# Bayesian Hierarchical Model for cross calibration

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TADML,
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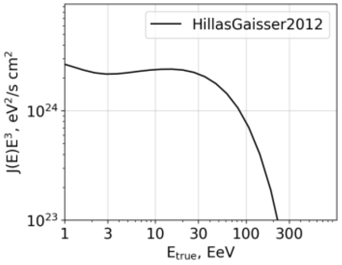
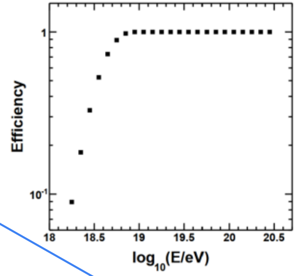
# Forward folding

CR spectrum



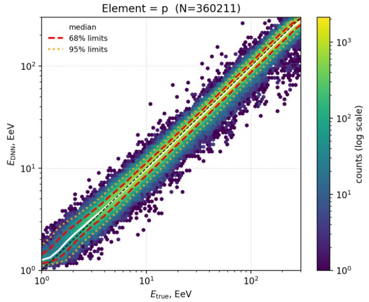

x

Efficiency




x

Response matrix/function

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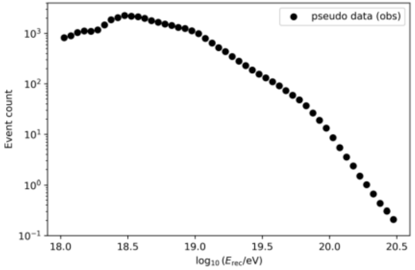
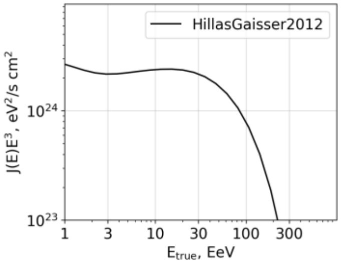
Event counts distribution




$$J^{\mathrm{raw}}(E_{\mathrm{SD}};\mathbf{s}) = \frac{\int \mathrm{d}\Omega \cos\theta \int \mathrm{d}E\, \epsilon(E,\theta) J(E;\mathbf{s}) \kappa(E_{\mathrm{SD}}|E;\theta)}{\int \mathrm{d}\Omega \cos\theta}$$

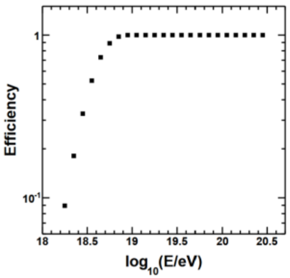
# Forward folding: log-likelihood minimization

CR spectrum




x

Efficiency




x

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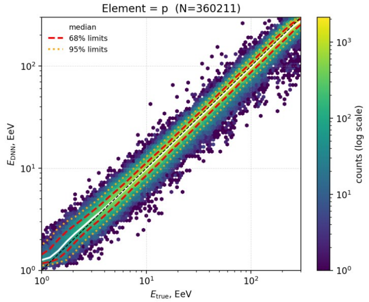


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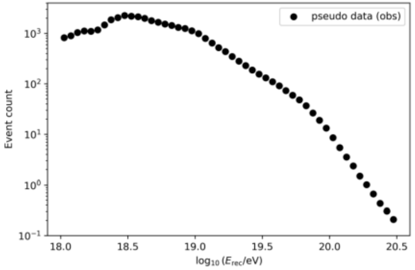



$$J^{\mathrm{raw}}(E_{\mathrm{SD}};\mathbf{s}) = \frac{\int \mathrm{d}\Omega \cos\theta \int \mathrm{d}E \epsilon(E,\theta) J(E;\mathbf{s}) \kappa(E_{\mathrm{SD}}|E;\theta)}{\int \mathrm{d}\Omega \cos\theta}$$

binning

$$R_{ij} = \frac{\int_{\Delta E_i} \mathrm{d}E_{\mathrm{SD}} \int_{\Delta E_j} \mathrm{d}E \int \mathrm{d}\Omega \cos\theta \kappa(E_{\mathrm{SD}}|E,\theta)\epsilon(E,\theta)J(E;\mathbf{s})}{\int_{\Delta E_j} \mathrm{d}E \int \mathrm{d}\Omega \cos\theta \; J(E;\mathbf{s})}$$

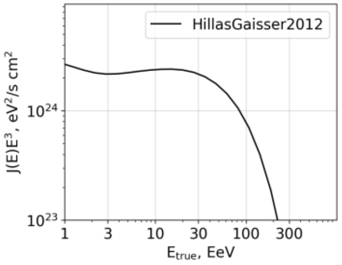
$$\nu_i = \sum_j R_{ij}\mu_j$$

Log Likelihood minimization:

$$\boxed{-\ln \mathcal{L}(\mathbf{s}) = \sum_i (\nu_i(\mathbf{s}) - N_i \ln \nu_i(\mathbf{s}))}$$

