**Workshop on recent developments from QCD to nuclear matter**

Dec. 17-20, 2025

# Crossover between ultracold atoms and nuclear matter

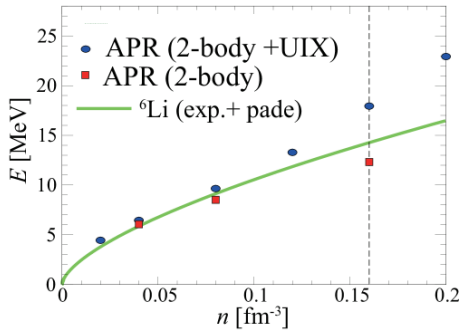
Hiroyuki Tajima

The University of Tokyo, Japan

**References**: HT, K. Iida, T. Kojo, and H. Liang, PRL **135**, 042701 (2025).

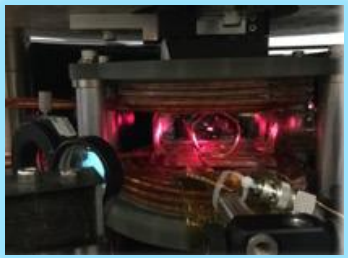
# Self-introduction: **Hiroyuki Tajima**
# Ultracold atoms to neutron star matter

**Unitary Fermi gas/**
**BEC-BCS crossover**




PRX, **7**, 041004 (2017).

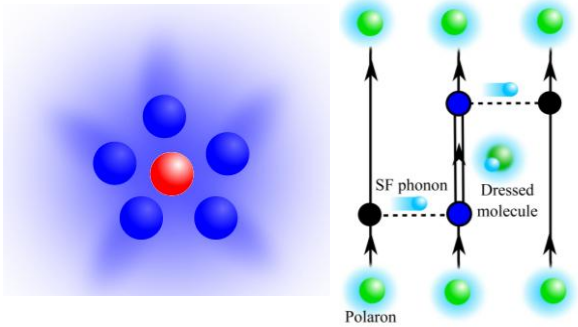
**Ultracold atoms**



- Tunable interaction
- Boson/fermion
- Gas/lattice
- (pseudo) spin and color

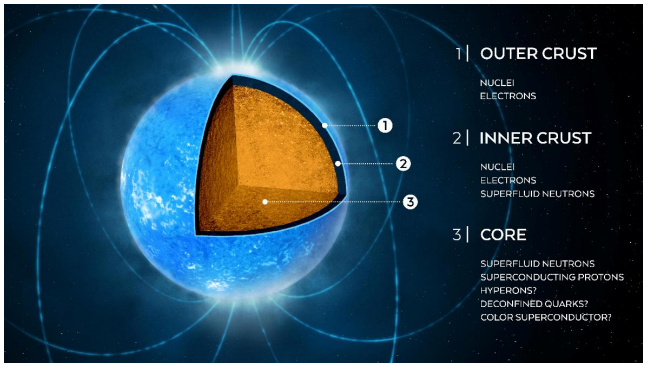
Photo: http://www.sci.osaka-cu.ac.jp/phys/laser/research_Li.html

**Polarons**




PLB **851**, 138567 (2024).
arXiv:2505.19117

**Neutron star matter**




A. L. Watts, *et al*., RMP **88**, 021001 (2016).

# Outline

- **Introduction**

  *Can we study a microscopic mechanism of hadron-quark crossover in cold atom physics?*

- **Formulation**

  *Tripling fluctuation theory*

- **Results**

  *Demonstration in a toy model*

- **Summary**

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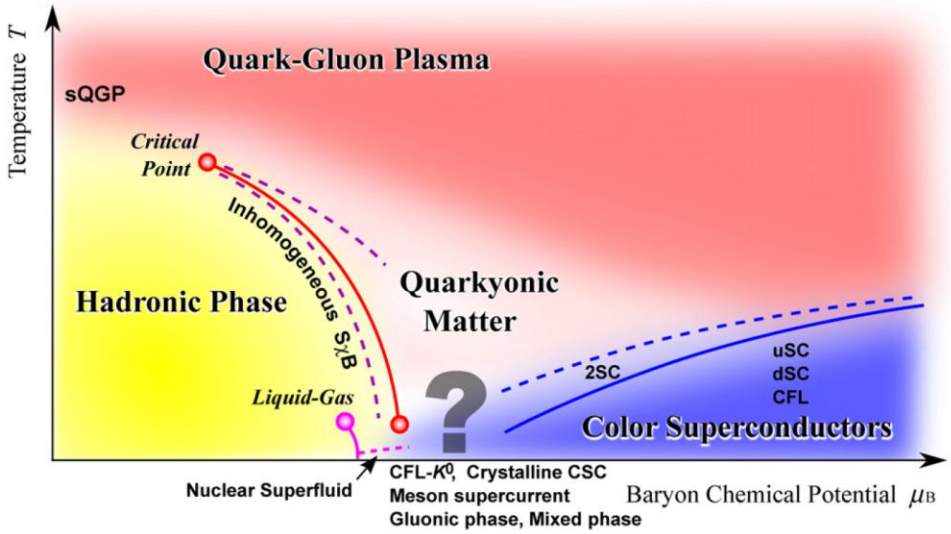
# Extremely dense matter

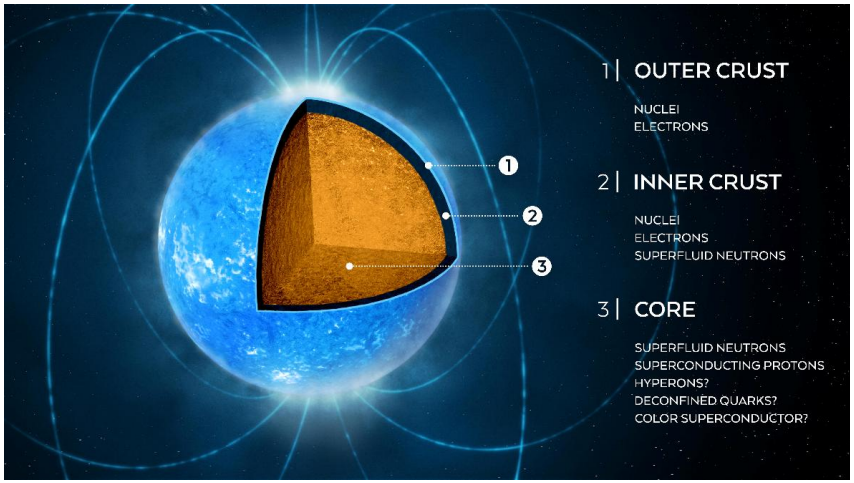
**How does the hadronic phase change into quark matter at finite densities?**

Dense QCD phase diagram




K. Fukushima, *et al.*, Rep. Prog. Phys. **74**, 014001 (2011).

Neutron star as a testing ground of dense matter



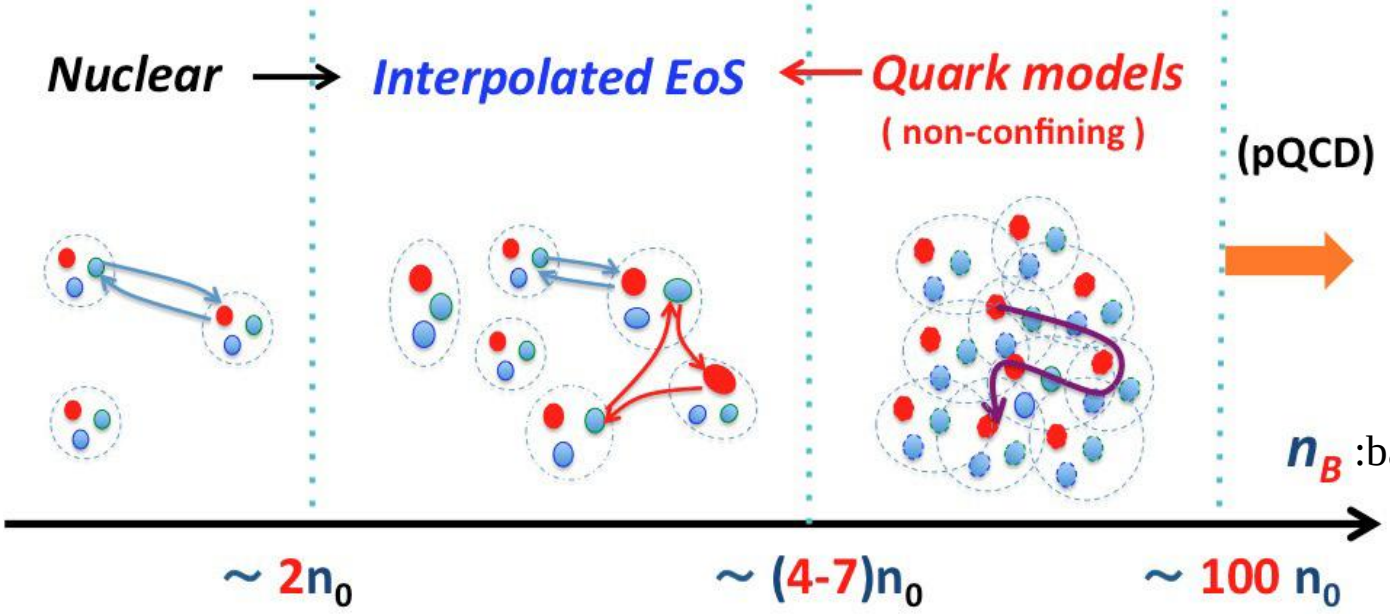

A. L. Watts, *et al.*, RMP **88**, 021001 (2016).

✕ **Sign problem in lattice QCD**

✕ **Limited information in the observation**

# Hadron-quark (HQ) crossover

G. Baym, *et al*., Rep. Prog. Phys. **81**, 056902 (2018).




normal nuclear density
$n_0 = 0.16\,\mathrm{fm}^{-3}$

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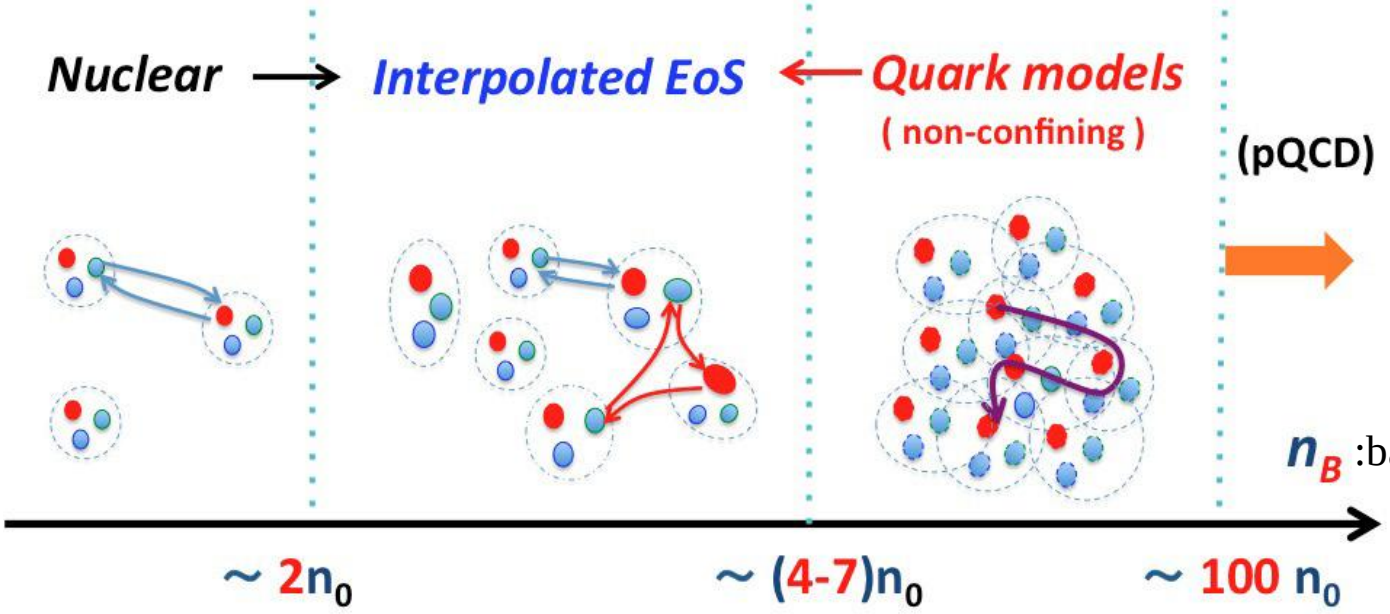
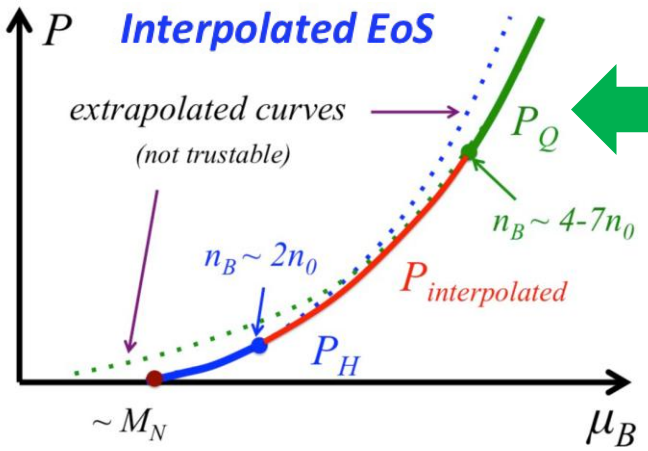



Ab initio simulation with realistic nuclear force, Chiral EFT, etc…




Quark matter EOS (e.g., NJL with mean field)

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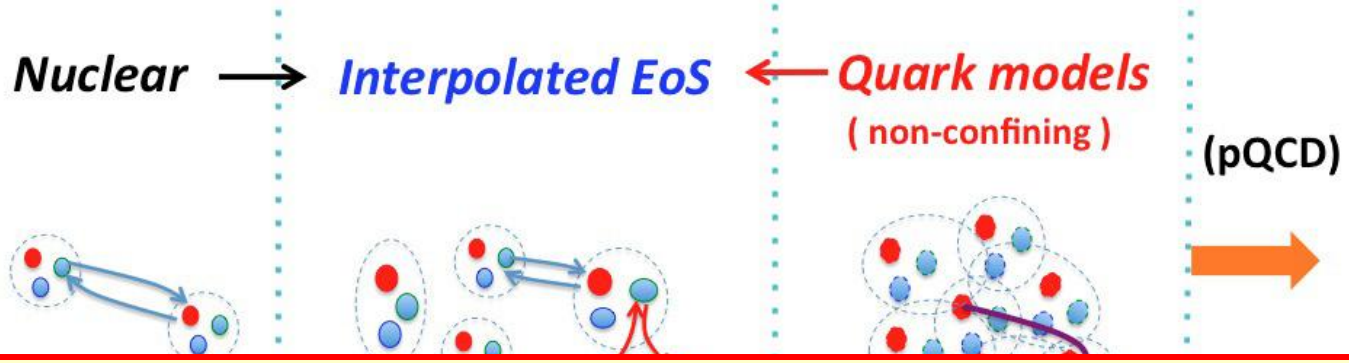
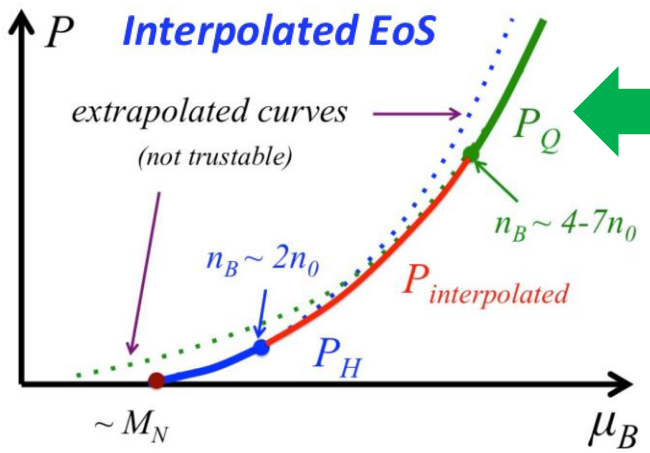



**Microscopic mechanism of the HQ crossover has been elusive**

$\sim 2n_0$ $\sim (4\text{-}7)n_0$ $\sim 100\ n_0$

Ab initio simulation with realistic nuclear force, Chiral EFT, etc…

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normal nuclear density
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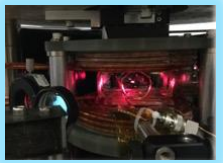
# Analogy with BEC-BCS crossover?

**Review**: Y. Ohashi, HT, and P. van Wyk, Prog. Part. Nucl. Phys. **111**, 103739 (2020).



**BEC-BCS crossover realized in ultracold Fermi gases**

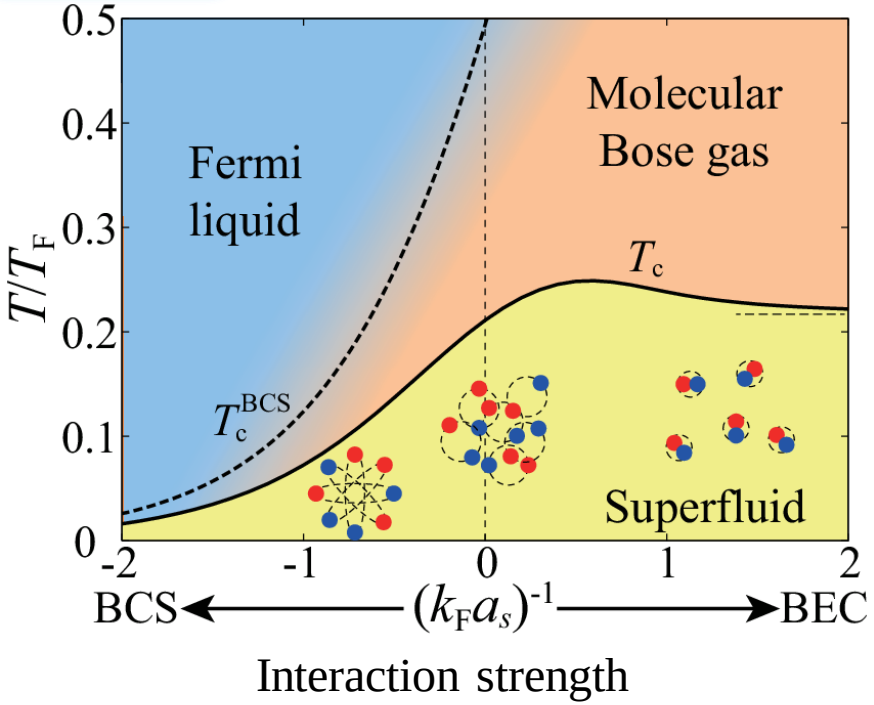
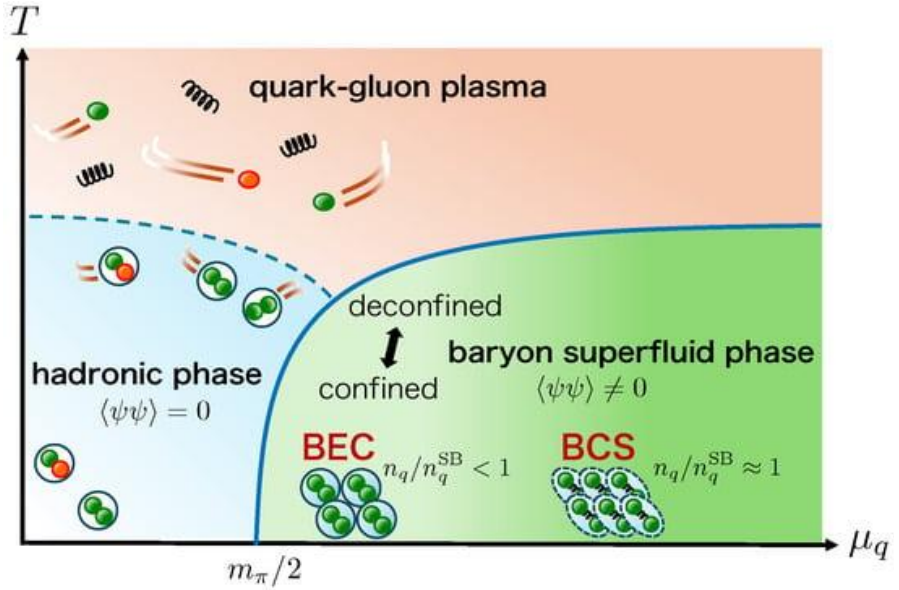



