Collective coordinate quantization of solitons with topological terms

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Introduction

θ^+ pentaquark $(uudd\bar{s})$ $m_{\theta^+} \approx 1530$ MeV?, $\Gamma_{\theta^+} \approx 15$ MeV?

- Predicted from Collective Coordinate Quantization (CCQ) in the Skyrme+WZW model

(Diakonov, Petrov, and Polyakov, 1997)

- Not found in experiments & Lattice

In a presence of **topological terms**,

CCQ needs modification (Adkins and Nappi, 1983), (Cherman et al, 2005)

Modifying CCQ

→ New constraints to the multiplets

→ θ^+ in fact do **<u>not</u>** exist

Review : Collective coordinate quantization

U_{sol}: soliton solution, $u \coloneqq U - U_{\mathrm{sol}}$: fluctuation

$$S[U] = S[U_{\mathrm{sol}}] + \int \mathrm{d}t \left(\frac{1}{2} (\dot{u}^T M \dot{u} - u^T K u) + A(u)^T \dot{u} \cdots \right)$$

Zero modes $\{\xi_a\} \in \mathrm{ker} K$

$$\Updownarrow$$

Collective coordinate $U_{\mathrm{sol}}(X^a)$, $\xi_a = \partial_a U_{\mathrm{sol}}$

Naïve CCQ : $U(t) = U\big(X^a(t)\big) \Leftrightarrow u = X^a(t) \xi_a$

$$S_{\mathrm{naive}}[X^a(t)] = \int \mathrm{d}t \left(\frac{1}{2} M_{ab} \dot{X}^a \dot{X}^b + \mathcal{A}_a \dot{X}^a \right)$$

particle with metric $M_{ab} = \xi_a^T M \xi_b$, vector potential $\mathcal{A}_a \coloneqq A^T \xi_a$

Review : Skyrme-WZW model

$SU(3)$ Skyrme+WZW model

$$S[U] = \int_4 \mathrm{tr}\left(-\frac{f_\pi^2}{16} L_\mu^2 + \frac{1}{32e^2}\left[L_\mu, L_\nu\right]^2\right) + \frac{iN_c}{240\pi^2}\int_5 \epsilon^{MNPQR}\,\mathrm{tr}\left(L_M L_N L_P L_Q L_R\right)$$

$L_\mu = \partial_\mu U U^\dagger$

$N_B = 1$ solution

$$U_{\mathrm{sol}}(\vec{x}) = \exp\left(2if(r)n^i\tau_i\right)$$

Zero modes : $u_{a\neq 8} = i[\tau_{a\neq 8}, U_{\mathrm{sol}}]$, $(\&\ \partial_i U_{\mathrm{sol}})$

$$\Updownarrow$$

$SU(3)/U(1)$ Collective coordinate

$$U_{\mathrm{sol}}(\vec{x}) \to V U_{\mathrm{sol}}(\vec{x}) V^\dagger, \qquad V \simeq V\exp(-i\theta^8\tau_8)$$

Review : Skyrme-WZW model

CCQ :

$$U(x) = V(t) U_{\mathrm{sol}}(\vec{x}) V^\dagger(t), \qquad V(t) \in SU(3) \qquad V^\dagger \dot{V} = i\Omega_R^a \tau_a$$

$$S_{\mathrm{naive}}[V] = \int \mathrm{d}t \left(\frac{I}{2} \left(\Omega_R^i\right)^2 + \frac{I'}{2} (\Omega_R^\alpha)^2 - \frac{N_c}{2\sqrt{3}} \Omega_R^8 \right)$$