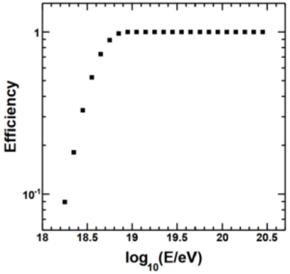
# Forward folding: MCMC

CR spectrum



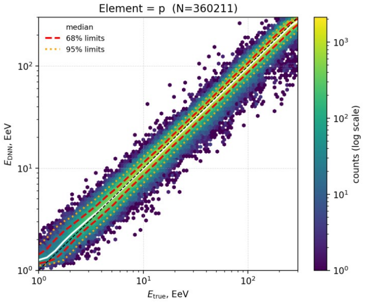

x

Efficiency




x

Response matrix/function

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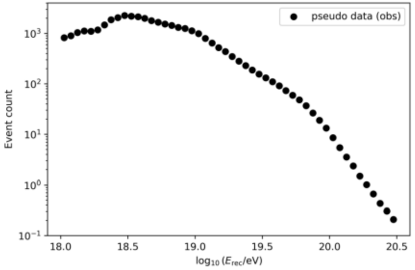
Event counts distribution




$$J^{\mathrm{raw}}(E_{\mathrm{SD}};\mathbf{s}) = \frac{\int \mathrm{d}\Omega\cos\theta \int \mathrm{d}E\,\epsilon(E,\theta)J(E;\mathbf{s})\kappa(E_{\mathrm{SD}}|E;\theta)}{\int \mathrm{d}\Omega\cos\theta}$$

binning

$$R_{ij} = \frac{\int_{\Delta E_i}\mathrm{d}E_{\mathrm{SD}}\int_{\Delta E_j}\mathrm{d}E\int \mathrm{d}\Omega\cos\theta\,\kappa(E_{\mathrm{SD}}|E,\theta)\epsilon(E,\theta)J(E;\mathbf{s})}{\int_{\Delta E_j}\mathrm{d}E\int\mathrm{d}\Omega\cos\theta\; J(E;\mathbf{s})}$$

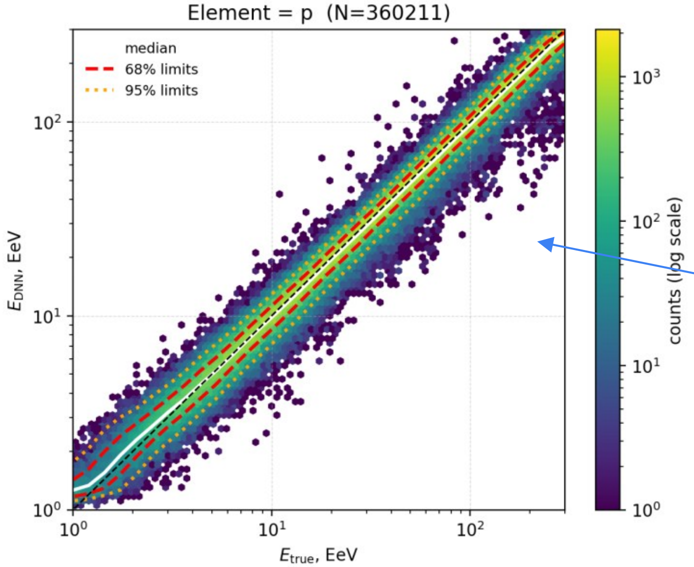
$$\nu_i = \sum_j R_{ij}\mu_j$$

Log posterior sampling using MCMC:

$$p(\theta \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \theta)\, p(\theta)}{p(\mathcal{D})}$$

$$\boxed{\log p(\theta \mid D) \propto \sum_{i=1}^{N_{E,\mathrm{rec}}} \left[N_i \log \nu_i(\theta) - \nu_i(\theta) - \log(N_i!)\right] + \log p(\theta)}$$

# Response (resolution) matrix




$$\nu_i = \int dE\, f(E) \int du\, \varepsilon(E,u) \int_{\hat{E}\in \mathrm{bin}\ i} \left[ \int ds\, p_{\mathrm{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u) \right] d\hat{E}.$$

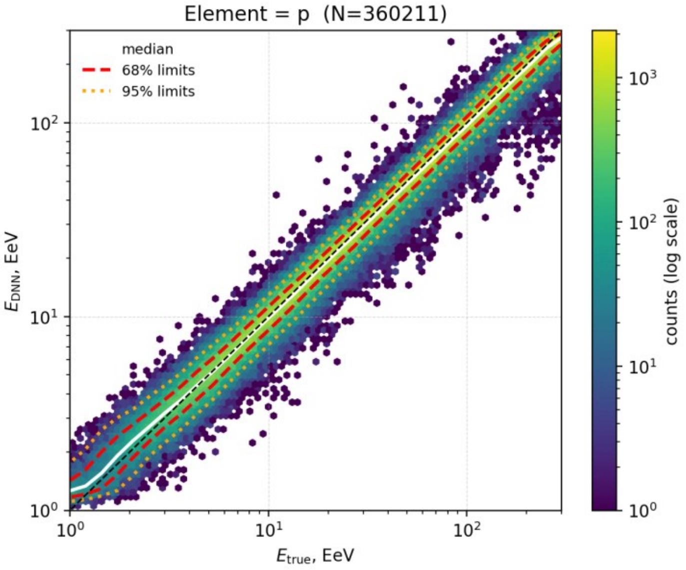
Resolution/smearing kernel:

$$\boxed{\kappa(\hat{E} \mid E, u) = \int ds\, p_{\mathrm{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u)}$$

$$\nu_i = \int dE\, f(E) \int du\, \varepsilon(E,u) \int_{\hat{E}\in \mathrm{bin}\ i} \kappa(\hat{E} \mid E, u)\, d\hat{E}$$