Photo: http://www.sci.osaka-cu.ac.jp/phys/laser/research_Li.html

**≃ HQ crossover in "2-color" QCD**

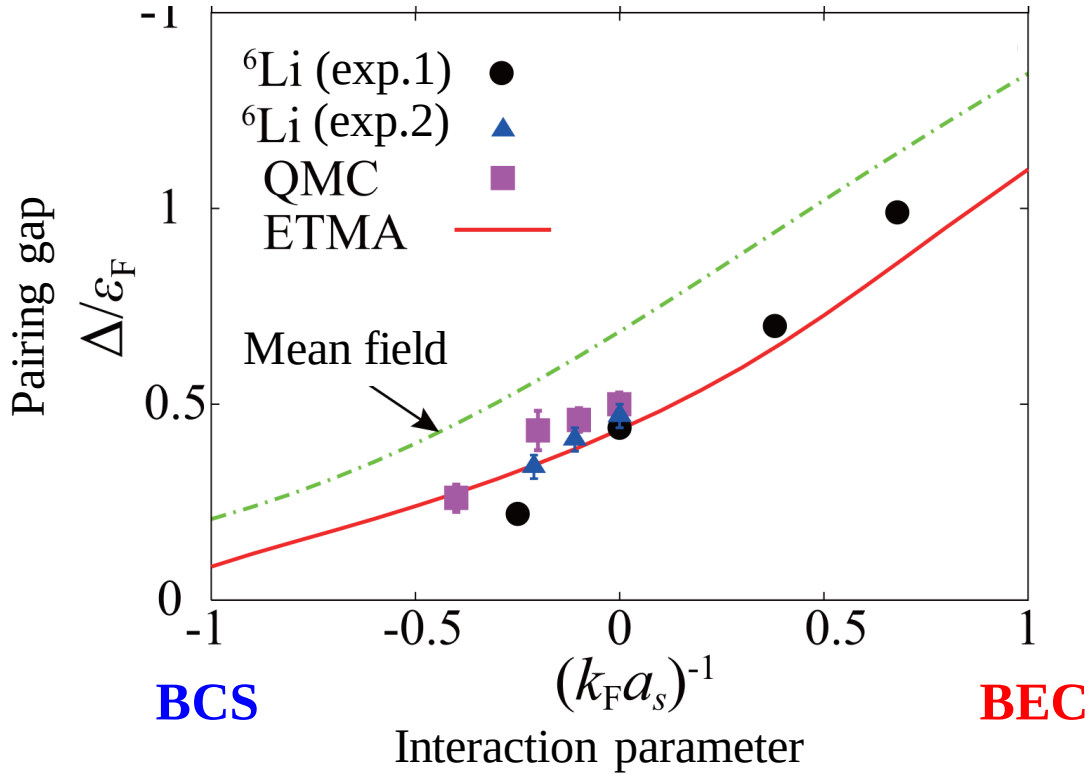



D. Suenaga, Symmetry **17**, 124 (2025).

# Many-body theory for the crossover

In the case of the BEC-BCS crossover, the mean-field (BCS-Eagles-Leggett) theory is "qualitatively" valid at zero temperatures.

**Mean field = Superfluid/superconducting order parameter Δ**

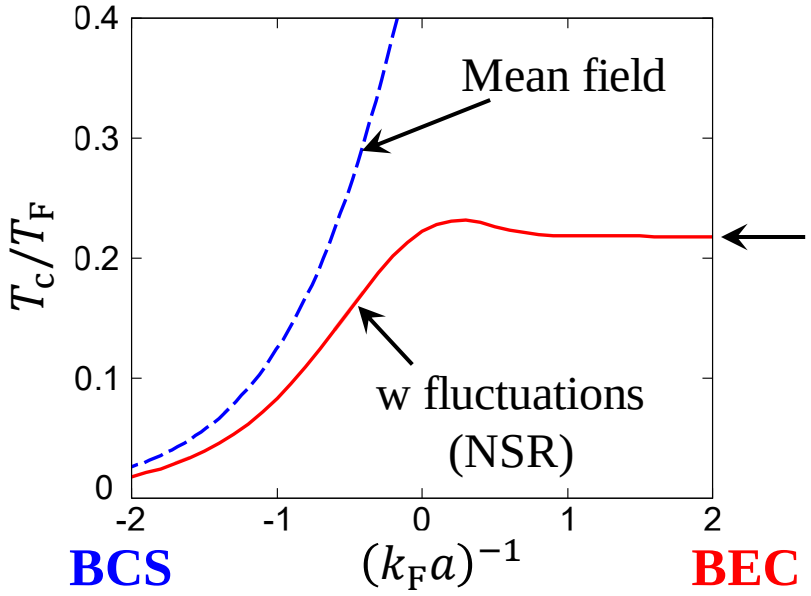



Y. Ohashi, HT, and P. van Wyk, Prog. Part. Nucl. Phys. **111**, 103739 (2020).

# Many-body theory for the crossover

In the absence of order parameters for crossover (e.g., hadron-quark crossover), the mean-field theory is **INVALID** even qualitatively

**Critical temperature**:
$\Delta(T = T_c) = 0$




0.218
(molecular BEC temperature)

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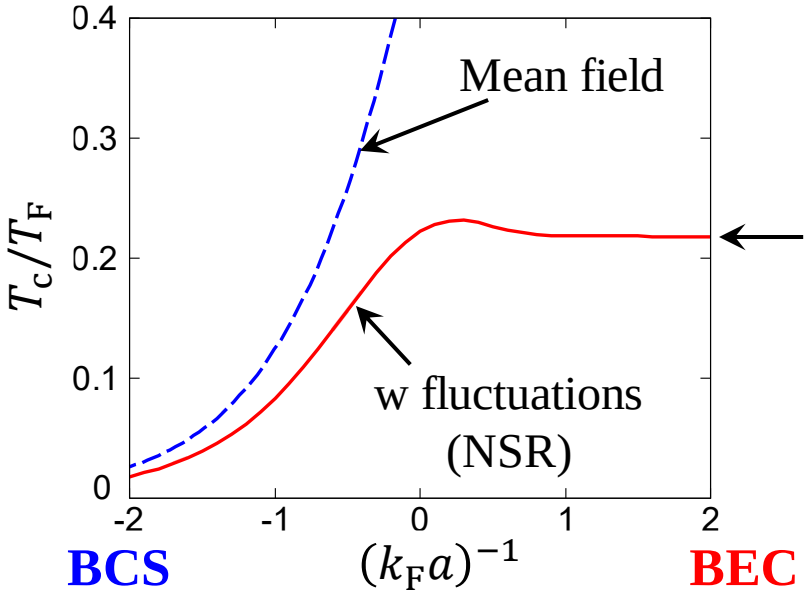


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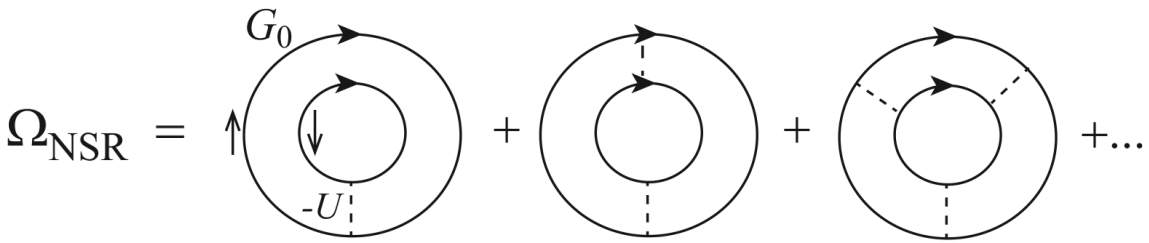
**Nozieres-Schmitt-Rink (NSR) approach to pairing fluctuations**

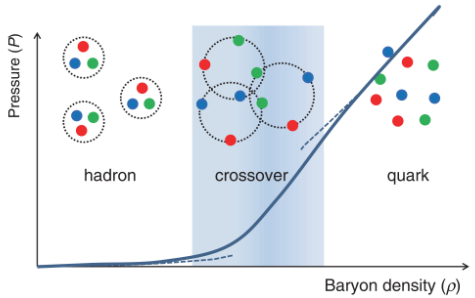



P. Nozières, and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).

➡ **Theory for "tripling" fluctuations is needed**

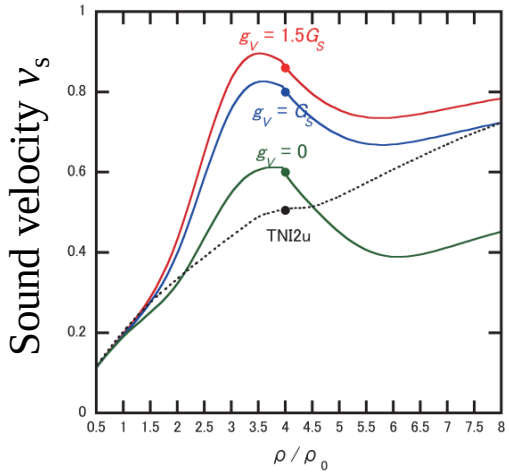
# Two key points to understand the hadron-quark crossover

**Peaked speed of sound**




Rapid increase of $P(\rho)$

➡ $v_s^2 = \frac{dP}{dE}$ **exhibits a peak**




K. Masuda, T. Hatsuda, and T. Takatsuka, PTEP **2013,** 073D01 (2013).

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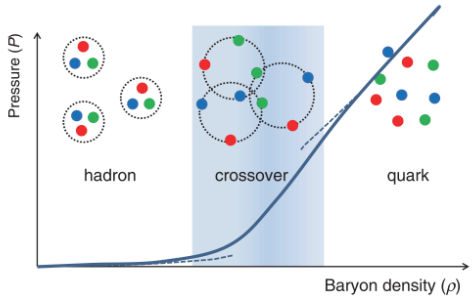
## Peaked speed of sound

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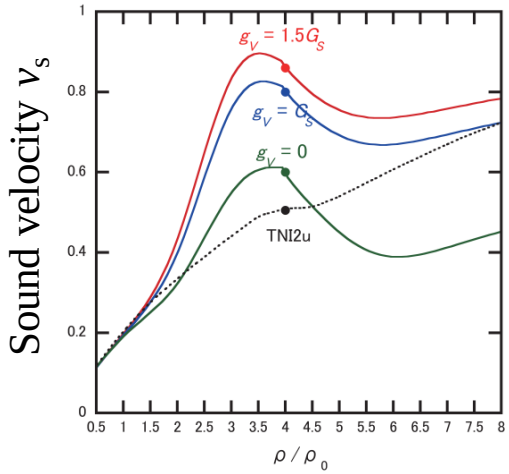
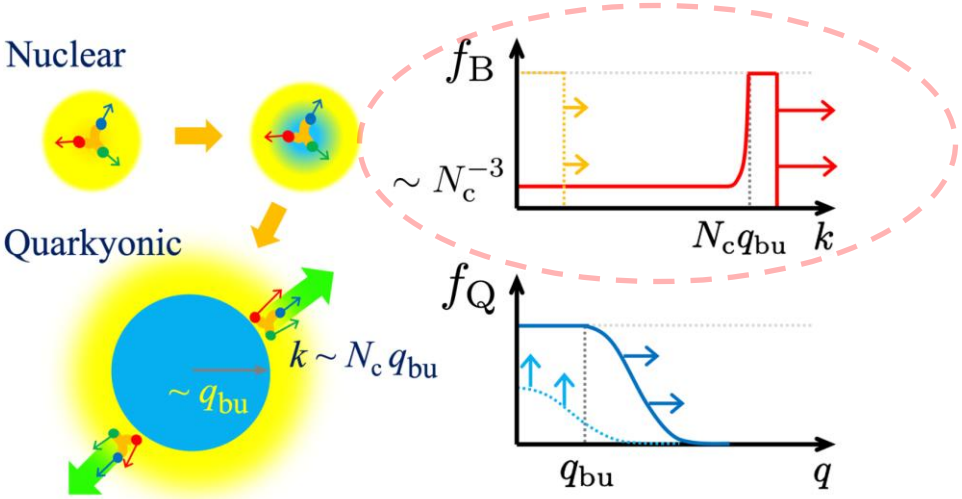



K. Masuda, T. Hatsuda, and T. Takatsuka, PTEP **2013,** 073D01 (2013).

## Non-monotonic behavior of baryon-momentum distribution

**Explicit duality model:**

Y. Fujimoto, T. Kojo, and L. D. McLerran, Phys. Rev. Lett. **132**, 112701 (2024).

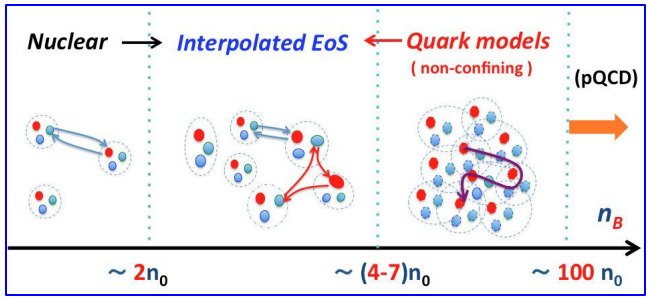


# In this talk…

- In analogy with the BEC-BCS crossover, we discuss the microscopic mechanism of the hadron-quark crossover.
- We show that tripling fluctuations explain two key points:

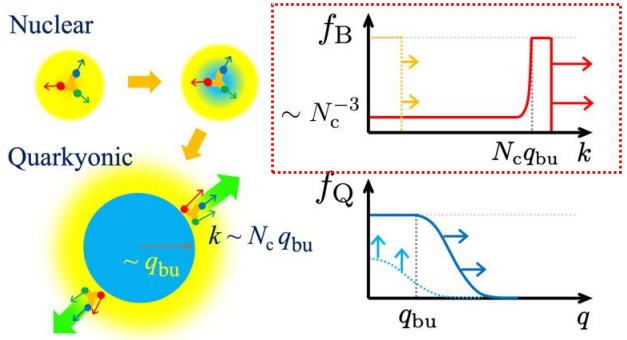
  **1**. baryon momentum shell structure, **2**. peaked speed of sound

HT, K. Iida, T. Kojo, and H. Liang, PRL **135**, 042701 (2025).



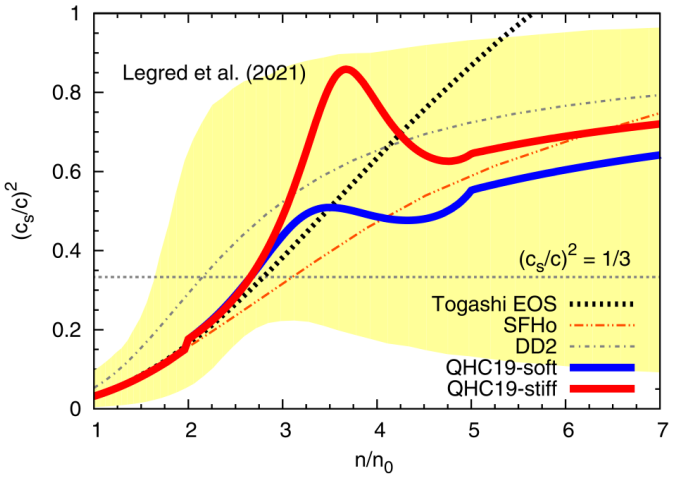

G. Baym, *et al.*, Rep. Prog. Phys. **81**, 056902 (2018).

**Momentum shell structure of baryons**




Y. Fujimoto, *et al.*, Phys. Rev. Lett. **132**, 112701 (2024).

**Peaked speed of sound**




Y.-J. Huang, *et al.*, Phys. Rev. Lett. **129**, 181101 (2022)

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# *N*-body clustering fluctuations

R. Dashen, S.-K. Ma, and H. J. Bernstein, Phys. Rev. **187**, 345 (1969).

**Clustering fluctuations on thermodynamic potential:**

$$\delta\Omega_N = T\sum_{\boldsymbol{K}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\ln\left(1+e^{-\omega/T}\right)\partial_\omega\varphi$$

***N*-body propagator and phase shift:** $\mathcal{G}/\mathcal{G}_0 = |\mathcal{G}/\mathcal{G}_0|e^{i\varphi}$

**Exact constraint**: $\varphi(\boldsymbol{K},\omega\to-\infty)=\varphi(\boldsymbol{K},\omega\to\infty)=0$

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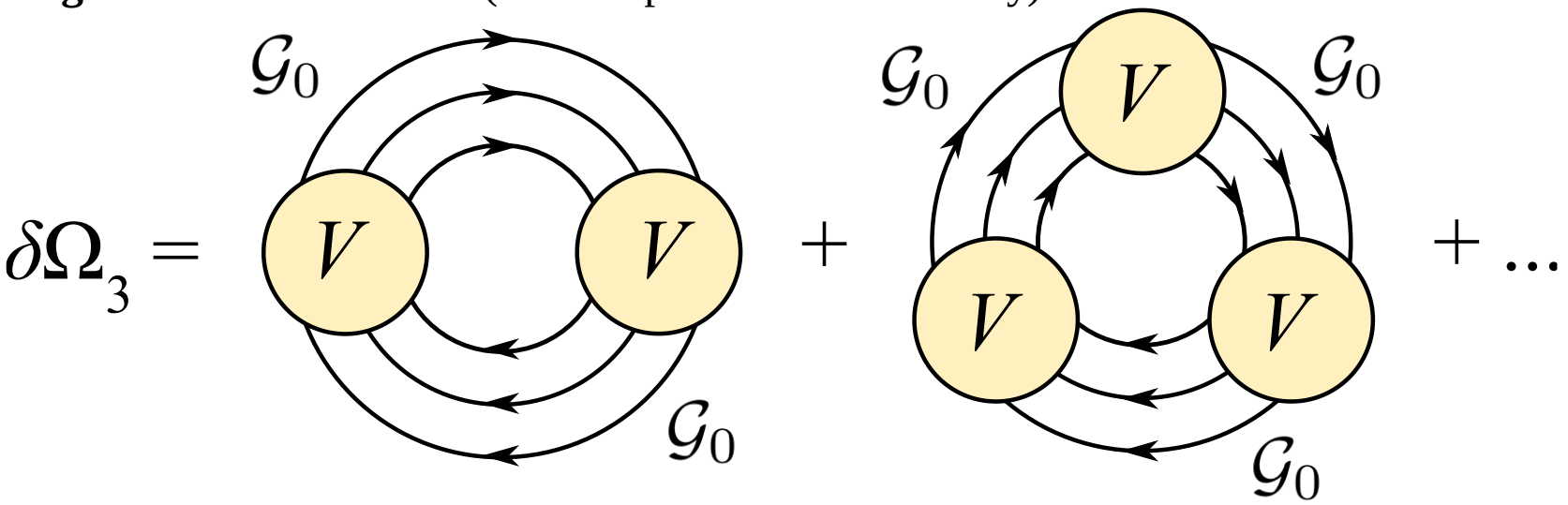
**Tripling fluctuations: $N = 3$** ($N = 2$ reproduces NSR theory)