$i = 1,2,3$ $\qquad$ $\alpha = 4,5,6,7$ $\qquad$ from the WZW term

Symmetries

$$SU(2)_R \subset SU(3)_R : \delta V = -iV\tau_i \,, SU(3)_V : \delta V = i\tau_a V$$

$$\psi(V) = \sum D^{(p,q)}_{I,I^3,Y:J,J^3,Y^R}(V)$$

$SU(3)$ Wigner D-matrix

Review : Skyrme-WZW model

1st class constraint ⇔ gauge d.o.f $\qquad V \simeq V\exp(-i\theta^8\tau_8)$

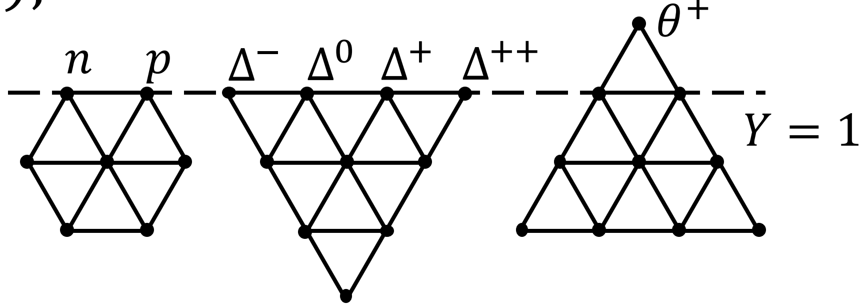
$$Y^R := -\frac{2}{\sqrt{3}}\frac{\partial L_{\mathrm{naive}}}{\partial \Omega_R^8} = \frac{N_c}{3} = 1$$

Representation (p,q) must include $Y = 1$ state

$$\therefore (p,q) = \underset{N}{(1,1)}, \underset{\Delta}{(3,0)}, (0,3), (2,2), \cdots$$

(0,3) has $Y = +2,\ I = 0$ state

⇒ θ^+ pentaquark : $(uudd\bar{s})$



(Diakonov, Petrov, and Polyakov, 1997)

However, θ^+ is not found in experiment & Lattice

Classification of zero modes

$$S[U] = S[U_{\mathrm{sol}}] + \int \mathrm{d}t \left(\frac{1}{2}(\dot{u}^T M \dot{u} - u^T K u) + \underbrace{\underline{\boldsymbol{A(u)^T \dot{u}}}}_{\text{From WZW term}} + \cdots \right)$$

EOM : $\delta S[U] \simeq M\ddot{u} - \underline{\boldsymbol{B\dot{u}}} + Ku = 0$

B : "magnetic field" from $A(u)$

$$u(t) = u_0(t) + \sum_m \alpha_m e^{-i\omega_m t} u_m + \mathrm{c.c.}$$

$$(-\omega_m^2 M + i\omega_m B + K)u_m = 0$$

For zero modes, **three types** of solutions

Classification of zero modes

① Dynamical zero modes

$$u_0 \simeq \left(X_0^i + V_0^i t\right)\xi_i + V_0^i \eta_i$$

- $K\eta_i = B\xi_i$ to cancel $B\dot{u}$ term (Adkins and Nappi, 1983),
- $B\xi_i \perp \mathrm{ker}K \Leftrightarrow \xi_i^T B\xi_a = 0, \forall \xi_a \in \mathrm{ker}K$ is necessary (Cherman et al, 2005)

② Cyclotron zero modes

$$u_0 \simeq \left(Z_0^A + W_0^A e^{-ib_A t}\right)\xi_A$$

$B\xi_A = -ib_A M\xi_A \quad (b_A \neq 0)$

③ Constraint zero modes

$$u_0 \simeq Y_0^\alpha \xi_\alpha$$

Only time-independent solutions

Classification of zero modes

Expansion of the action

$$S[U] = S_{\text{zero}} + S_{\text{massive}}$$

$$S_{\text{zero}}[X^a] = \int \mathrm{d}t \left(\frac{1}{2} G_{ab} \dot{X}^a \dot{X}^b + \mathcal{A}_a \dot{X}^a \right)$$

$$G_{ab} = \begin{pmatrix} M_{ij} + \boldsymbol{\eta}_i^T \boldsymbol{K} \boldsymbol{\eta}_j & & \\ & M_{AB} & \\ & & \mathbf{0} \end{pmatrix} \neq M_{ab}$$