# Response (resolution) matrix

**Simulated** resolution kernel

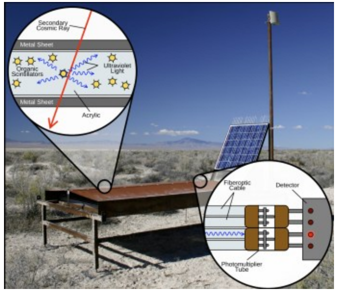



**Real** resolution/smearing kernel:

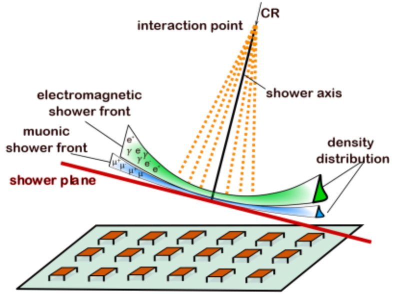
$$\kappa(\hat{E} \mid E, u) = \int ds\, p_{\mathrm{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u)$$

Detector fluctuations/smearing:

- sampling/LDF
- Scintillator/PMT

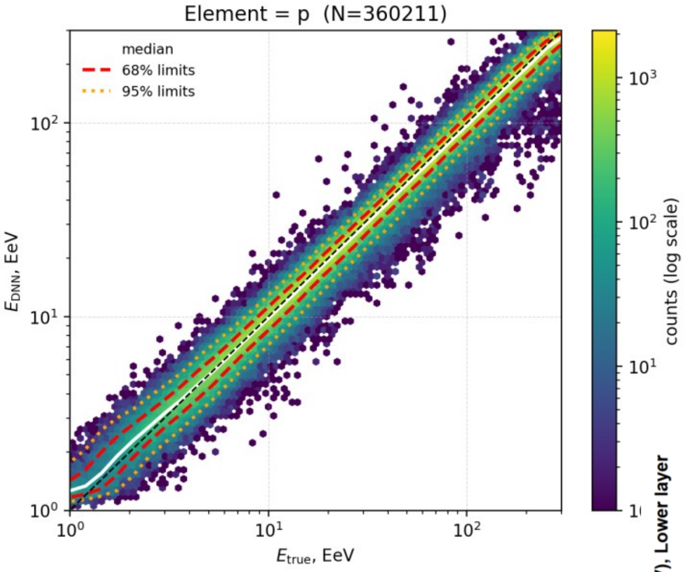


Shower size (s) fluctuations



# Response (resolution) matrix

**Simulated** resolution kernel



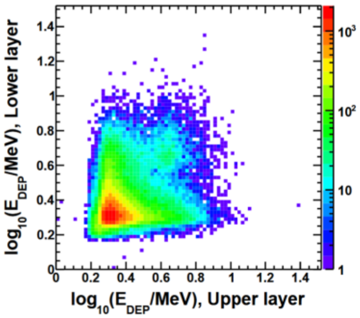

**Simulated** resolution/smearing kernel:

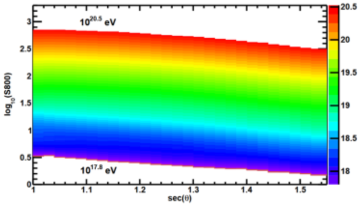
$$\boxed{\kappa(\hat{E} \mid E, u) = \int ds\, p_{\mathrm{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u)}$$

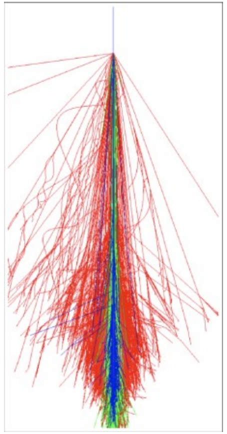
Detector fluctuations/smearing:

- Geant4
- Reconstruction (Std or DNN)

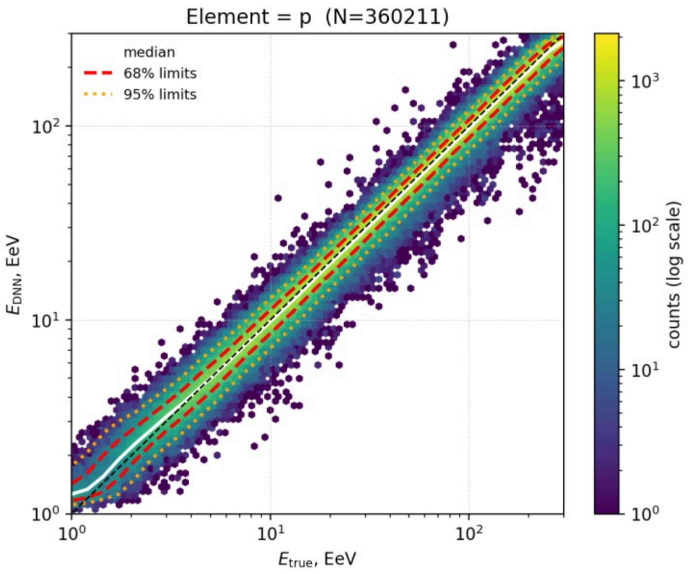
Shower size (s) fluctuations (CORSKA MC)