$V$: short-range interaction responsible for $N$-body cluster formation

# Bound state v.s. Scattering state

**Tripling fluctuation contribution to the particle number density**

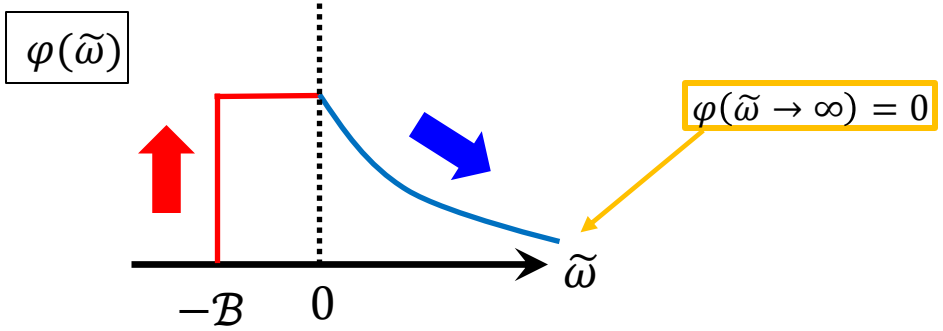
$$\delta n = -\frac{\partial \delta\Omega_3}{\partial \mu} = 3\sum_{\boldsymbol{K}} \int_{-\infty}^{\infty} d\tilde{\omega}\, \frac{1}{e^{(\omega + E_{\mathrm{B}}^{\mathrm{kin}}(K) - \tilde{\mu}_{\mathrm{B}})/T} + 1} \frac{\partial_{\tilde{\omega}} \varphi(\tilde{\omega})}{\pi}$$

$\tilde{\omega} = \omega - E_{\mathrm{B}}^{\mathrm{kin}}(K) + \tilde{\mu}_{\mathrm{B}}$ : subtracted three-body energy

**Bound state** **v.s.** **Scattering state**

$$\frac{\partial_{\tilde{\omega}} \varphi(\tilde{\omega})}{\pi} = \delta(\tilde{\omega} + \mathcal{B}) + \Theta(\tilde{\omega}) \frac{\partial_{\tilde{\omega}} \varphi_{\mathrm{scatt.}}(\tilde{\omega})}{\pi}$$

$> 0$ $< 0$




**Baryon kinetic energy:**

$$E_{\mathrm{B}}^{\mathrm{kin}}(K) = K^2/2M_{\mathrm{B}}$$

**Baryon chemical potential:**

$$\tilde{\mu}_{\mathrm{B}} = 3\tilde{\mu} \equiv 3(\mu - \Sigma_{\mathrm{HF}})$$

$\Sigma_{\mathrm{HF}}$: Hartree-Fock shift

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**Bound state** **v.s.** **Scattering state**

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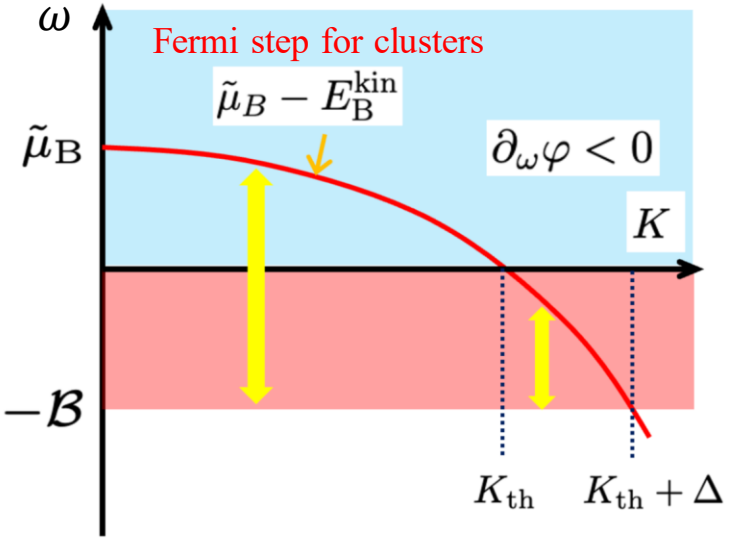
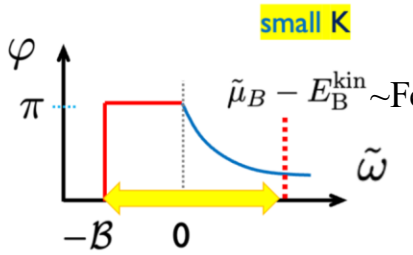
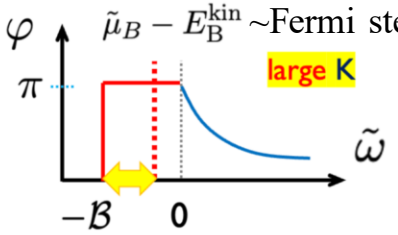
**$K-\omega$ plane**




**"Strong cancellation"**




**"No cancellation"**

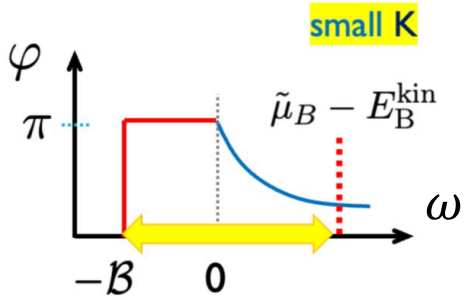
# Baryon momentum distribution

Number density: $n = -\frac{\partial \Omega_{\rm HF}}{\partial \mu} - \frac{\partial \delta\Omega_3}{\partial \mu} \equiv 3\sum_{\boldsymbol{k}} f_Q(\boldsymbol{k}) + 3\sum_{\boldsymbol{K}} f_B(\boldsymbol{K})$

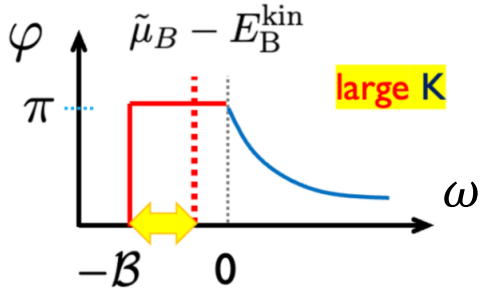
$$f_B(\boldsymbol{K}) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, \partial_\omega \varphi(\omega) f(\omega - \tilde{\mu}_B + E_B^{\rm kin}(K))$$


**Strong cancellation**

**No cancellation**




Non-trivial CoM momentum ($\boldsymbol{K}$) dependence arises via interplay of bound and scattering states



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# How to demonstrate the crossover physics?
## -1D nonrelativistic three-color fermions-

- Hamiltonian density: $\hat{H} = \hat{H}_0 + \hat{V}_3$

**One-body kinetic term**

$$\hat{H}_0 = \sum_{a=\mathrm{r,g,b}} \psi_a^\dagger \left( -\frac{\partial_x^2}{2m} - \mu \right) \psi_a$$

$m$: mass
$\mu$: chemical potential
$a = \mathrm{r, g, b}$: pseudo-color (hyperfine states)
$\psi_a^\dagger, \psi_a$: fermionic field operator

**Three-body interaction** (involving quantum anomaly with asymptotic freedom)

J. Drut, et al., PRL **120**, 243002 (2018).

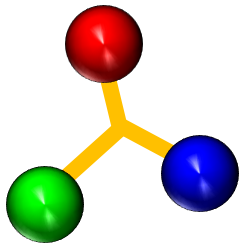
$$\hat{V}_3 = V(\psi_\mathrm{r}^\dagger \psi_\mathrm{r})(\psi_\mathrm{g}^\dagger \psi_\mathrm{g})(\psi_\mathrm{b}^\dagger \psi_\mathrm{b})$$

$V < 0$ : three-body attraction

**Three-body binding energy**

$$\boldsymbol{\mathcal{B}} = \frac{\Lambda^2}{m} \exp\left( \frac{2\sqrt{3}\pi}{mV} \right)$$

$\Lambda$: UV cutoff scale



# Phase shift of three-body propagator

**Three-body propagator**

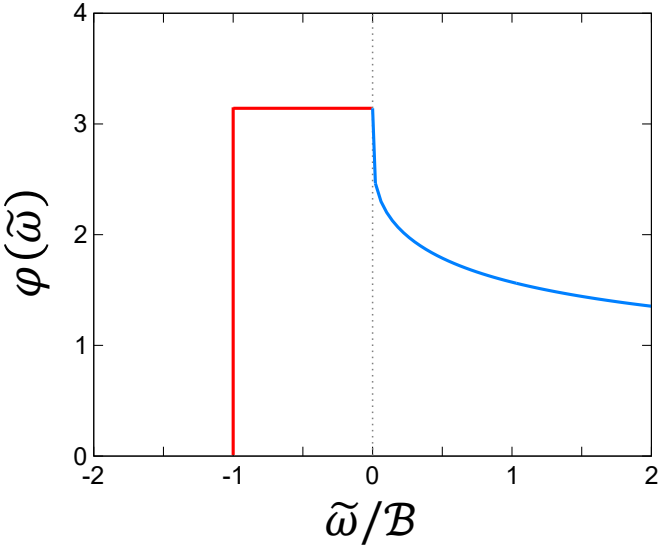
$$\mathcal{G}(K,\omega) = \frac{\mathcal{G}_0(K,\omega)}{1 - V\mathcal{G}_0(K,\omega)} \qquad \mathcal{G}_0(K,\omega) \simeq -\frac{m}{2\sqrt{3}\pi}\ln\left(\frac{-\tilde{\omega} - i\delta + \Lambda^2/m}{-\tilde{\omega} - i\delta}\right)$$

*Mott effect (medium suppression of bound state at small $K$) is neglected

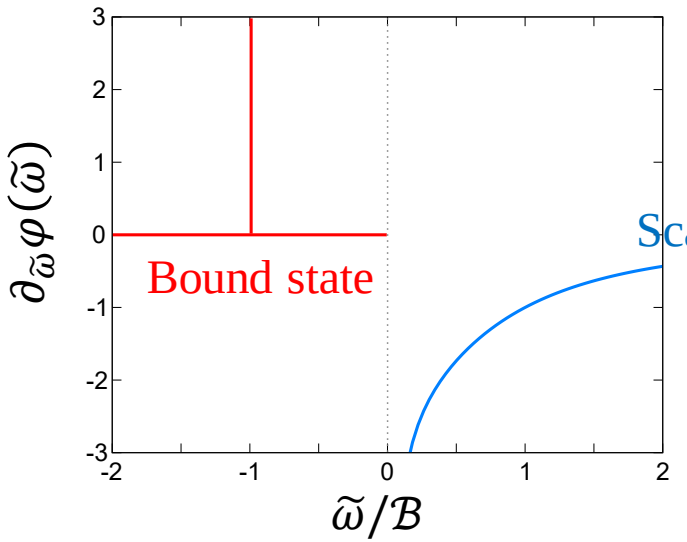
**Derivative of the phase shift**

$$\partial_{\tilde{\omega}}\varphi(\tilde{\omega}) = \pi\delta(\tilde{\omega} + \mathcal{B}) - \Theta(\tilde{\omega})\frac{\pi}{\tilde{\omega}[\ln^2(\tilde{\omega}/\mathcal{B}) + \pi^2]}$$

Phase shift




Derivative of phase shift

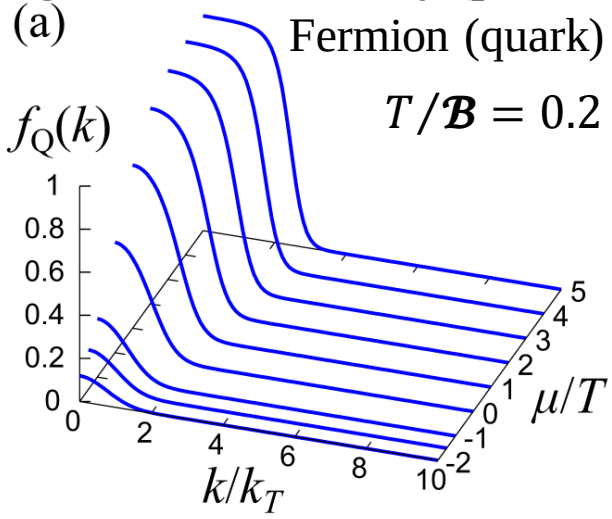


# Momentum distributions

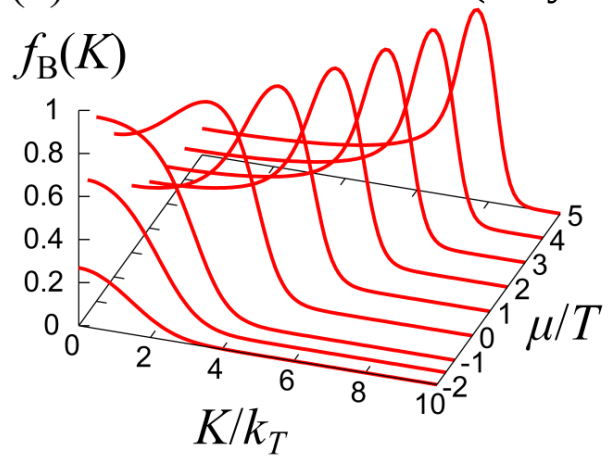
Model: 1D non-relativistic three-color fermions with color-singlet three-body interaction

**Tripling fluctuation theory (present work)**

(a) Fermion (quark)




(b) cluster (baryon)




$k_T = \sqrt{2mT}$: Thermal momentum scale

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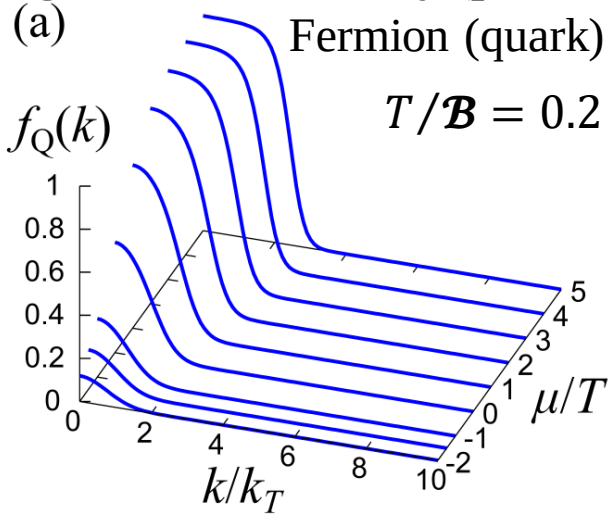


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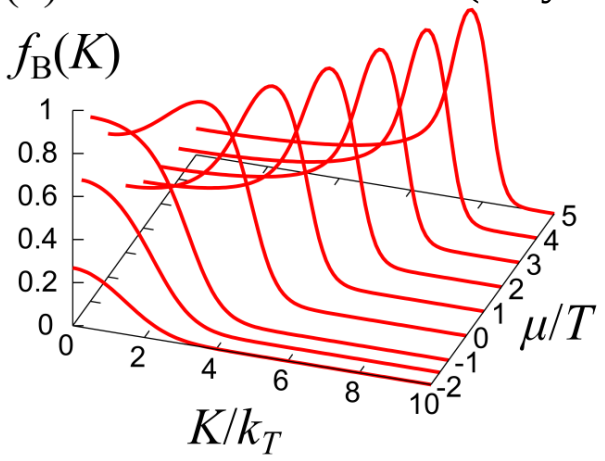


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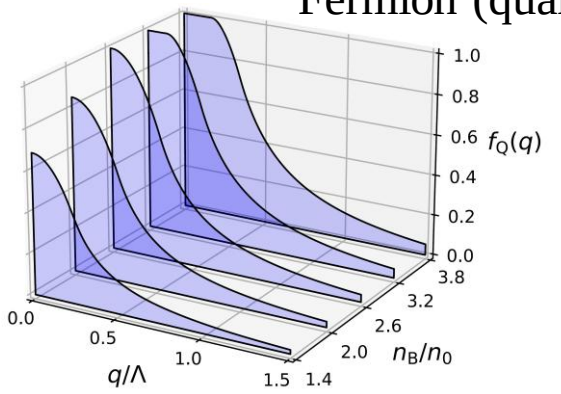
**Explicit Duality model**

PRL **132**, 112701 (2024).

Fermion (quark)

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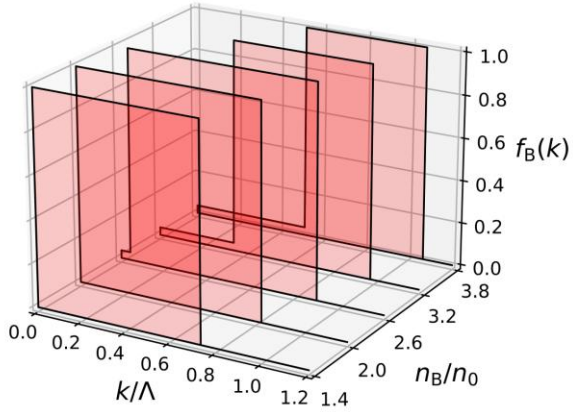
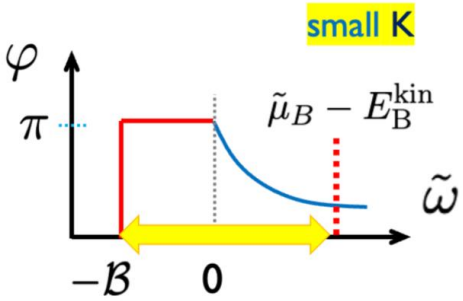
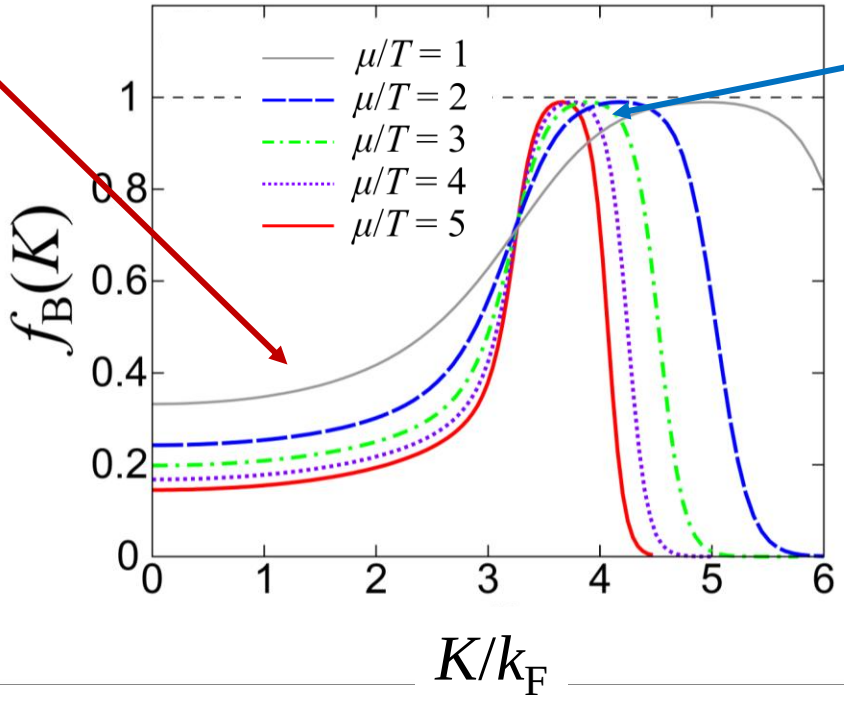


# Quarkyonic momentum shell

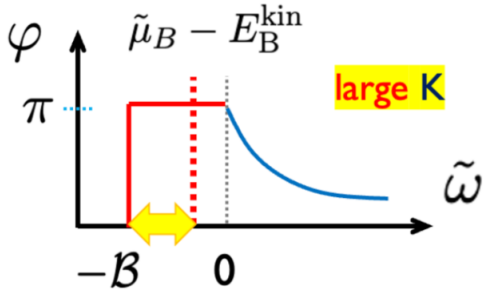
**Momentum distribution of baryonic fluctuations**

**Strong cancellation**




**No cancellation**



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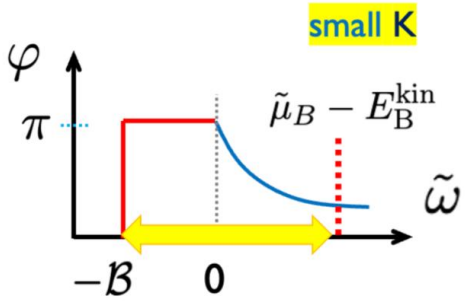