① η_i contributes to the metric

② No kinetic terms for constraint zero modes

⇒Produce **constraints**

Generalized CCQ

① Classify zero modes

- Dynamical zero modes : $\xi_a^T B \xi_i = 0$
- Cyclotron zero modes : $\mathrm{B}\xi_a = -ib_A M \xi_a$
- Constraint zero modes : otherwise

② Replace the kinetic term from the naïve CCQ

- Dynamical : $M_{ij} \to G_{ij} = M_{ij} + \eta_i^T K \eta_j$
- Cyclotron : unchanged
- Constraint : remove the kinetic term

Skyrme-WZW model revisited

In Skyrme-WZW model, $\xi_a^T B \xi_b = -\frac{N_c}{2\sqrt{3}} f_{ab8}$

$i = 1,2,3$ are dynamical zero modes

$$M\xi_a = \begin{cases} \rho_I(r)\tau_i \\ \rho_{I'}(r)\tau_\alpha \end{cases}, B\xi_a = \begin{cases} 0 & \eta_i \equiv 0 \quad i = 1,2,3 \\ -\frac{N_c}{2\sqrt{3}} \rho_{N_B}(r) f_{\alpha\beta 8}\tau_\beta & \alpha,\beta = 4,5,6,7 \end{cases}$$

$\alpha = 4,5,6,7$ are constraint zero modes $\quad \because \rho_{I'} \not\propto \rho_{N_B}$

$$\therefore S_{\text{zero}}[V] = \int \mathrm{d}t \left(\frac{I}{2} \left(\Omega_R^i\right)^2 + \cancel{\frac{I'}{2} (\Omega_R^\alpha)^2} - \frac{N_c}{2\sqrt{3}} \Omega_R^8 \right)$$

Constraints

$$Y^R = 1, Q_\alpha^R \coloneqq \frac{\partial L_{\text{zero}}}{\partial \Omega_R^\alpha} = 0$$

Skyrme-WZW model revisited

$$Y^R = 1, Q^R_\alpha \coloneqq \frac{\partial L_{\mathrm{zero}}}{\partial \Omega^\alpha} = 0$$

$$[Q^R_\alpha, Q^R_\beta] \approx -i\frac{\sqrt{3}}{2} f_{\alpha\beta 8}$$

$Q^R_\alpha = 0$ are 2nd class

How to treat 2nd class constraints?

$$\text{Let } e_1^\pm \coloneqq Q_4^R \mp i Q_5^R,\ e_2^\pm \coloneqq Q_6^R \mp i Q_7^R$$