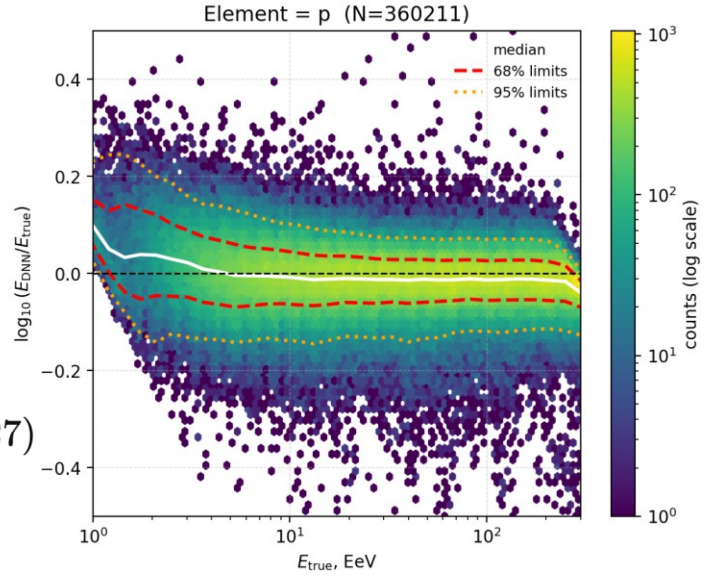



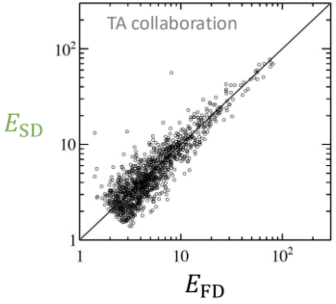



# Standard calibration



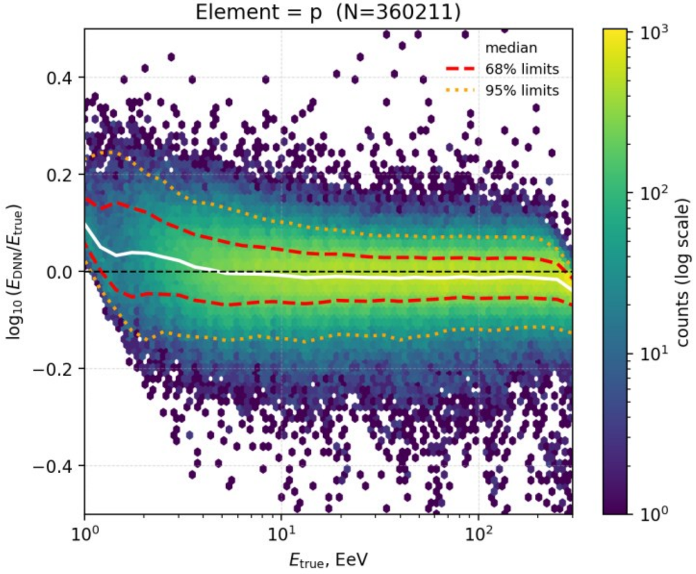

$\log_{10}(1.27)$

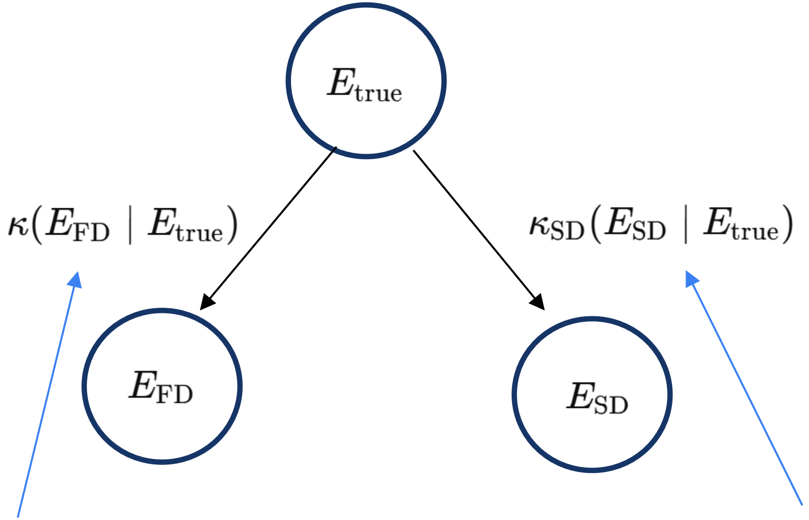