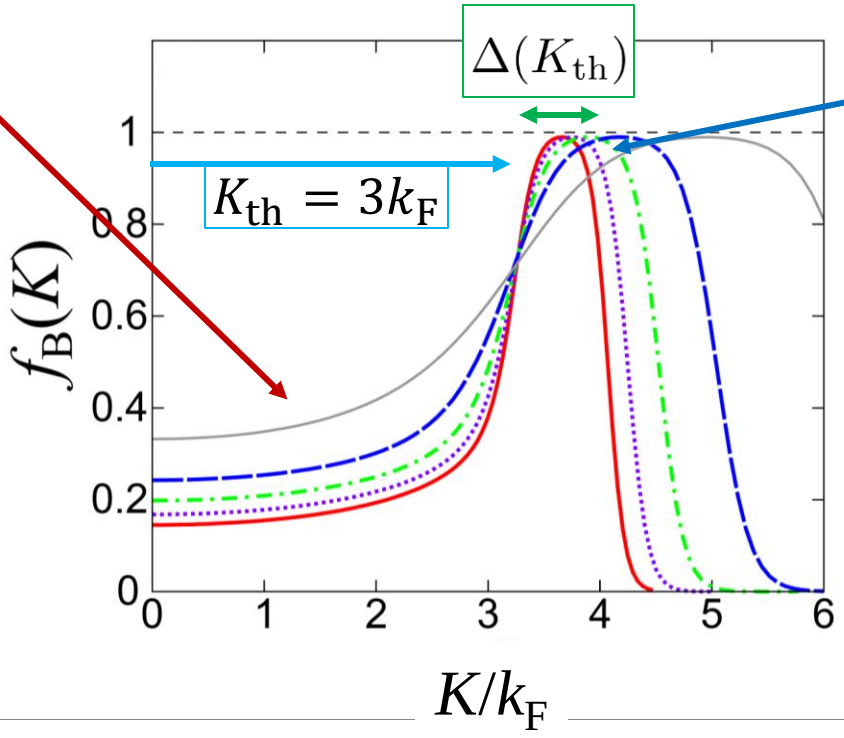


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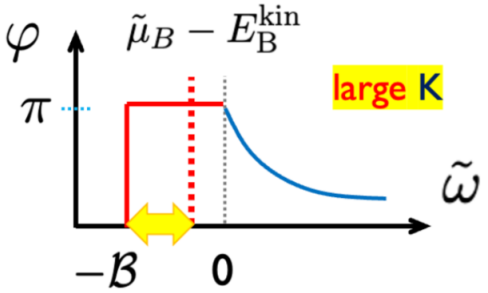
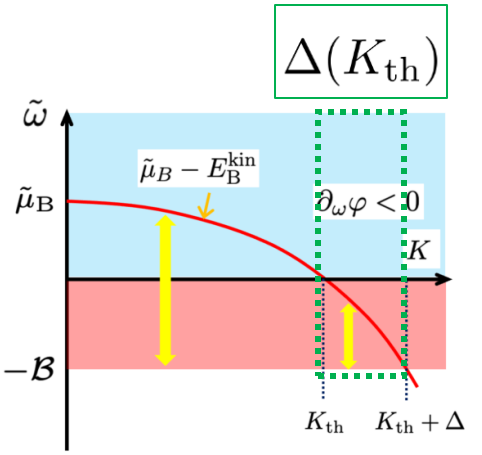



**Momentum-shell width:** $\Delta(K_{\rm th}) \equiv \sqrt{K_{\rm th}^2 + 2M_{\rm B}\mathcal{B}} - K_{\rm th}$

Analytical expression: $f_{\rm B}^{T=0}(K) = \Theta(K_{\rm th} + \Delta - K)\Theta(K - K_{\rm th})$

($T \to 0$) $\quad + \Theta(K_{\rm th} - K)\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left[\frac{\ln\frac{K_{\rm th}^2 - K^2}{2M_{\rm B}\mathcal{B}}}{\pi}\right]\right)$



# Peaked speed of sound

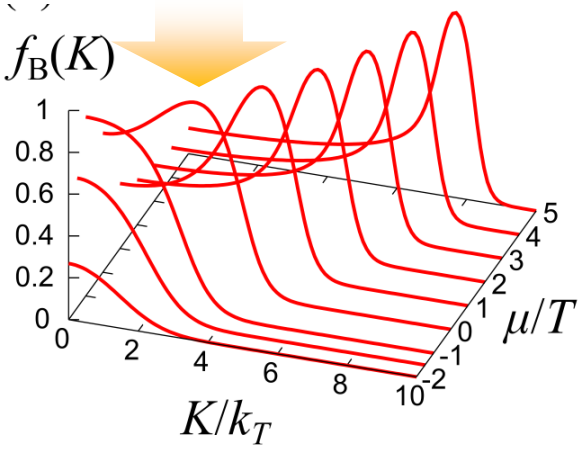
**Squared speed of sound:**

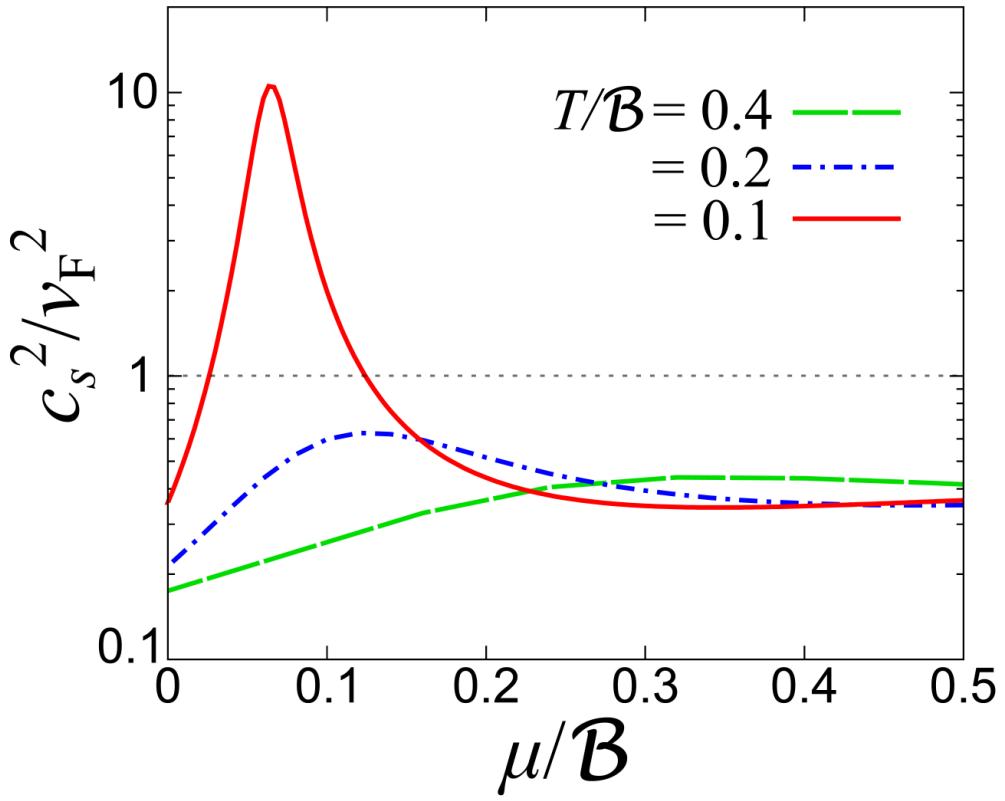
$$c_s^2 = \frac{n}{m}\left(\frac{\partial n}{\partial \mu}\right)^{-1}$$

**Density susceptibility:**

$$\frac{\partial n}{\partial \mu} = \frac{\partial n_{\mathrm{Q}}}{\partial \mu} + \frac{\partial n_{\mathrm{B}}}{\partial \mu}$$

$$\frac{\partial n_{\mathrm{B}}}{\partial \mu} = 3\sum_K \frac{\partial f_{\mathrm{B}}(K)}{\partial \mu} < 0$$



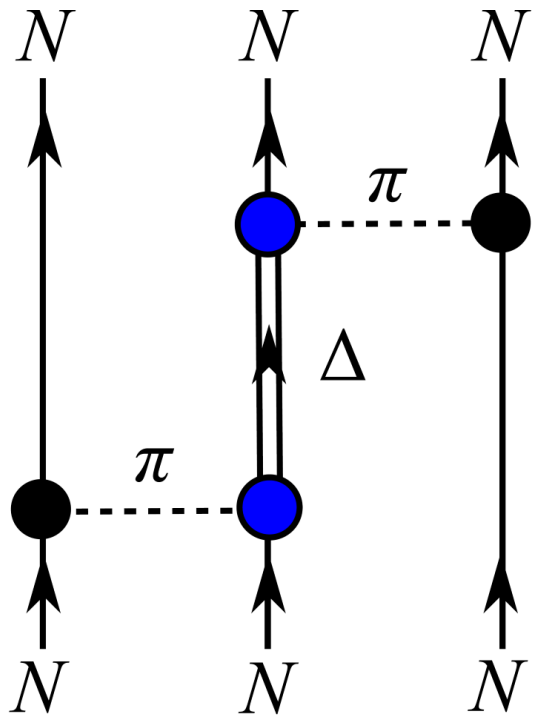

Peaked speed of sound is induced by suppressed baryon distributions at low momenta

# Three-body force in ultracold atoms?

HT, E. Nakano, and K. Iida, arXiv:2505.19117

**Nucleon ⇔**
**Pion ⇔**
**Δ resonance ⇔**

**Fujita-Miyazawa three-body force**



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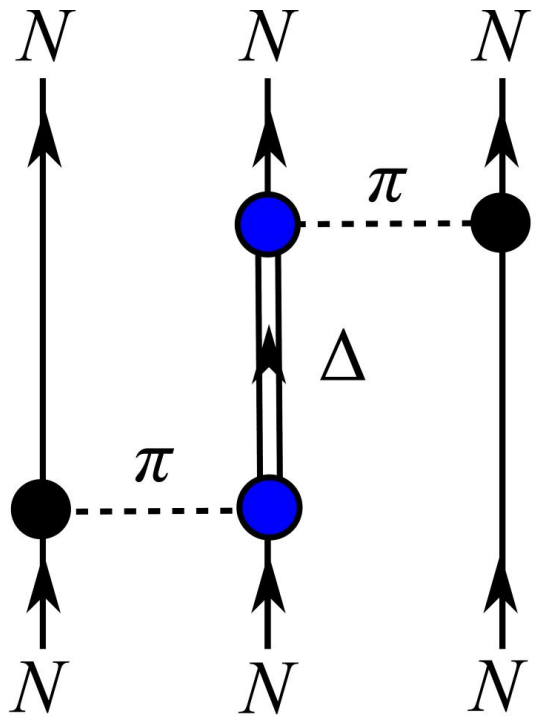
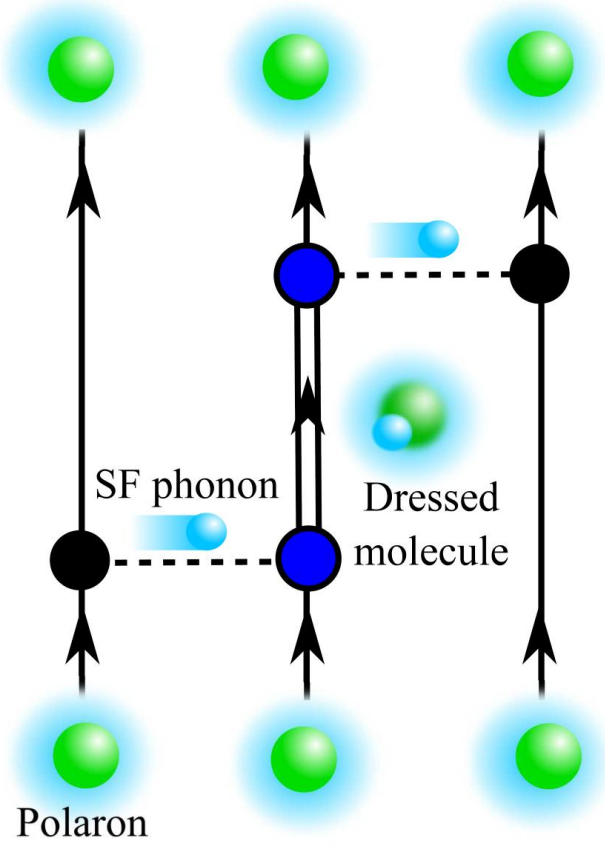



**Tunable counterpart in ultracold atoms**



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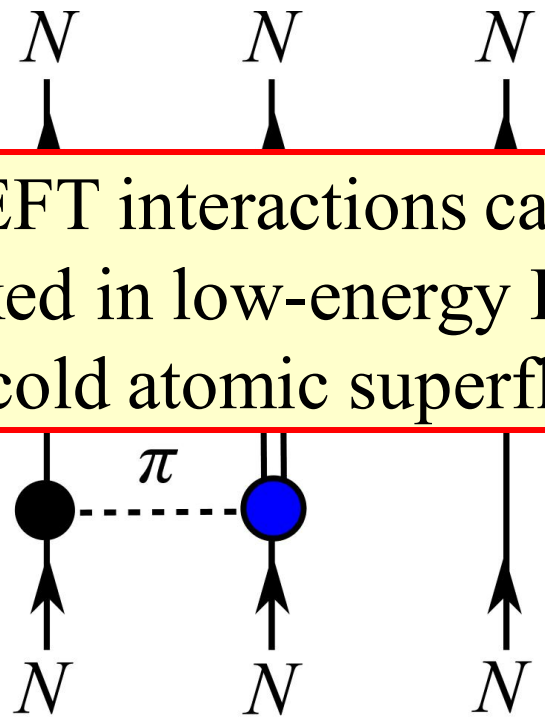
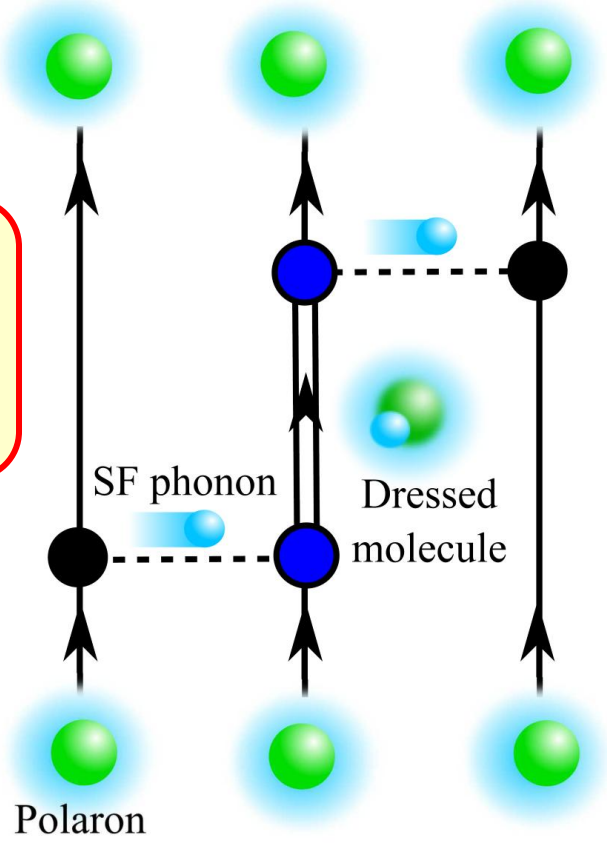



Chiral EFT interactions can be mimicked in low-energy EFT of ultracold atomic superfluid!

**Tunable counterpart in ultracold atoms**



# Outline

- **Introduction**

  *Can we study a microscopic mechanism of hadron-quark crossover in cold atom physics?*

- **Formulation**

  *Tripling fluctuation theory*

- **Results**

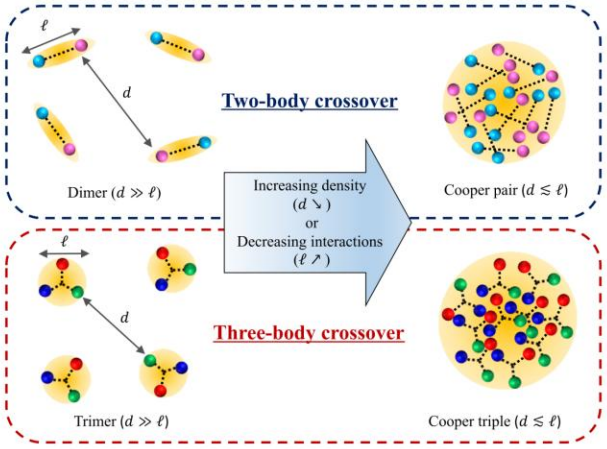
  *Demonstration in a toy model*

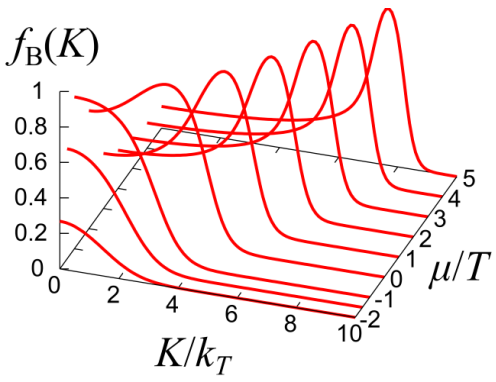
- **Summary**

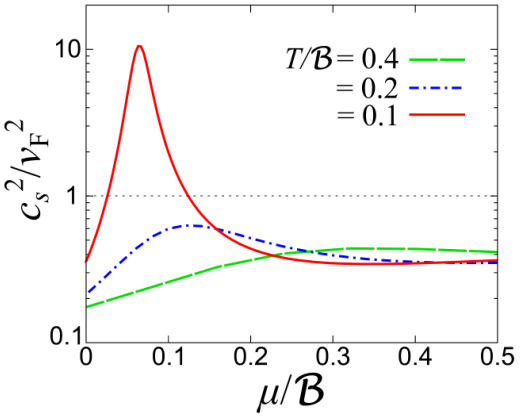
# Summary

HT, K. Iida, T. Kojo, and H. Liang, PRL **135**, 042701 (2025).

- In analogy with the BEC-BCS crossover in two-component Fermi gases, we have discussed the three-body counterpart in three-color fermions, where bound trimer gases change into degenerate Fermi state with tripling fluctuations.
- It is found that tripling fluctuations can induce a peaked speed of sound as well as quarkyonic-like momentum distributions.