$$\Rightarrow [e_1^+, e_2^+] \approx [e_1^-, e_2^-] \approx 0$$

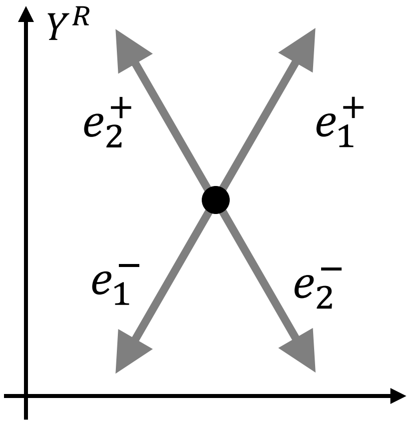
Assume physical states

$$e_1^+\psi = e_2^+\psi = 0, \text{ or } e_1^-\psi = e_2^-\psi = 0$$

Skyrme-WZW model revisited

Physical states

$$e_1^+\psi = e_2^+\psi = 0, \text{ or } e_1^-\psi = e_2^-\psi = 0$$



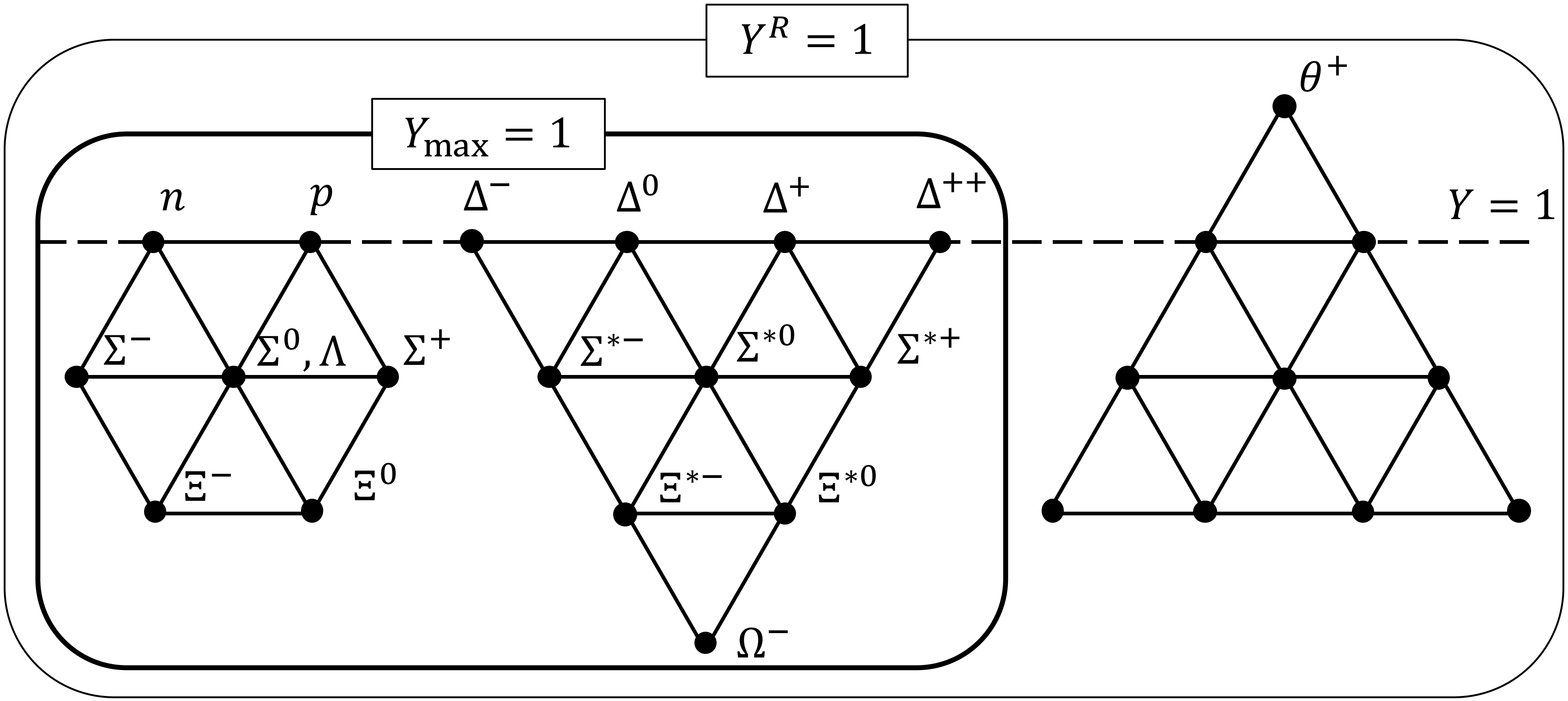
$$[Y^R, e_1^\pm] = \pm e_1^\pm$$
$$[Y^R, e_2^\pm] = \pm e_2^\pm$$

$$(e_1^\pm, e_2^\pm) \text{ raise (lower) } Y^R \Rightarrow Y^R = Y_{\mathrm{max}}, \text{or } Y_{\mathrm{min}}$$

$$\underline{\therefore Y^R = 1 = Y_{\mathrm{max}}}$$

Only representation with $Y_{\mathrm{max}} = 1$ is allowed

Skyrme-WZW model revisited



(1,1) and (3,0) are $Y_{\max} = 1$

(0,3) is $Y_{\max} = 2 \Rightarrow$ **Do not appear as physical state**

$\therefore$ $\boldsymbol{\theta^+}$ do not exist!!