$$E = \frac{E_{\mathrm{SD}}}{1.27}$$

# Calibration with Hierarchical MCMC



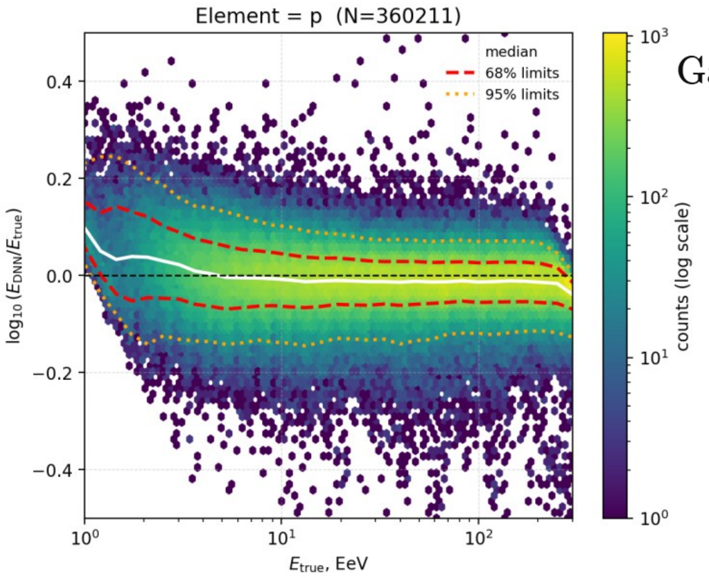



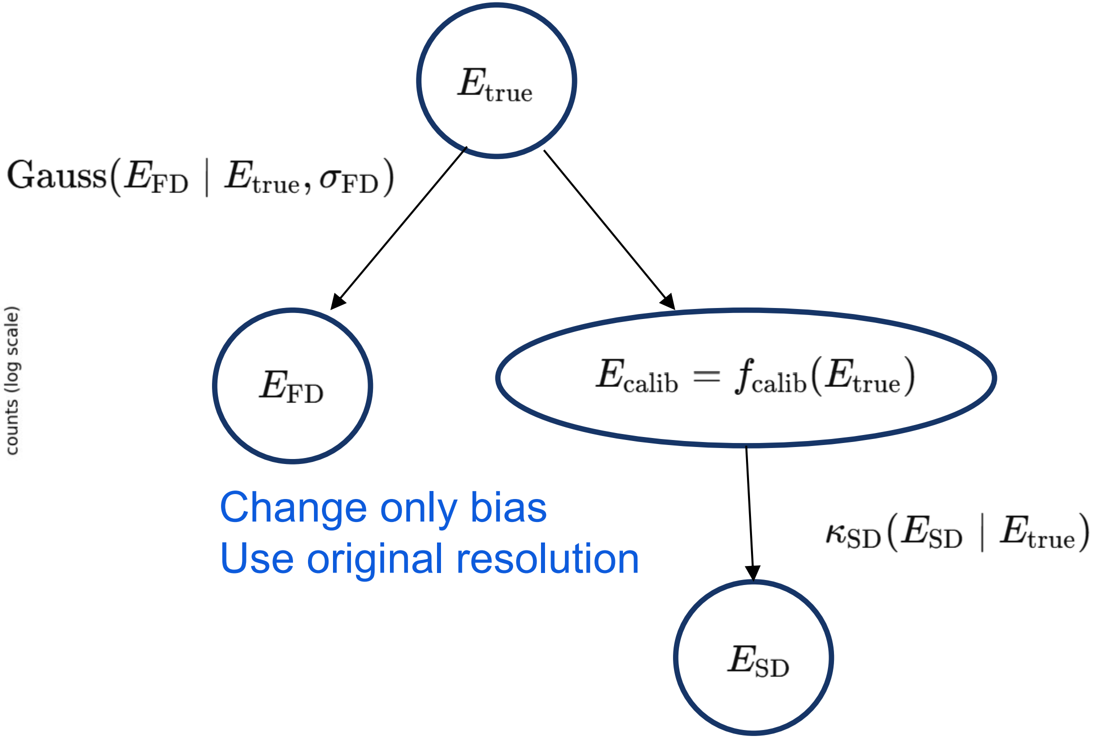

Assume to be known:

Should be fixed/calibrated

$$\mathrm{Gauss}(E_{\mathrm{FD}} \mid E_{\mathrm{true}}, \sigma_{\mathrm{FD}})$$

# Calibration with Hierarchical MCMC



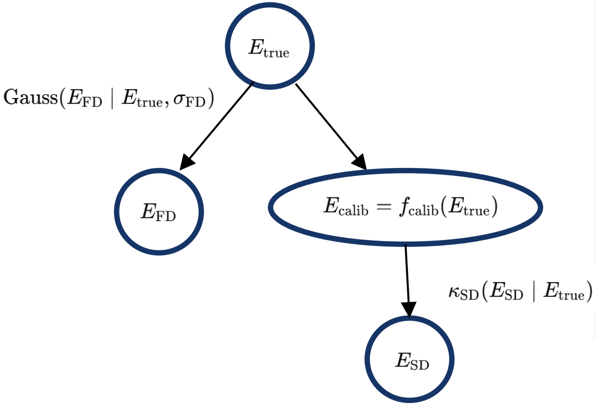
# Parametrization

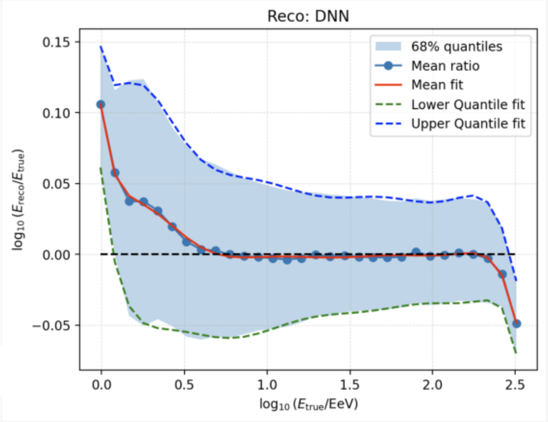
## Gauss (or Student-t) mixture model

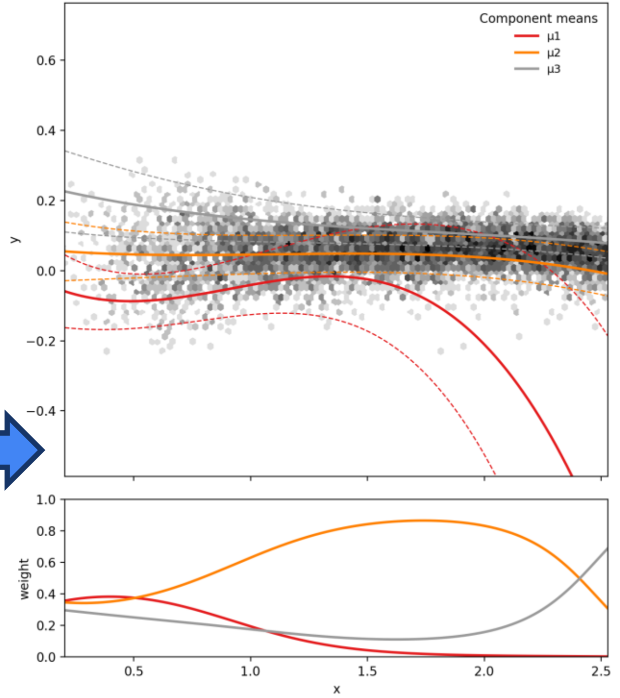
$$\kappa_{\mathrm{SD}}(E_{\mathrm{SD}} \mid E_{\mathrm{true}}) = \sum_{n=1}^{N} w_n(x)\, G(r \mid \mu_n(x), \sigma_n(x))$$

$$r = \log_{10}(E_{\mathrm{SD}}/E_{\mathrm{true}}) \qquad x = \log_{10} E_{\mathrm{true}}$$

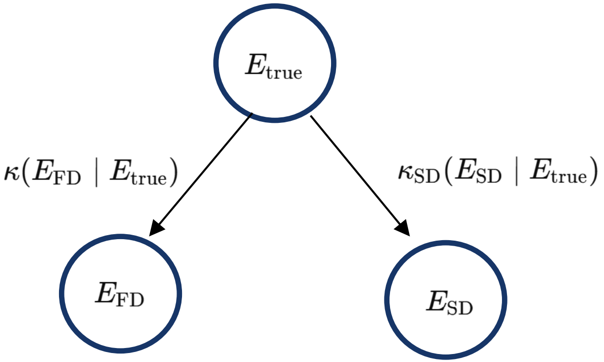
$$f_{\mathrm{calib}} = a\,x^2 + b\,x + c$$