**Future perspectives:** Application to more realistic systems relevant to neutron-star matter and heavy ion collision, and quantitative comparison with Monte Carlo simulation

# Appendix

# Difference btw pairing and tripling fluctuations

P. Nozieres and S. Schmitt-Rink, JLTP **59**, 195 (1985).

$$\Omega - \Omega_f = -\frac{1}{\pi}\sum_{\mathbf{q}} \int_{-\infty}^{+\infty} d\omega\, g(\omega)\delta(q,\omega)$$

$$\frac{1}{2}(N - N_f) = -\frac{\partial}{\partial\mu}(\Omega - \Omega_f) = \sum_{\mathbf{q}} \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} g(\omega)\frac{\partial}{\partial\mu}\delta(q,\omega)$$

**Pairing fluctuation case**

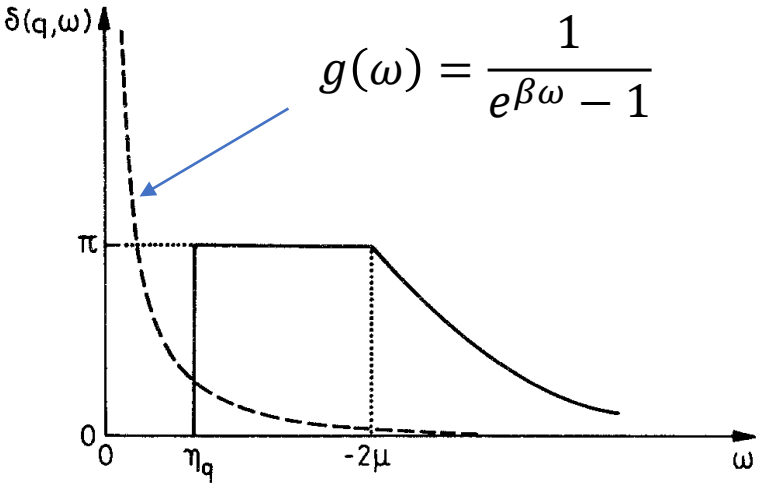
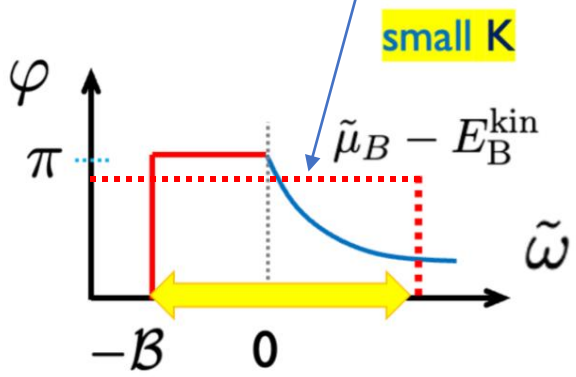



Fig. 7. The phase shift $\delta(q,\omega)$ in the dilute strong coupling limit. $\eta_q$ corresponds to the bound state, $-2\mu$ to the continuum threshold. The dashed curve is the Bose factor $g(\omega)$.

$$\eta_q = \frac{q^2}{4m} - \mathfrak{B} \qquad \delta_b(q,\omega) = \pi\theta(\omega - \eta_q)$$

**Tripling fluctuation case**

$$f(\omega - \tilde{\mu}_B + E_B^{\mathrm{kin}}(K)) = \frac{1}{e^{\beta(\omega - \tilde{\mu}_B + E_B^{\mathrm{kin}}(K))} + 1}$$




Even for pairing fluctuations, the cancellation can occur but masked by enhanced Bose distribution.

# Role of the phase shift

$$\delta\Omega_{N=3} = T\sum_{\boldsymbol{K}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\ln\left(1+e^{-\omega/T}\right)\partial_\omega\,\varphi(\boldsymbol{K},\omega)$$

$$= \sum_{\boldsymbol{K}}\int_{-\infty}^{\infty}\frac{d\tilde{\omega}}{\pi}\frac{1}{e^{\left(\omega+E_{\mathrm{B}}^{\mathrm{kin}}(K)-\tilde{\mu}_{\mathrm{B}}\right)/T}+1}\varphi(\tilde{\omega})$$

**Integration by part & Changing variable**

subtracted three-body energy: $\omega \to \tilde{\omega} = \omega - E_{\mathrm{B}}^{\mathrm{kin}}(K) + \tilde{\mu}_{\mathrm{B}}$

**Baryon kinetic energy:**

$$E_{\mathrm{B}}^{\mathrm{kin}}(K) = K^2/2M_{\mathrm{B}}$$

**Baryon chemical potential:**

$$\tilde{\mu}_{\mathrm{B}} = 3\tilde{\mu} \equiv 3(\mu - \Sigma_{\mathrm{HF}})$$

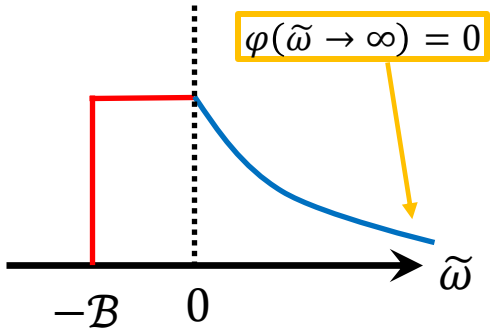
$\Sigma_{\mathrm{HF}}$: Hartree-Fock shift

**Structure of the phase shift**

**Bound state** **Scattering state**

$$\varphi(\tilde{\omega}) = \pi\Theta(\tilde{\omega}+\mathcal{B})\Theta(-\tilde{\omega}) + \Theta(\tilde{\omega})\varphi_{\mathrm{scatt.}}(\tilde{\omega})$$

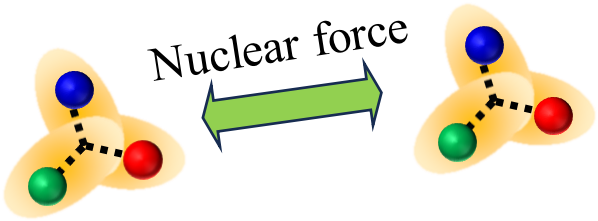
$\mathcal{B}$ :Cluster binding energy



# Theoretical approaches to nuclear equation of state (EOS)

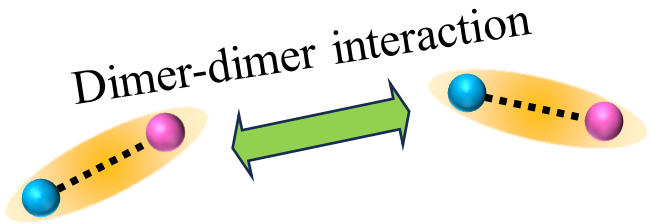
**Conventional nuclear EOS**
**= Effective theory of nucleons**
(Nucleons + nuclear force)

$$H = H_0 + V_{NN} + V_{NNN} + \cdots$$




**In the sense of BEC-BCS crossover…**

**Effective theory of dimers**
➡ **never describe the crossover regime**




Molecular BEC EOS

N. Navon et al., Science **328**, 729 (2010).

$$E = \frac{N}{2}E_b + N\frac{\pi\hbar^2 a_{dd}}{2m} \times n\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{na_{dd}^3} + \ldots\right)$$

$a_{dd}$: dimer-dimer scattering length

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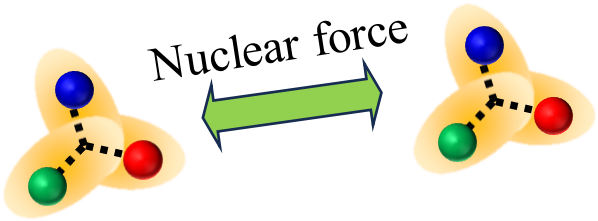
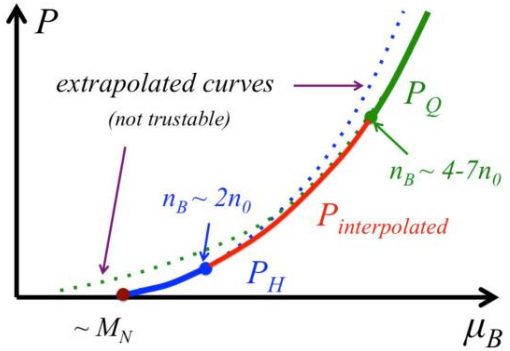



## Interpolated EOS based on the hadron-quark crossover




G. Baym, *et al.*, Rep. Prog. Phys. **81**, 056902 (2018).

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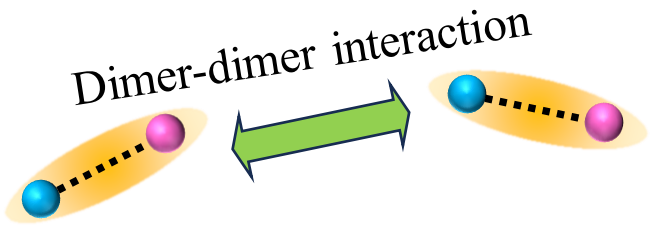


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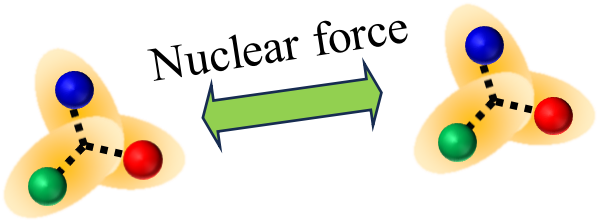


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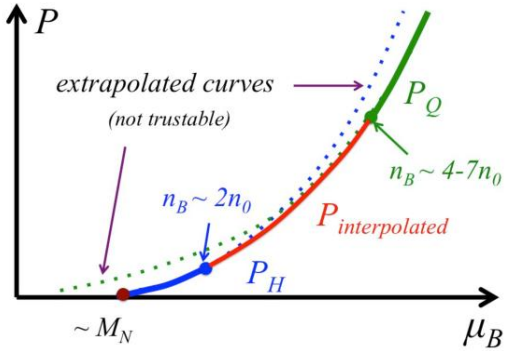


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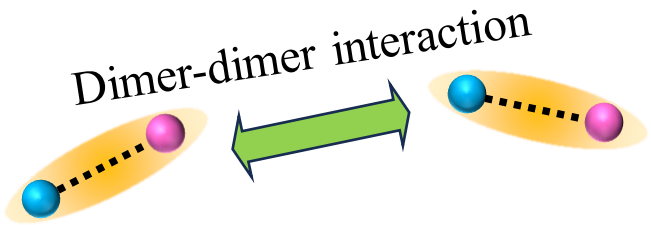


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## Phenomenologically interpolating BCS and BEC EOS

➡ **no microscopic foundation**

Molecular BEC EOS

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Interpolate!

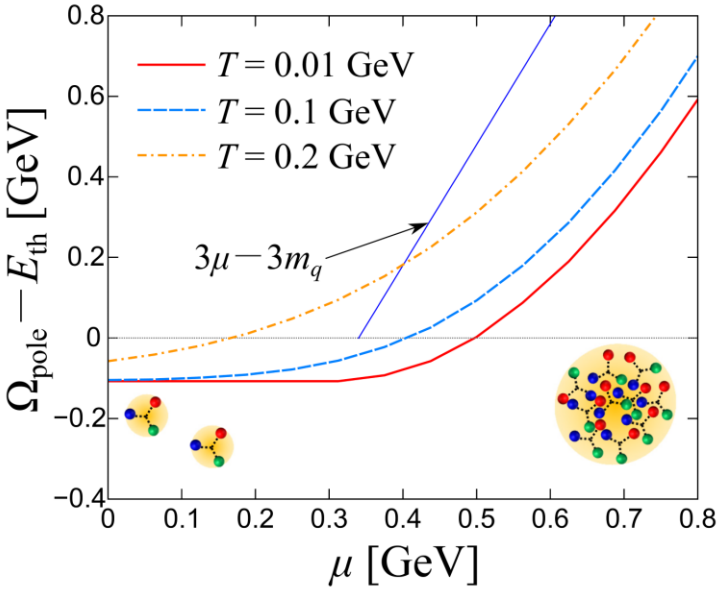
Fermi gas EOS at weal coupling

$$E = \frac{3}{5}NE_F\left(1 + \frac{10}{9\pi}k_F a + 0.18(2)(k_F a)^2 + 0.03(2)(k_F a)^3 + \ldots\right)$$

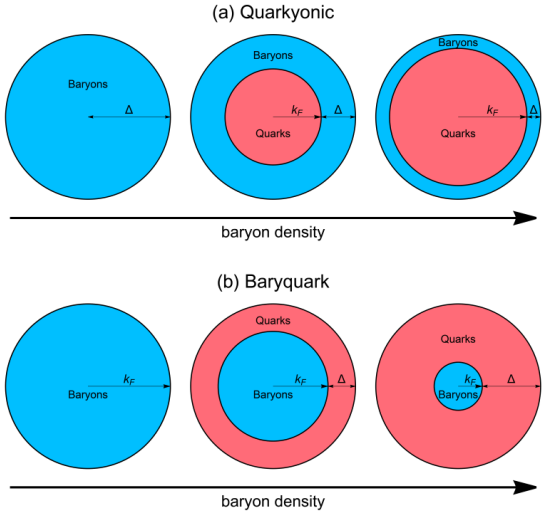
# Three-body pole in relativistic system

HT, S. Tsutsui, T. M. Doi, and K. Iida, Symmetry **15**, 333 (2023).

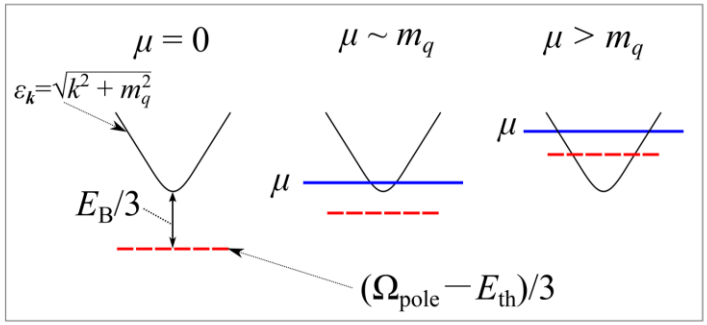
$$H = \sum_{p}\sum_{j}\varepsilon_p \psi^{\dagger}_{p,j}\psi_{p,j} + \sum_{k,q,k',q',P} V_{k,q,k',q',P}\psi^{\dagger}_{k,\mathrm{r}}\psi^{\dagger}_{q,\mathrm{g}}\psi^{\dagger}_{P-k-q,\mathrm{b}}\psi_{P-k'-q',\mathrm{b}}\psi_{q',\mathrm{g}}\psi_{k',\mathrm{r}},$$



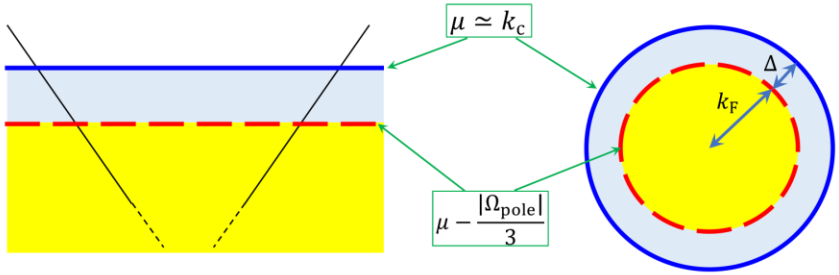

**Quarkyonic or Baryquark?**




arXiv:2211.14674




**Our scenario is close to quarkyonic**

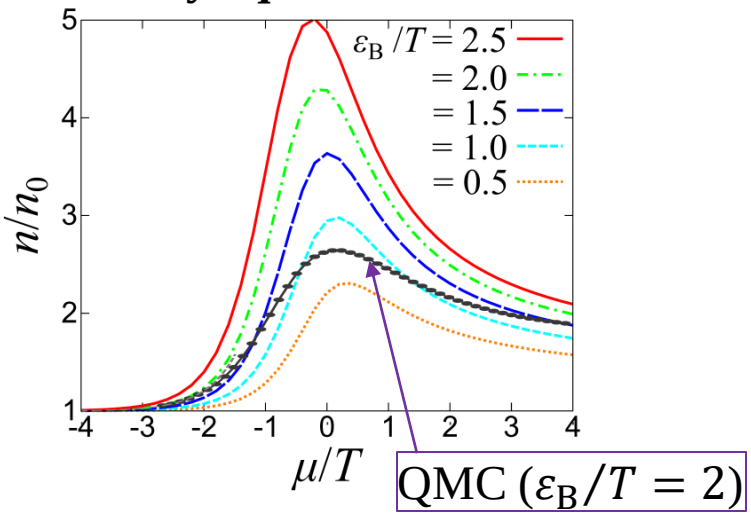



$m_q = 0.34$ GeV, $M_B = 0.91$ GeV

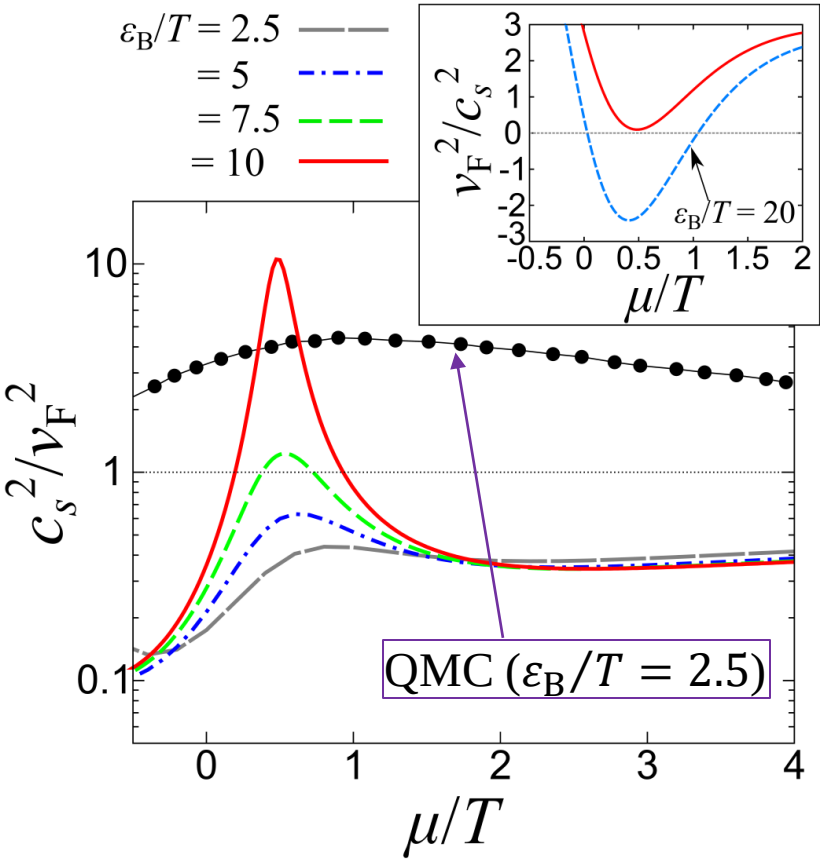
# Comparison with QMC

Qualitatively OK, but better approximation is needed for quantitative calculation