8 and 10 are the only allowed multiplets

Conclusion

- There are 3 types of zero modes
- One of them produces constraints
- θ^+ is prohibited from the constraints

Future works

- Application to other models
- The origin of constraints

Backup : Non-zero modes

Non-zero modes :

$$u(t) = \sum_m \alpha_m e^{-i\omega_m t} \xi_m$$

$$(-\omega_m^2 M + i\omega_m B + K)\xi_m = 0$$

Features :

- $\xi_m^* \neq \xi_m$: $N - K$ interaction$\neq$ $N - \bar{K}$ interaction
- $\xi_m \not\angle \ker K$

General solution

$$\begin{aligned} &u(t) \\ &= \left(X_0^i + V_0^i t\right)\xi_i + Y_0^\alpha \xi_\alpha + \left(Z_0^A + W_0^A e^{-ib_A t}\right)\xi_A \\ &+ \sum_m \alpha_m e^{-i\omega_m t} \xi_m + \text{c.c.} \end{aligned}$$

Backup : Mode expansion

Expand the action in Hamiltonian formalism

$$S[u,\pi] = \int \mathrm{d}t\left(\pi^T \dot{u} - H(u,\pi)\right)$$

$$H(u,\pi) = \frac{1}{2}((\pi - A)^T M^{-1}(\pi - A) + u^T K u)$$

$$u(t) = X^i(t)\xi_i + V^i(t)\eta_i + \ Y^\alpha(t)\xi_\alpha + Z^A(t)\xi_A + \sum_m \alpha_m(t)\xi_m + \mathrm{c.c.}$$

$$\pi(t) = V^i(t)M\xi_i + W^A(t)M\xi_A + \sum_m -i\omega_m\alpha_m M\xi_m + \mathrm{c.c.} + \mathrm{A(u)}$$

$$S[u,\pi] = S_{\mathrm{zero}} + \sum_m S[\alpha^m]$$

$$S[\alpha^m] \propto \int \mathrm{d}t\left(-\mathrm{Im}\left(\alpha_m^\dagger \dot{\alpha}_m\right) - \omega_m \alpha_m^\dagger \alpha_m\right)$$

Backup : Mode expansion

Expand the action in Hamiltonian formalism

$$S_{\mathrm{zero}} = \int \mathrm{d}t \left(\left(g_{ij} V^i + \mathcal{A}_i \right) \dot{X}^j - \frac{1}{2} g_{ij} V^i V^j \right) + \int \mathrm{d}t \left((M_{AB} W^A + \mathcal{A}_A) \dot{Z}^B - \frac{1}{2} M_{AB} W^A W^B \right) + \int \mathrm{d}t \mathcal{A}_\alpha \dot{Y}^\alpha$$

Integrating out V^i, W^A,

$$S_{\mathrm{zero}} = \int \mathrm{d}t \left(\frac{1}{2} g_{ij} \dot{X}^i \dot{X}^j + \frac{1}{2} M_{AB} W^A W^B + \mathcal{A}_i \dot{X}^i + \mathcal{A}_\alpha \dot{Y}^\alpha + \mathcal{A}_A \dot{Z}^A \right)$$

Backup : Other approach

Other possibility :

$\theta^+ = N - K$ bound or resonance state?

Analysis of oscillating modes

⇒ already done in bound state approach

Neither bound states nor resonance are found

Callan, Hornbostel, and Klebanov (1988), Itzhaki et al (2004)

Backup : meson mass

Including meson mass term

$$L_{\text{mass}} \propto \mathrm{Tr}\big(\mathrm{Re}(MU)\big), M = \mathrm{diag}(m_\pi^2, m_\pi^2, m_K^2)$$

① $SU(3)$ coordinates

- Mass term⇒ potential term of $V(t)$
  & interaction with massive modes
- Structure of the kinetic term unchanged

The constraints are remained valid

② $SU(2)$ coordinates

$S \neq \mathbf{0}$ baryons ⇔ massive bound states

Discussed in Bound state approach