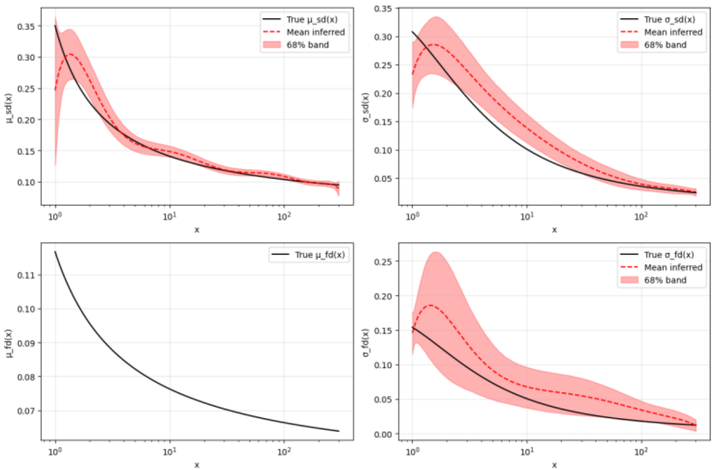


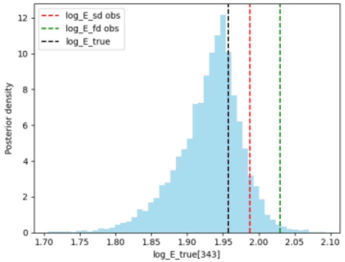



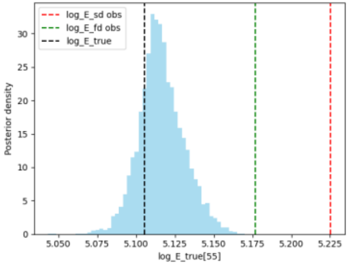



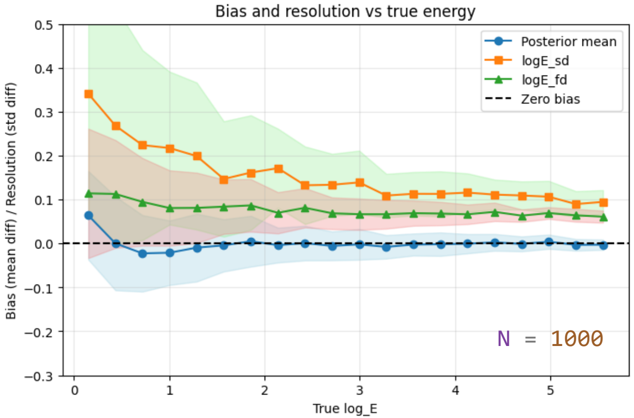
# Test: full resolution reconstruction

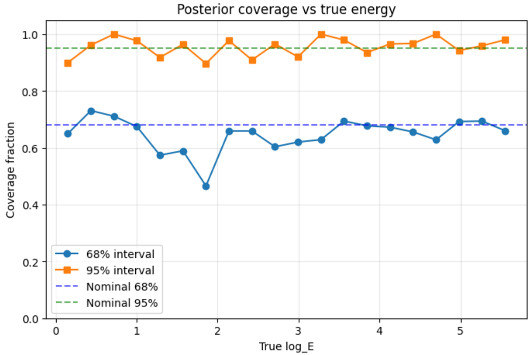


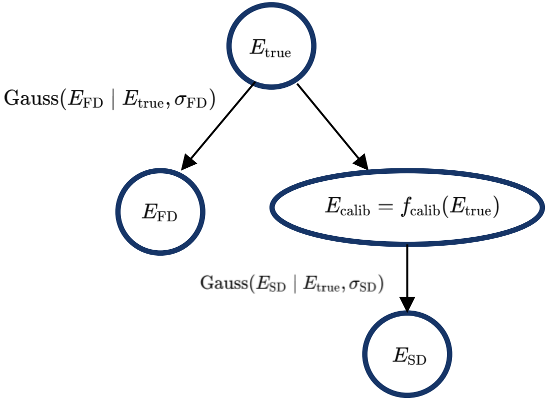



# Test: Gaussian, no energy dependence

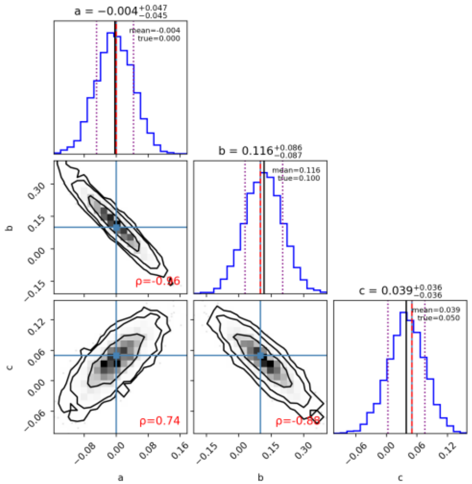


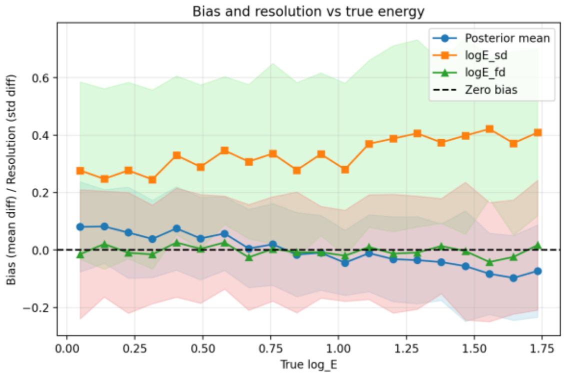

```
Nsample = 1100

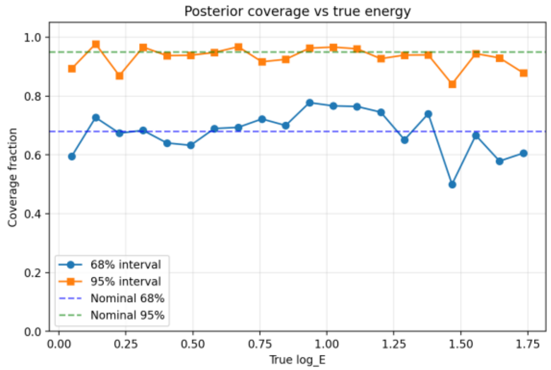
a_true = 0
b_true = 0.1
c_true = 0.05

mu_resid = 0.2
sigma_resid = 0.3
sigma_fd_frac = 0.2
```

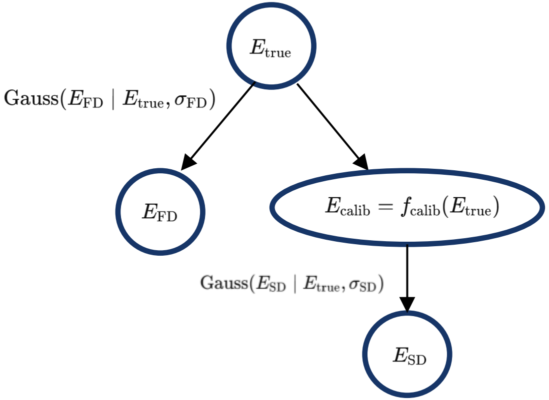
$$f_{\mathrm{calib}} = a\,x^2 + b\,x + c$$