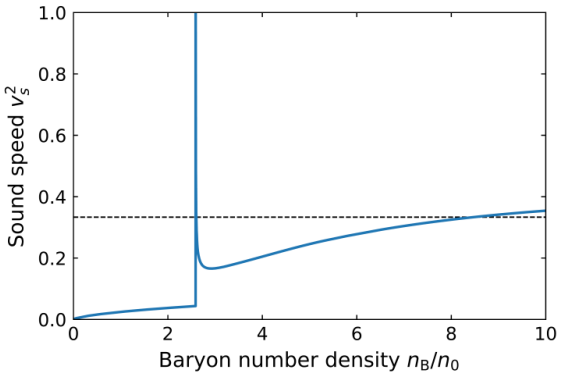
**Density equation of state**




**Speed of sound**




**Speed of sound in the duality model**



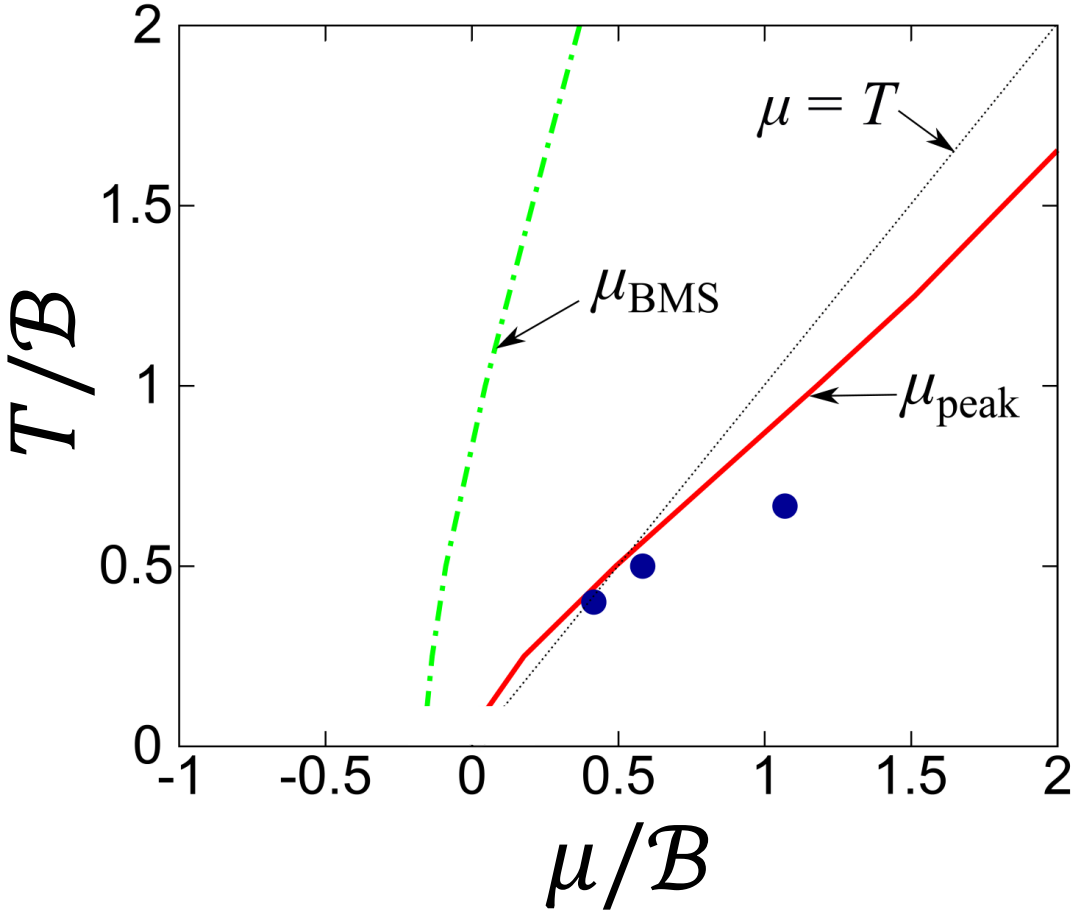

Y. Fujimoto, PRL **132**, 112701 (2024).

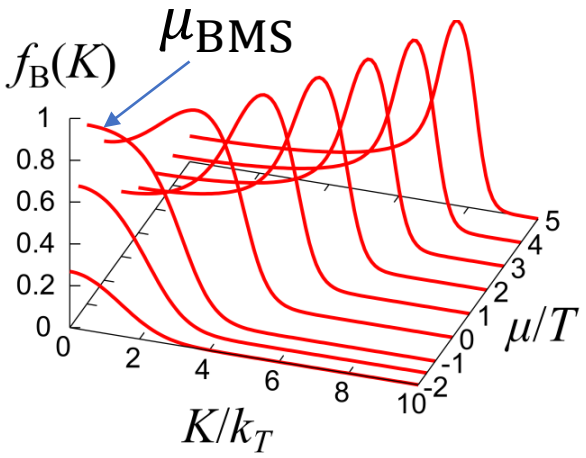
QMC: J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

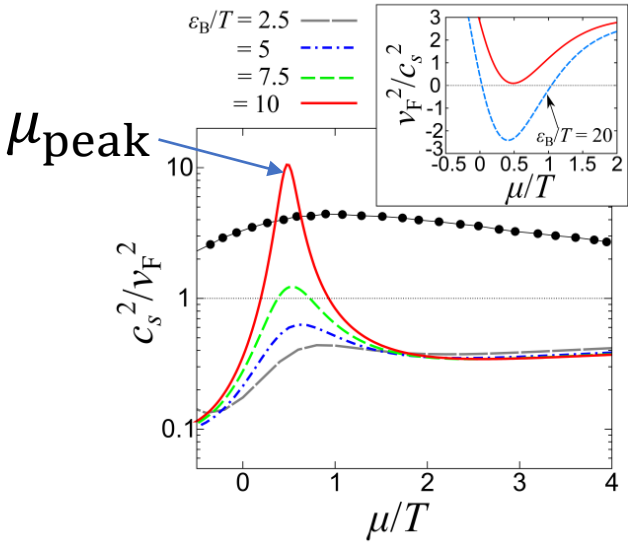
# Finite-temperature phase diagram

$\mu_{\rm BMS}$ : baryonic momentum shell starts to appear

$\mu_{\rm peak}$ : sound velocity is peaked




● QMC: J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

# Crossover equation of state and baryonic distribution functions

$\Omega = \Omega_{\rm HF} + \delta\Omega_3$ $\Omega_{\rm HF}$: Hartree-Fock contribution

**Tripling fluctuations:** $\delta\Omega_3 = -T\sum_K \ln\left[1+e^{-(-\mathcal{B}+E_{\rm B}^{\rm kin}-\tilde{\mu}_{\rm B})/T}\right]$

$$+T\sum_K \int_0^\infty \frac{d\omega}{\omega}\frac{\ln\left[1+e^{-(\omega+E_{\rm B}^{\rm kin}-\tilde{\mu}_{\rm B})/T}\right]}{\ln^2(\omega/\mathcal{B})+\pi^2}$$

**Baryonic distribution:** $f_{\rm B}(K) = f(-\mathcal{B}+E_{\rm B}^{\rm kin}-\tilde{\mu}_{\rm B})$ **:Bound state**

$$-\frac{\partial\delta\Omega_3}{\partial\mu} = 3\sum_K f_{\rm B}(K)$$

$$-\int_0^\infty \frac{d\omega}{\omega}\frac{f(\omega+E_{\rm B}^{\rm kin}-\tilde{\mu}_{\rm B})}{\ln^2(\omega/\mathcal{B})+\pi^2}$$ **:Scattering state**

**Baryon kinetic energy:**

$E_{\rm B}^{\rm kin}(K) = K^2/2M_{\rm B} \equiv K^2/(6m)$

**Baryon chemical potential:**

$\tilde{\mu}_{\rm B} = 3\tilde{\mu} \equiv 3(\mu - \Sigma_{\rm HF})$

# How to demonstrate the crossover physics?
## -1D nonrelativistic three-color fermions-

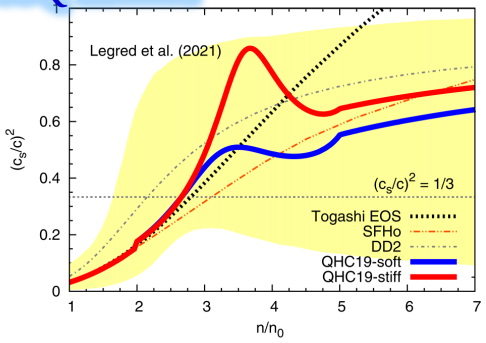
**Latter section: 1D nonrelativistic (1DNR) three-color Fermi gases with three-body attraction**

**Why?**

- Sign problem free Quantum Monte Carlo
- Similarity with HQ crossover
- Possible realization in future atomic experiments

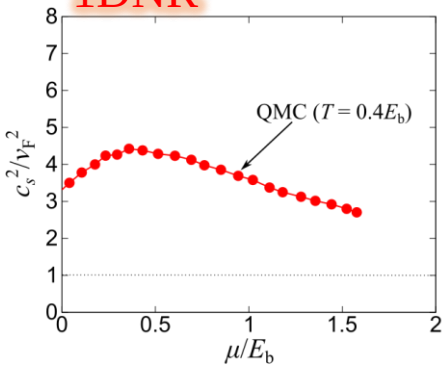
### Peaked speed of sound

HQ matter




Y.-J. Huang, *et al.*, Phys. Rev. Lett. **129**, 181101 (2022)

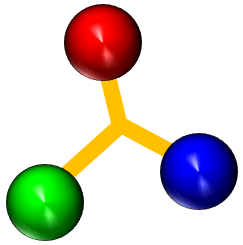
1DNR




J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

Both systems exhibit a characteristic peaked behavior in the crossover regime

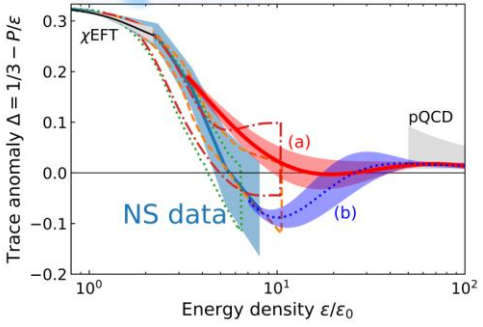
### Asymptotic freedom and trace anomaly



$$\frac{\partial g_3}{\partial \ln \Lambda} = \frac{m}{\sqrt{3\pi}} g_3^2$$

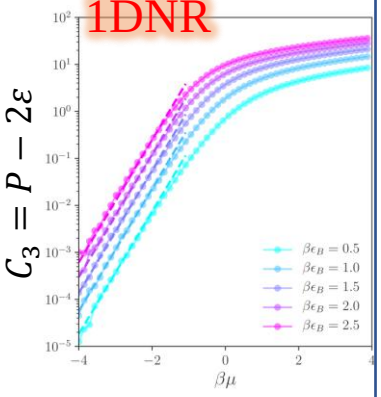
J. Drut, et al., Phys. Rev. Lett. **120**, 243002 (2018).

HQ matter




Y. Fujimoto, *et al.*, Phys. Rev. Lett. **129**, 252702 (2022).

1DNR




J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

Trace anomaly would influence EOS

# Realization of tunable three-body interaction in cold atoms

A. Hammond, *et al.*, Phys. Rev. Lett. **128**, 083401 (2022)

EOS in Rabi-coupled 2-com. 1D BEC

$$\frac{E_{\mathrm{MF}}}{V} = -\frac{\hbar\Omega}{2}(\phi_\uparrow^*\phi_\downarrow + \phi_\downarrow^*\phi_\uparrow) + \frac{\hbar\delta}{2}(|\phi_\uparrow|^2 - |\phi_\downarrow|^2) + \sum_{\sigma\sigma'}\frac{g_{\sigma\sigma'}}{2}|\phi_\sigma|^2|\phi_{\sigma'}|^2.$$

Low-energy EFT

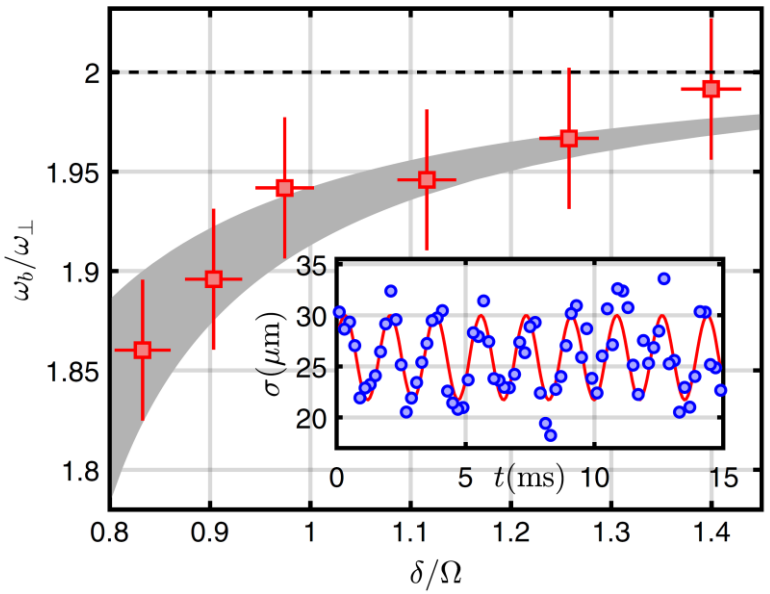
$$\frac{E_{\mathrm{MF}}}{N} \approx \epsilon_- + g_2\frac{n}{2} + g_3\frac{n^2}{3}$$

with $g_2 = g - \dfrac{\bar{g}}{1+\delta^2/\Omega^2}$

and $g_3 = -\dfrac{3\bar{g}^2}{\hbar\Omega}\dfrac{\delta^2/\Omega^2}{(1+\delta^2/\Omega^2)^{5/2}}$

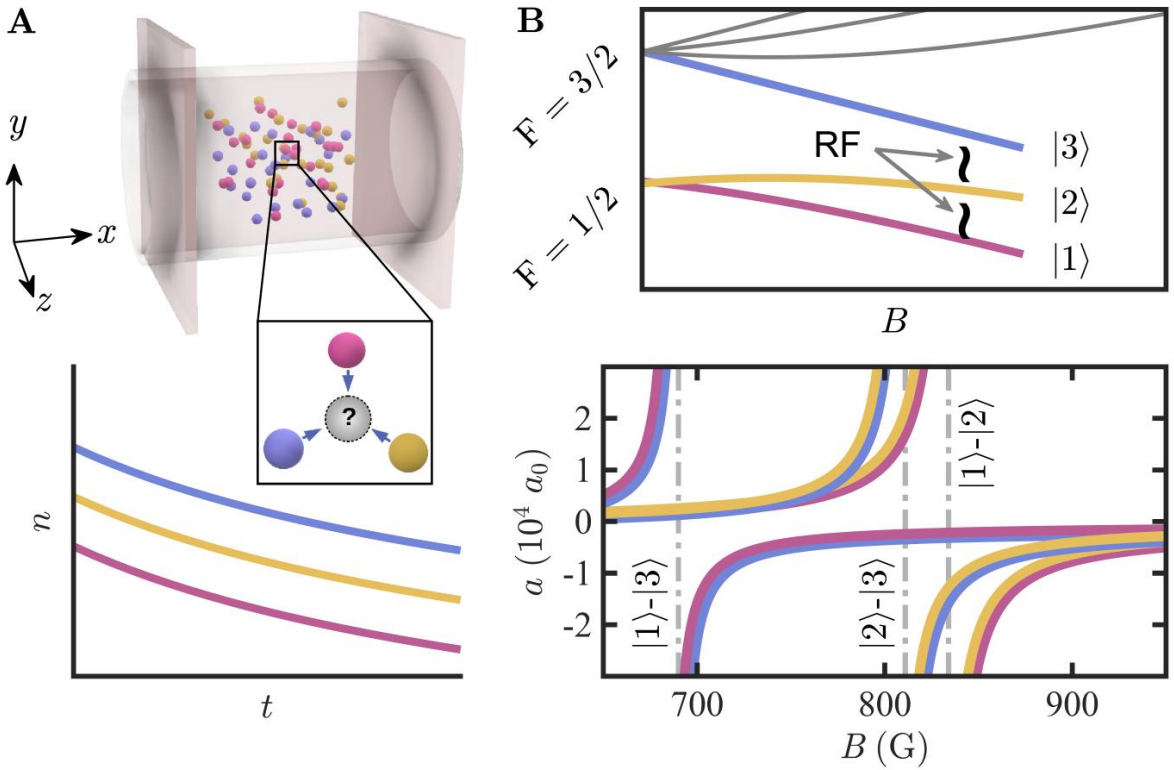
Breezing mode frequency

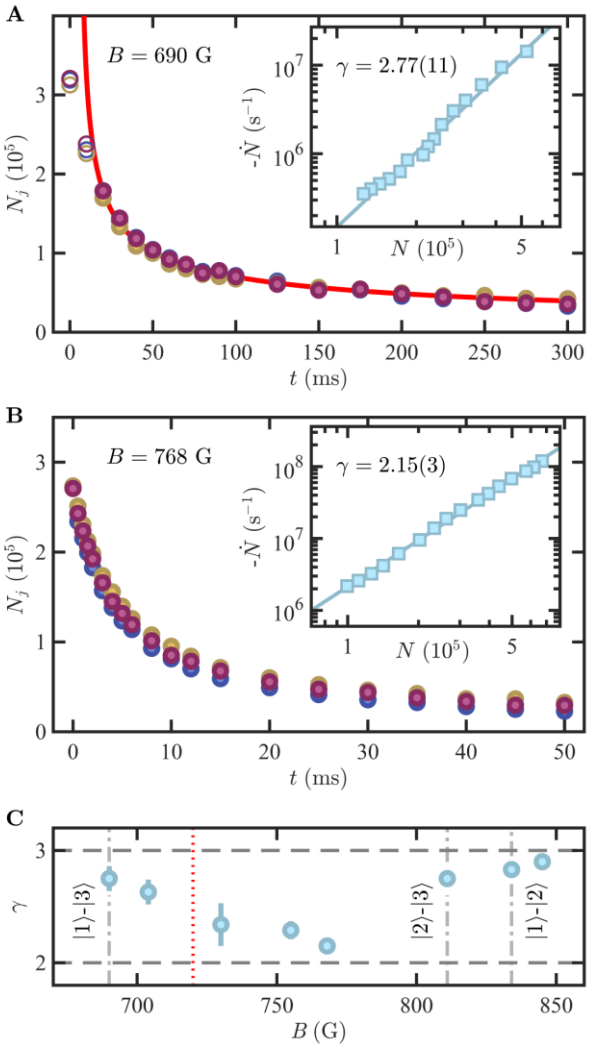
$$\omega_b = 2\omega_\perp\sqrt{1 + E_3/E_{\mathrm{pot}}}$$



# Recent experiments of three-component Fermi gases

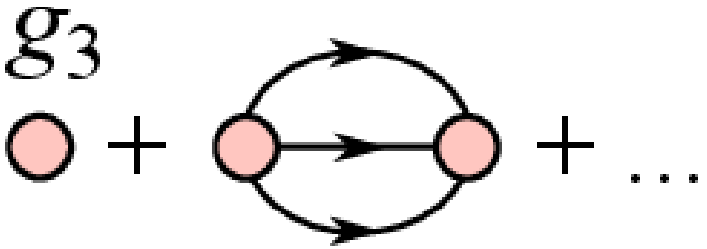
G. L. Schumacher, et al., arXiv:2301.92237



# Three-body $T$-matrix for three-body interaction

**Three-body coupling constant $g_3$ can be represented by the three-body binding energy $\varepsilon_B$**

$T_3$ = 


$$T_3(P,\Omega_+) = \left[\frac{1}{g_3} - \Xi_0(P,\Omega_+)\right]^{-1}$$

$\Xi_0$: Three-body propagator in vacuum

$$\Xi_0(P,\Omega_+) = \sum_{k,q} \frac{1}{\Omega_+ - \varepsilon_{\frac{P}{3}+k-\frac{q}{2}} - \varepsilon_{\frac{P}{3}+q} - \varepsilon_{\frac{P}{3}-k-\frac{q}{2}}} = -\frac{m}{2\sqrt{3}\pi} \ln\left(\frac{\Lambda^2 + P^2/6 - m\Omega_+}{P^2/6 - m\Omega_+}\right)$$

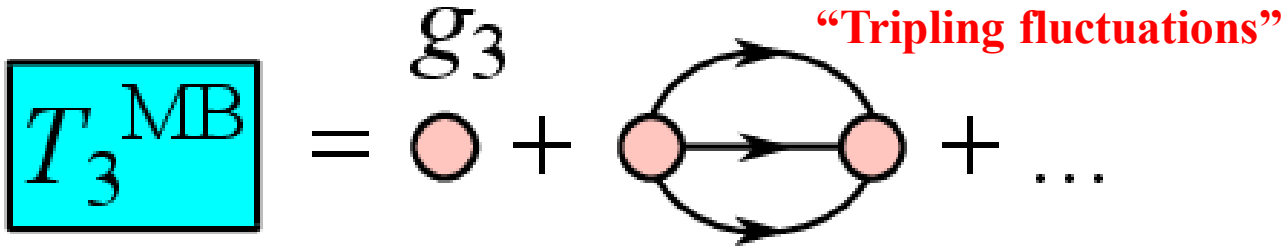
$\Lambda$: cutoff

**Three-body binding energy**

$$\frac{1}{g_3} - \Xi_0(0, \Omega = -\varepsilon_B) = 0 \quad \Rightarrow \quad \varepsilon_B = \frac{\Lambda^2}{m} \exp\left(\frac{2\sqrt{3}\pi}{m g_3}\right)$$

# In-medium three-body $T$-matrix

HT, S. Tsutsui, T. M. Doi, and K. Iida, Phys. Rev. Research **4**, L012021 (2022).




$$T_3^{\mathrm{MB}}(\boldsymbol{P}, i\Omega_n) = \left[\frac{1}{g_3} - \Xi(\boldsymbol{P}, i\Omega_n)\right]^{-1}$$

$\Xi$: In-medium three-particle (three-hole) propagator

$\Omega_n = (2n+1)\pi T$: Fermion Matsubara frequency

## In-medium three-body equation

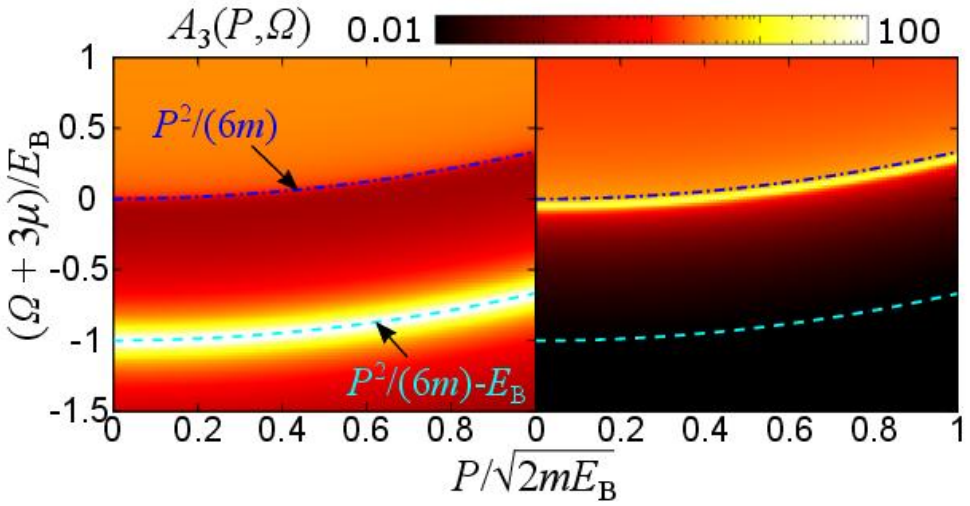
$$\frac{1}{g_3} - \Xi(\boldsymbol{P} = 0, \Omega = -E_{\mathrm{B}}^{\mathrm{M}}) = 0$$

# Three-body spectral function

HT, S. Tsutsui, T. M. Doi, and K. Iida, Phys. Rev. Research **4**, L012021 (2022).

