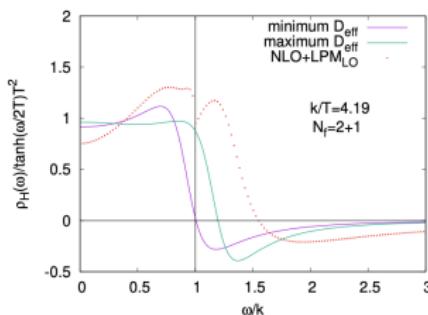
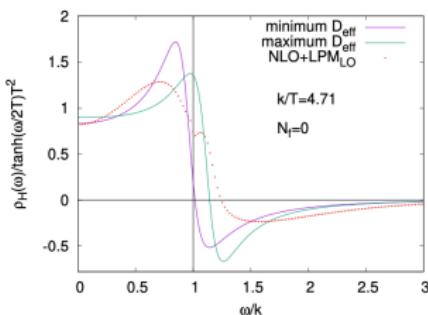


Spectral reconstruction - χ^2 -fits

(a) Polynomial fits

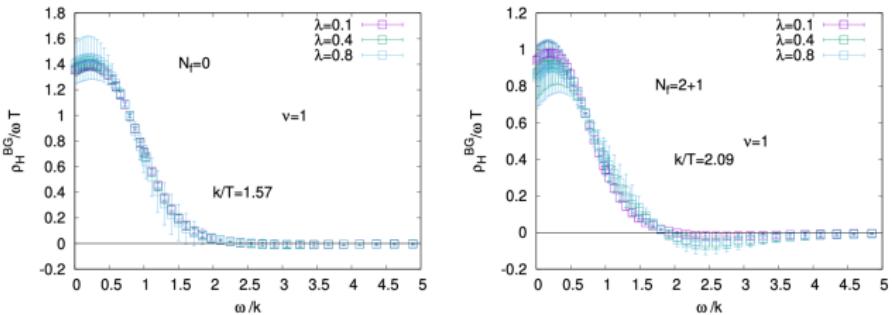


(b) Padé fits

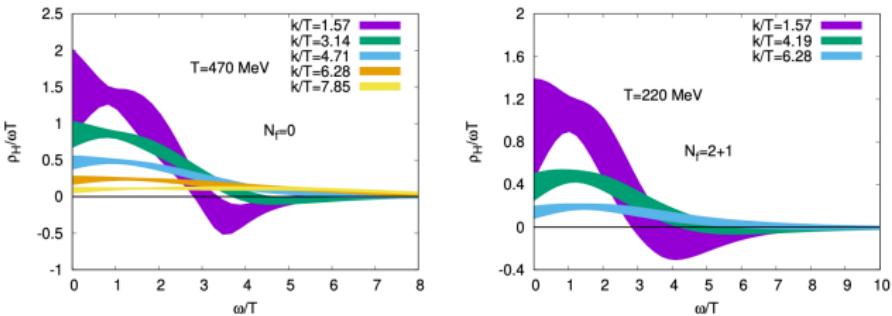


Spectral reconstruction - BGM and GPR

Backus-Gilbert method



Gaussian processes



Photon rate from the lattice - $D_{\text{eff}} T$ comparison