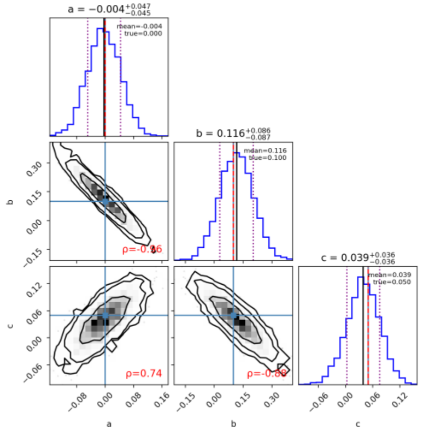


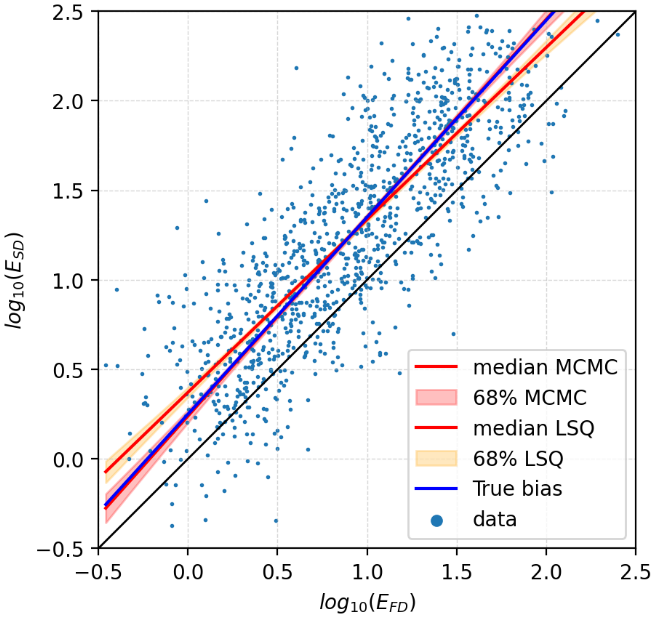



# MCMC vs least square minimization (LSQ)

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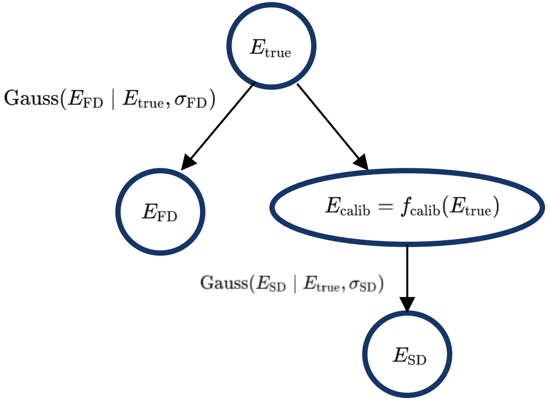
mu_resid = 0.2
sigma_resid = 0.3
sigma_fd_frac = 0.2
```

$$f_{\rm calib} = ax^2 + bx + c + x$$

```
MCMC:
a = -0.004 ± 0.045
b = 0.117 ± 0.085
c = 0.038 ± 0.035
```

```
LSQ:
a = -0.001 ± 0.033
b = -0.034 ± 0.064
c = 0.372 ± 0.028
```

# MCMC vs least square minimization (LSQ)

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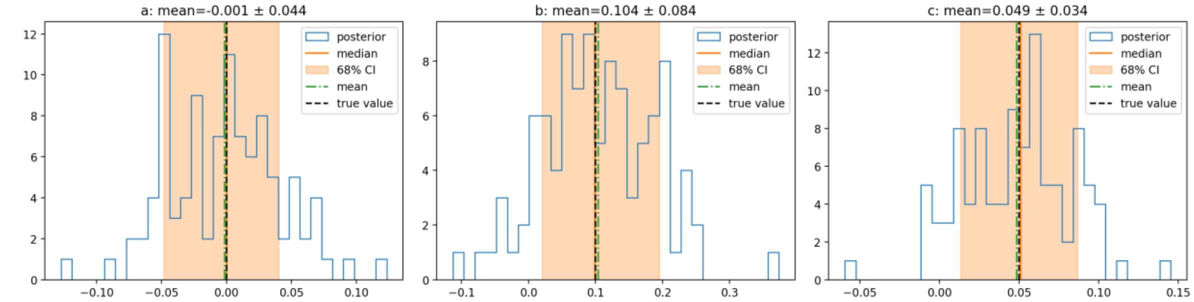
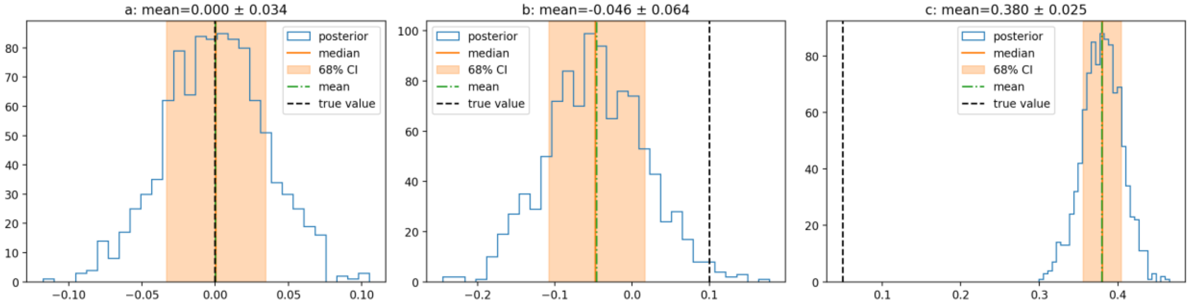
MCMC




LSQ



# Conclusions

- Calibration of the resolution function is required to correctly perform forward folding
- Auger used log-likelihood calibration on hybrid data; early work by Hans Dembinski showed that least-squares fits are biased
- MCMC is a powerful analog of the log-likelihood approach, as it does not require explicit multidimensional integration
- Bayesian MCMC calibration allows a natural extension of the MCMC forward folding method