In-medium three-body spectra

$$A_3(P,\Omega) = -\mathrm{Im}T_3^{\mathrm{MB}}(P,\Omega_+)$$

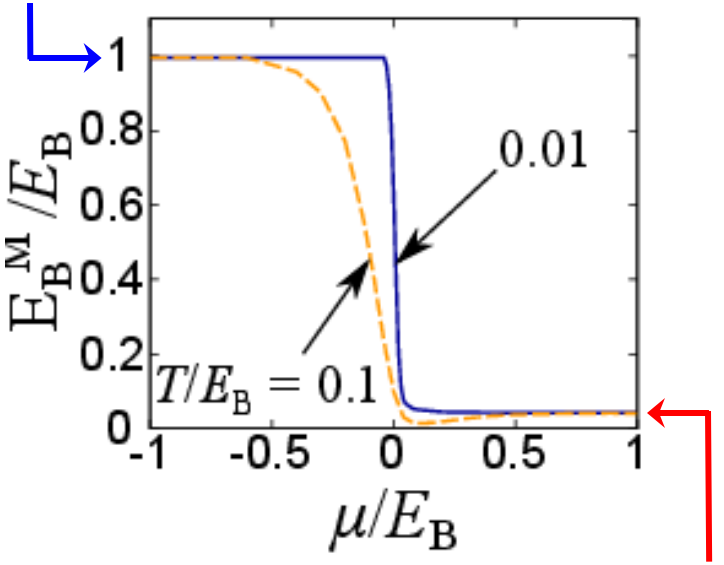



Low density
$(\mu/E_B = -1)$

High-density
$(\mu/E_B = 2)$

In-medium three-body binding energy

**Three-body problem**




$E_B^M/E_B \simeq 0.04$

**The three-body pole survives even at high density**

# Non-relativistic trace anomaly

Trace anomaly equation

$$2\hat{H} - \hat{T}_{xx} = -\frac{g_3^2}{\sqrt{3}\pi}(\psi_{\mathrm{r}}^{\dagger}\psi_{\mathrm{r}})(\psi_{\mathrm{g}}^{\dagger}\psi_{\mathrm{g}})(\psi_{\mathrm{b}}^{\dagger}\psi_{\mathrm{b}})$$

$\hat{T}_{ij}$: energy-momentum tensor

W. S. Dasa, et al., Mod. Phys. Lett. A **34**, 1950291 (2019).

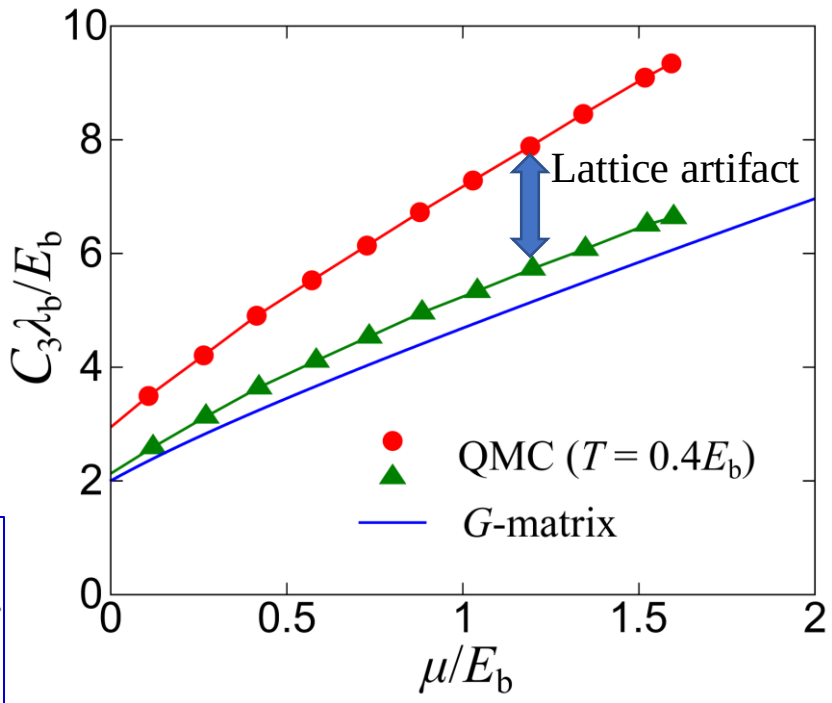
Three-body contact

Statistical average: $2E - P = C_3$

$$C_3 = \frac{8\sqrt{3}}{3\pi}\rho E_{\mathrm{F}}\frac{3E_{\mathrm{F}}/E_{\mathrm{b}}}{\left(1+\frac{3E_{\mathrm{F}}}{E_{\mathrm{b}}}\right)\left[\ln\left(1+\frac{3E_{\mathrm{F}}}{E_{\mathrm{b}}}\right)\right]^2}$$

$E$: energy density  $P$: pressure

Comparison with QMC results

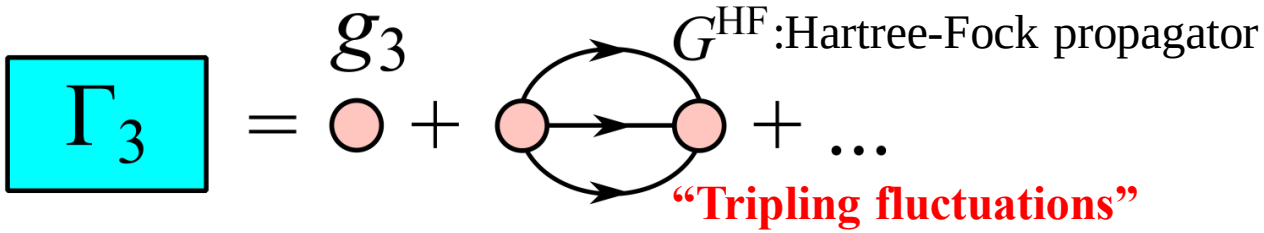



$\lambda_{\mathrm{b}} = \sqrt{2\pi/mE_{\mathrm{b}}}$: length scale associated eith $E_b$

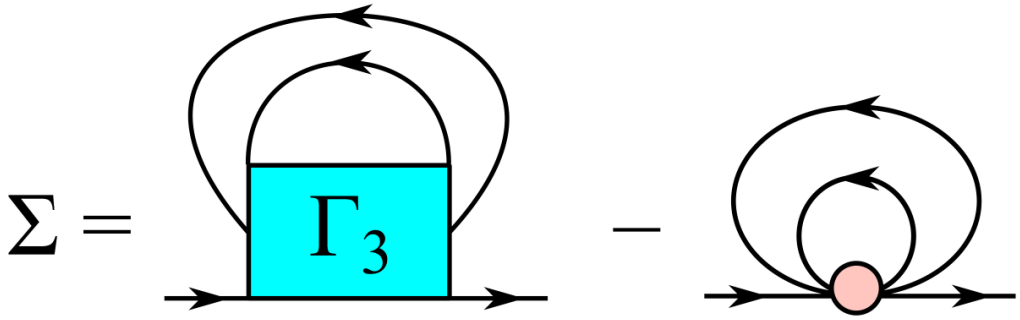
QMC: J. McKenny, *et al.*, PRA **102**, 023313 (2020).

# Nozières-Schmitt-Rink-type approach for the three-body crossover

**In-medium three-body *T*-matrix**




**Self-energy for tripling fluctuations**




**Dyson equation**

$$G_k(i\omega_\ell) = G_k^{\mathrm{HF}}(i\omega_\ell) + G_k^{\mathrm{HF}}(i\omega_\ell)\Sigma_k(i\omega_\ell)G_k(i\omega_\ell)$$

$$\simeq G_k^{\mathrm{HF}}(i\omega_\ell) + \left[G_k^{\mathrm{HF}}(i\omega_\ell)\right]^2 \Sigma_k(i\omega_\ell)$$

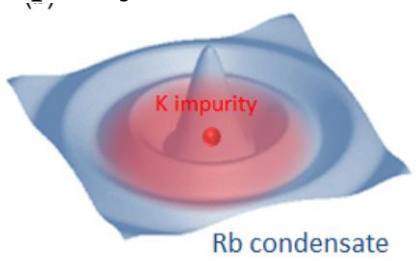
Truncated at O(Σ) (NSR approximation)  PPNP **111**, 103739 (2020).

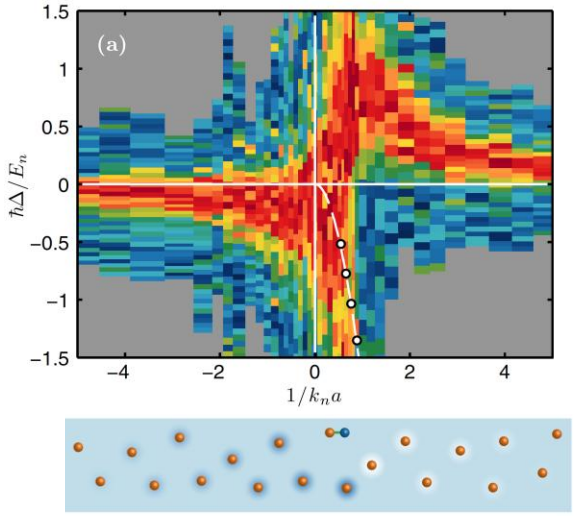
# Yukawa interaction in cold atoms

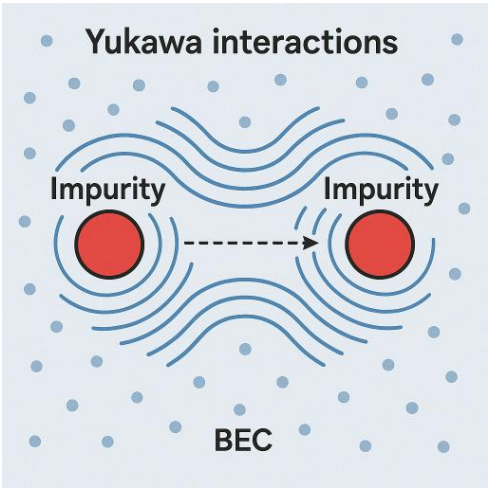
Two-polaron interaction in BEC is induced by exchange of superfluid phonons (analogous to pion exchange)

**“Bose” polaron**

Impurity immersed in BEC




by Chat GPT画伯

$$V_{2b}(r) = -\frac{\alpha}{r} e^{-r/\xi}$$

$\xi$: BEC healing length

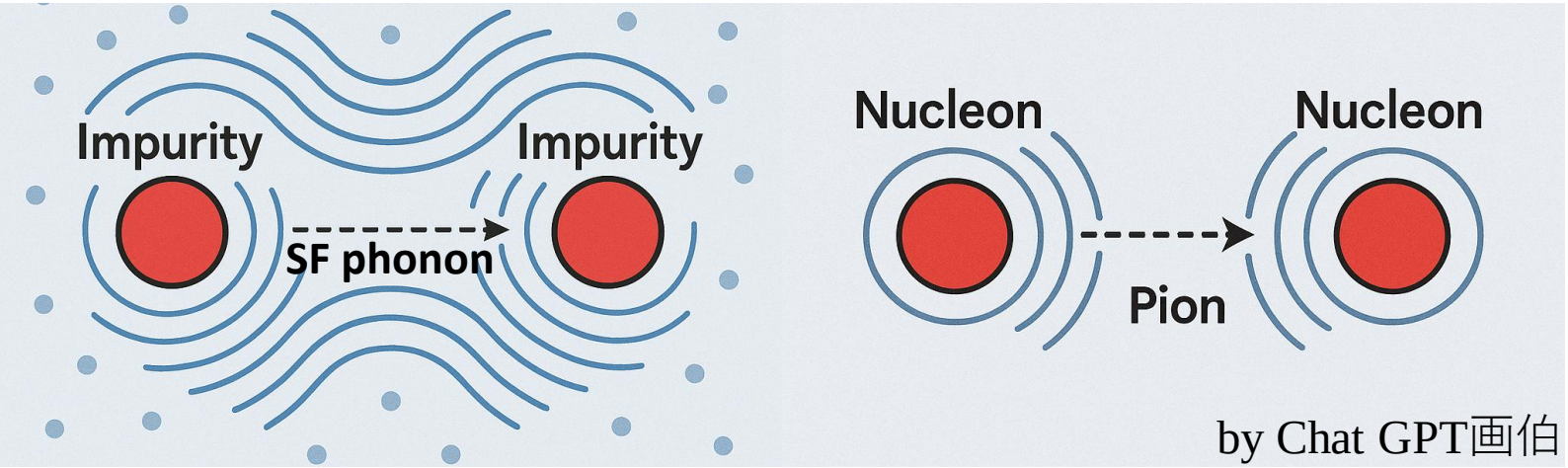
Nils B. Jørgensen, et al., PRL **117**, 055302 (2016).

M. J. Mijslma, et al., PRA **61**, 053601 (2000).

# Analogy between polaron and nucleon

**Inter-polaron force**

**Nuclear force**




$$V_{2\mathrm{b}}(r) = -\frac{\alpha}{r} e^{-r/\xi}$$

$\xi$: BEC healing length

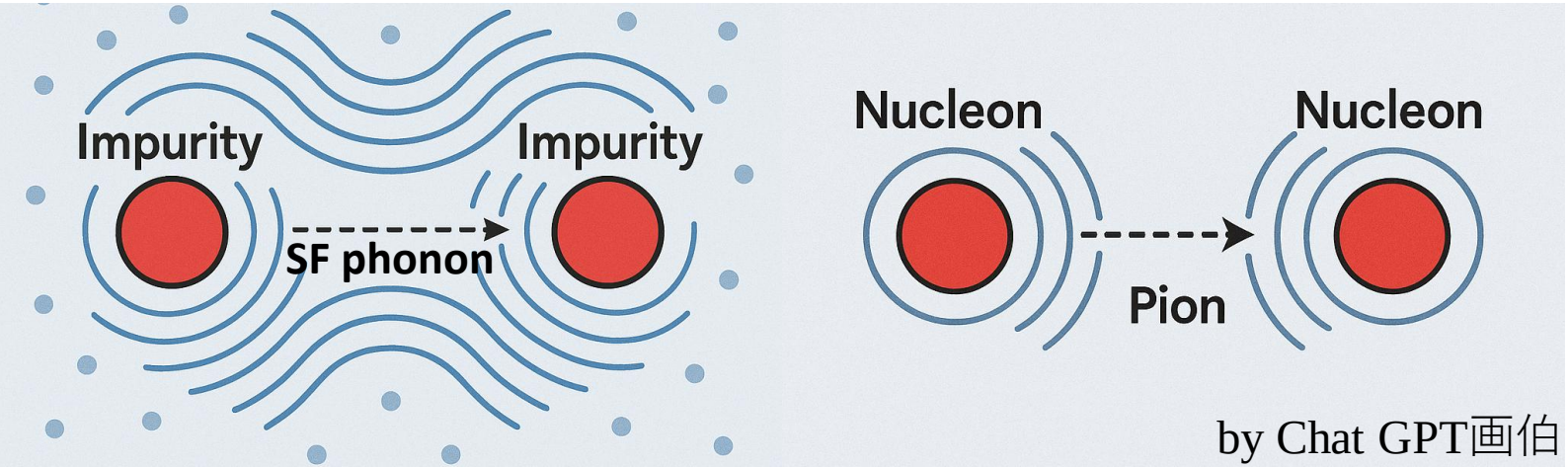
$$V_{2\mathrm{b}}(r) = -\frac{\alpha'}{r} e^{-m_\pi r}$$

$m_\pi^{-1}$: inverse pion mass

# Analogy between polaron and nucleon

**Inter-polaron force**

**Nuclear force**




$$V_{2\mathrm{b}}(r) = -\frac{\alpha}{r} e^{-r/\xi}$$

$\xi$: BEC healing length

$$V_{2\mathrm{b}}(r) = -\frac{\alpha'}{r} e^{-m_\pi r}$$

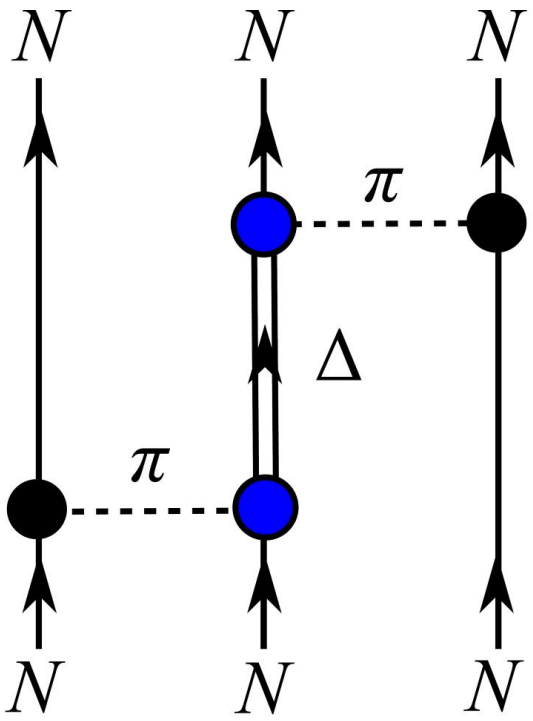
$m_\pi^{-1}$: inverse pion mass

**Fujita-Miyazawa counterpart in three-polaron force?**

# Fujita-Miyazawa-type three-body force among polarons

HT, E. Nakano, and K. Iida, arXiv:2505.19117

**Nucleon**
**Pion**
**Δ resonance**



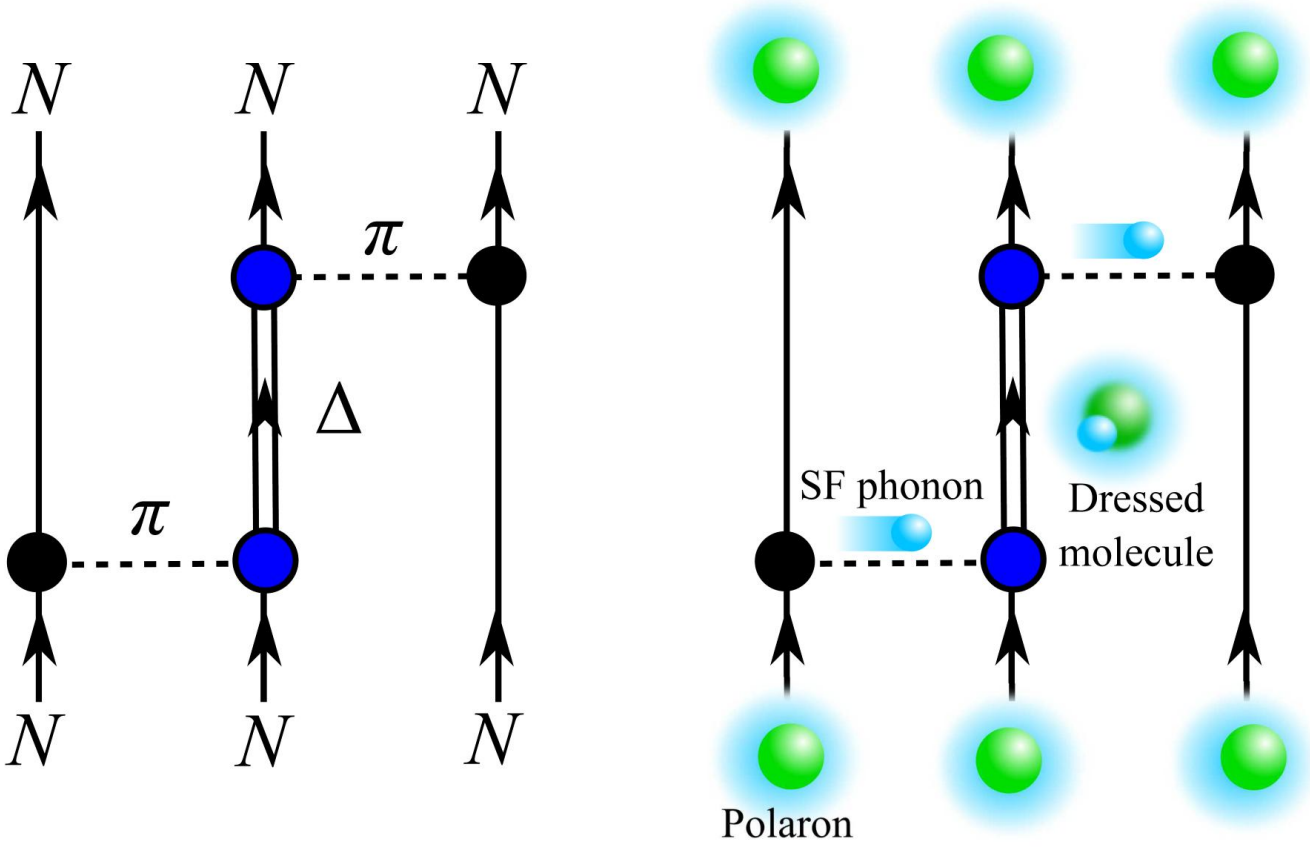
# Fujita-Miyazawa-type three-body force among polarons

HT, E. Nakano, and K. Iida, arXiv:2505.19117

**Nucleon ⇔ polaron**

**Pion ⇔ superfluid phonon**

**Δ resonance ⇔ Feshbach molecule (closed-channel)**



# Two-channel model of cold atoms near the Feshbach resonance

$$\hat{H} = \sum_{\boldsymbol{k}} \left[ \xi_{\boldsymbol{k},b} \hat{b}^\dagger_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}} + \xi_{\boldsymbol{k},c} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \right] + \sum_{\boldsymbol{P}} \xi_{\boldsymbol{P},A} \hat{A}^\dagger_{\boldsymbol{P}} \hat{A}_{\boldsymbol{P}}$$

$$+ \frac{U_{bb}}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^\dagger_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}^\dagger_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}_{-\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \hat{b}_{\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}}$$

$$+ U_{bc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^\dagger_{\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}^\dagger_{-\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} \hat{c}_{-\boldsymbol{k}'+\frac{M_c}{M_A}\boldsymbol{P}} \hat{b}_{\boldsymbol{k}'+\frac{M_b}{M_A}\boldsymbol{P}}$$

$$+ g \sum_{\boldsymbol{P},\boldsymbol{k}} \left[ \hat{A}^\dagger_{\boldsymbol{P}} \hat{b}_{-\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}_{\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} + \text{h.c.} \right]$$

Kinetic energies:

$$\xi_{\boldsymbol{k},b} = k^2/(2M_b) - \mu_b$$
$$\xi_{\boldsymbol{k},c} = k^2/(2M_c) - \mu_c$$
$$\xi_{\boldsymbol{P},A} = P^2/(2M_A) - \mu_b - \mu_c + \nu$$

$\mu_{b,c}$: chemical potential
$M_{b,c,A}$: mass
$\nu(B)$: closed-channel energy

# Two-channel model of cold atoms near the Feshbach resonance

**Medium boson** **Impurity** **Closed-channel molecule**

$$\hat{H} = \sum_{\boldsymbol{k}} \left[ \xi_{\boldsymbol{k},b} \hat{b}^\dagger_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}} + \xi_{\boldsymbol{k},c} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \right] + \sum_{\boldsymbol{P}} \xi_{\boldsymbol{P},A} \hat{A}^\dagger_{\boldsymbol{P}} \hat{A}_{\boldsymbol{P}}$$

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# Two-channel model of cold atoms near the Feshbach resonance

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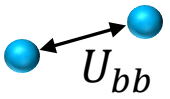


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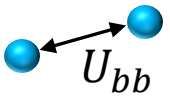


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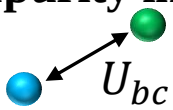


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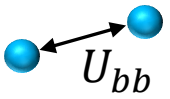


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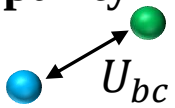


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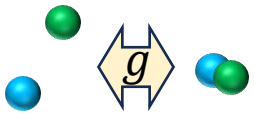


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# Polaron in Bose-Einstein condensate

$$b_{\boldsymbol{k}} = \sqrt{n_0}\delta_{\boldsymbol{k},\boldsymbol{0}} + \hat{\pi}_{\boldsymbol{k}}(1 - \delta_{\boldsymbol{k},\boldsymbol{0}}),$$

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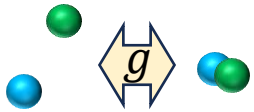

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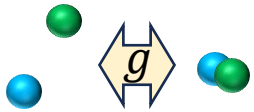
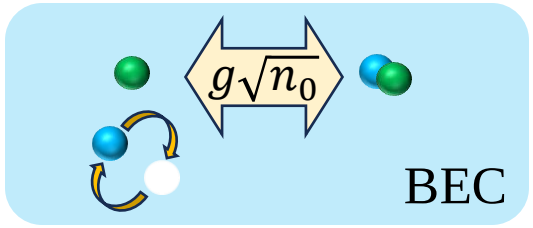



**Coherent atom-molecule mixing**

$$g\sqrt{n_0}\sum_{\boldsymbol{k}}\left(c^{\dagger}_{\boldsymbol{k}}A_{\boldsymbol{k}} + A^{\dagger}_{\boldsymbol{k}}c_{\boldsymbol{k}}\right)$$



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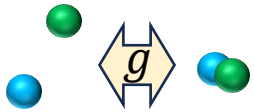


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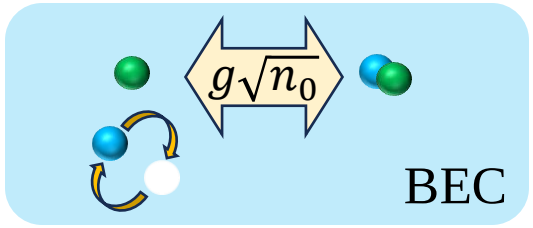



**Nucleon-like and Δ-like polarons as diagonalized eigenstates**

$$\hat{H} = \hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi + \hat{V}$$

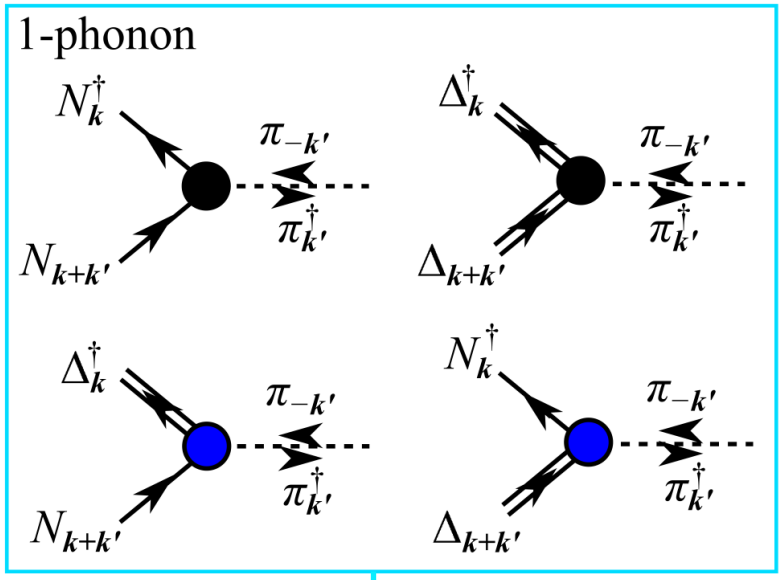
Ground state: $\hat{H}_N = \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k},N}\hat{N}^{\dagger}_{\boldsymbol{k}}\hat{N}_{\boldsymbol{k}}$

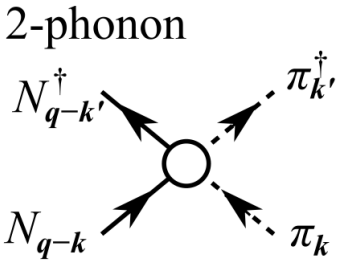
Excited state: $\hat{H}_\Delta = \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k},\Delta}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}$

Bogoliubov Hamiltonian for pion-like boson excitation

$$\hat{H}_\pi = \sum_{\boldsymbol{k}}(\xi_{\boldsymbol{k},b} + 2U_{bb}n_0)\hat{\pi}^{\dagger}_{\boldsymbol{k}}\hat{\pi}_{\boldsymbol{k}} + \frac{U_{bb}n_0}{2}\sum_{\boldsymbol{k}}\left[\hat{\pi}^{\dagger}_{\boldsymbol{k}}\hat{\pi}^{\dagger}_{-\boldsymbol{k}} + \hat{\pi}_{-\boldsymbol{k}}\hat{\pi}_{\boldsymbol{k}}\right]$$

# Absorption and emission of pion-like boson excitations

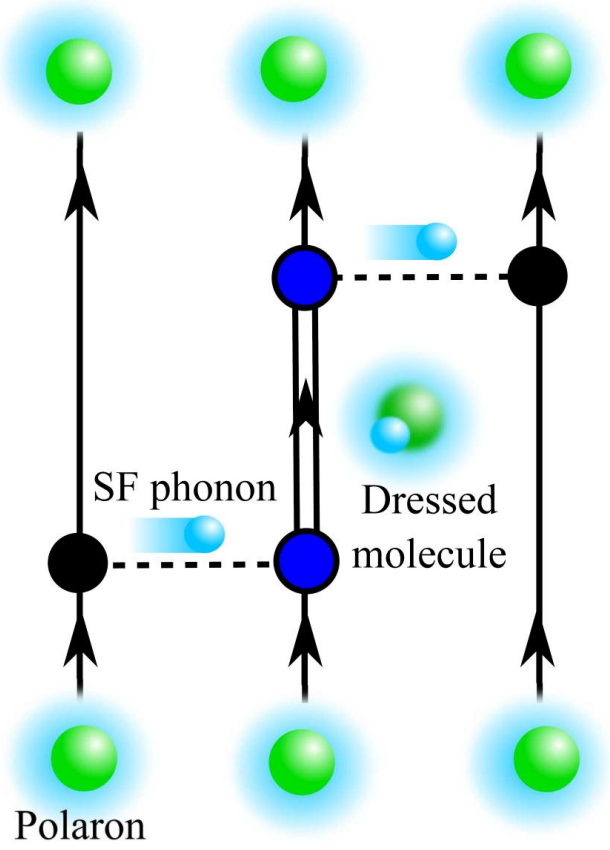



*Higher order at $n_0 \gg 1$

$$\hat{V} = \sum_{\boldsymbol{k},\boldsymbol{k}'} \Big[ f^{NN\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{N}^\dagger_{\boldsymbol{k}} \hat{N}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} + f^{\Delta\Delta\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Delta}^\dagger_{\boldsymbol{k}} \hat{\Delta}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} + f^{\Delta N\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Delta}^\dagger_{\boldsymbol{k}} \hat{N}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} + f^{N\Delta\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{N}^\dagger_{\boldsymbol{k}} \hat{\Delta}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} \Big] + \text{h.c.}$$

# Fujita-Miyazawa three-body force




$$\hat{V}_{\mathrm{FM}} = \frac{1}{6} \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{q}_1,\boldsymbol{q}_2} U_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}(\boldsymbol{q}_1,\boldsymbol{q}_2) \times \hat{N}^\dagger_{\boldsymbol{k}_1} \hat{N}^\dagger_{\boldsymbol{k}_2} \hat{N}^\dagger_{\boldsymbol{k}_3} \hat{N}_{\boldsymbol{k}_3-\boldsymbol{q}_1} \hat{N}_{\boldsymbol{k}_2+\boldsymbol{q}_1-\boldsymbol{q}_2} \hat{N}_{\boldsymbol{k}_1+\boldsymbol{q}_2}$$

**$2\pi$-exchange-like form of coupling strength**

$$U_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}(\boldsymbol{q}_1,\boldsymbol{q}_2) = -6 \mathcal{G}^{\Delta}_{\boldsymbol{k}_2+\boldsymbol{q}_1} \mathcal{G}^{N\pi\Delta}_{\boldsymbol{k}_1,\boldsymbol{q}_2,\boldsymbol{k}_2+\boldsymbol{q}_1} \mathcal{G}^{\Delta\pi N}_{\boldsymbol{k}_2,\boldsymbol{q}_1,\boldsymbol{k}_3}$$

$\Delta$ prop. $\pi$-like SF phonon prop. with form factors

At $g \ll U_{bc}\sqrt{n_0}$

$$U_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}(\boldsymbol{q}_1,\boldsymbol{q}_2) \propto \frac{1}{(\boldsymbol{q}_1^2+\xi^{-2})(\boldsymbol{q}_2^2+\xi^{-2})}$$

$\xi$: BEC healing length

# How to measure?

**Interaction energy in the impurity equation of state**

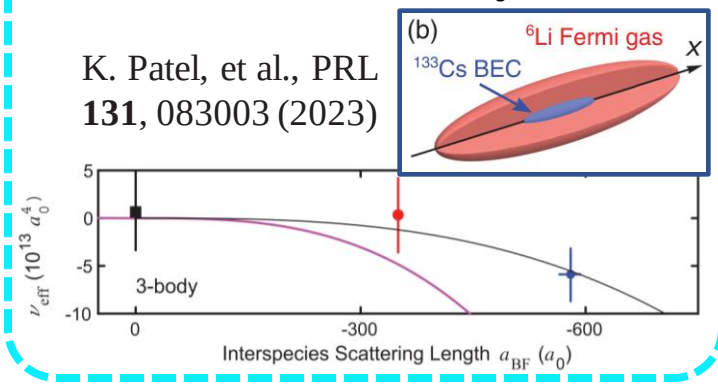
$$\delta E_3 \propto U_{0,0,0}(\mathbf{0},\mathbf{0}) n_N^3$$

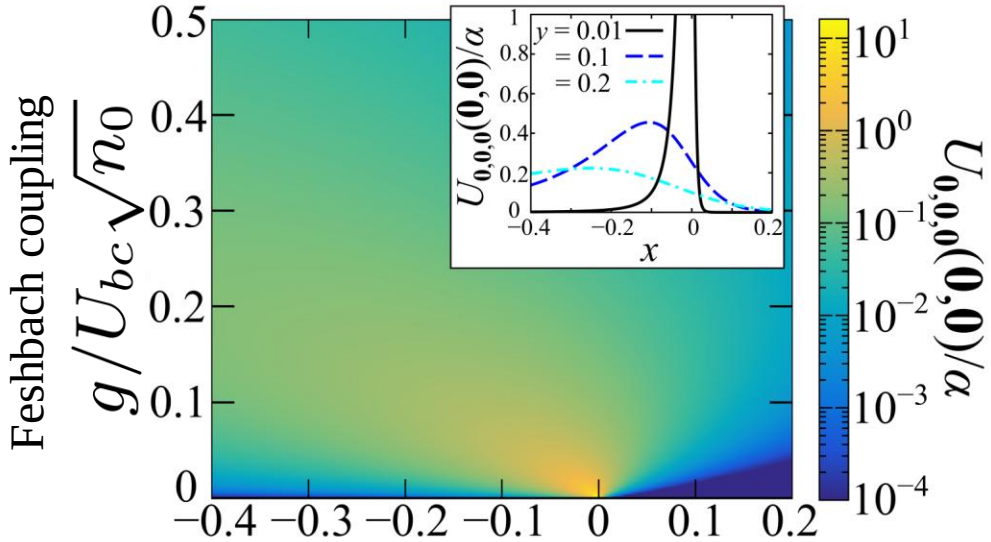
$n_N$: ground-state polaron density

$$U_{0,0,0}(\mathbf{0},\mathbf{0}) = \frac{\alpha y^2 (1 - x/2)^2}{(x^2 + 4y^2)^{3/2}} \left( \frac{1}{2} - \frac{y^2 + x/2}{2\sqrt{x^2 + 4y^2}} \right)^2$$

**Observation of fermion-mediated three-body force**

K. Patel, et al., PRL **131**, 083003 (2023)




**Tunable via $\nu(B)$!**

Closed-channel energy level: $\{\nu - (U_{bc} + U_{bb})n_0\}/U_{bc}n_0$

# Hamiltonian effective field theory based on the open-system description

We do not have to resort to path integral formalism

**Grand-canonical partition function**

$$Z = \mathrm{Tr}\left[e^{-\beta(\hat{H}_N+\hat{H}_\Delta+\hat{H}_\pi+\hat{V})}\right]$$

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**Equation for effective interaction**

$$e^{-\beta\hat{V}_{\mathrm{eff}}} = \mathrm{Tr}_{\Delta\pi}\left[e^{-\beta(\hat{H}_\Delta+\hat{H}_\pi)}\hat{S}(\beta)\right]$$

**S-matrix operator**

$$\hat{S}(\beta) = T_\tau \exp\left[-\int_0^\beta d\tau\, \hat{V}(\tau)\right]$$

Interaction representation in the imaginary time formalism

$$\hat{V}(\tau) = e^{\tau(\hat{H}_N+\hat{H}_\Delta+\hat{H}_\pi)}\hat{V}e^{-\tau(\hat{H}_N+\hat{H}_\Delta+\hat{H}_\pi)}$$

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**Perturbative expression of the effective interaction**

$$\hat{V}_{\mathrm{eff}} = \sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}}{\ell!\beta}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_\ell\,\langle T_\tau[\hat{V}(\tau_1)\cdots\hat{V}(\tau_\ell)]\rangle$$

$$\langle\cdots\rangle = \mathrm{Tr}_{\pi\Delta}[e^{-\beta(\hat{H}_\Delta+\hat{H}_\pi)}\cdots]/\mathrm{Tr}_{\pi\Delta}[e^{-\beta(\hat{H}_\Delta+\hat{H}_\pi)}]$$