Linear Analysis and Nonlinear Asymptotics of Neutrino Flavor Conversions: Recent Developments

Jiabao Liu

Shoichi Yamada Group, Waseda University

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Collective Neutrino Oscillations in Supernovae and Neutron Star Mergers @Taiwan

What oscillation physicists care about?

Linear Analysis

- SFI, FFI, CFI
- Eigenvalue problem, but non-selfadjoint
- $\omega(\boldsymbol{k})$ relation

→

- Full QKE
- Attenuation
- Coarse-Graining, for example BGK
- Quantum closure
- Machine Learning
- Mathematics?

Asymptotics

- Quasistationary
- Effective erasing of instabilities
- Universal law?
- Really steady state?

Linear analysis

$$\Pi^{\mu\nu}(k) = \eta^{\mu\nu} + \sqrt{2}G_F \int dP \frac{\Delta f(P)\, v^\mu v^\nu}{\omega - \mathbf{v}\cdot\mathbf{k} + i\Gamma(P)}$$

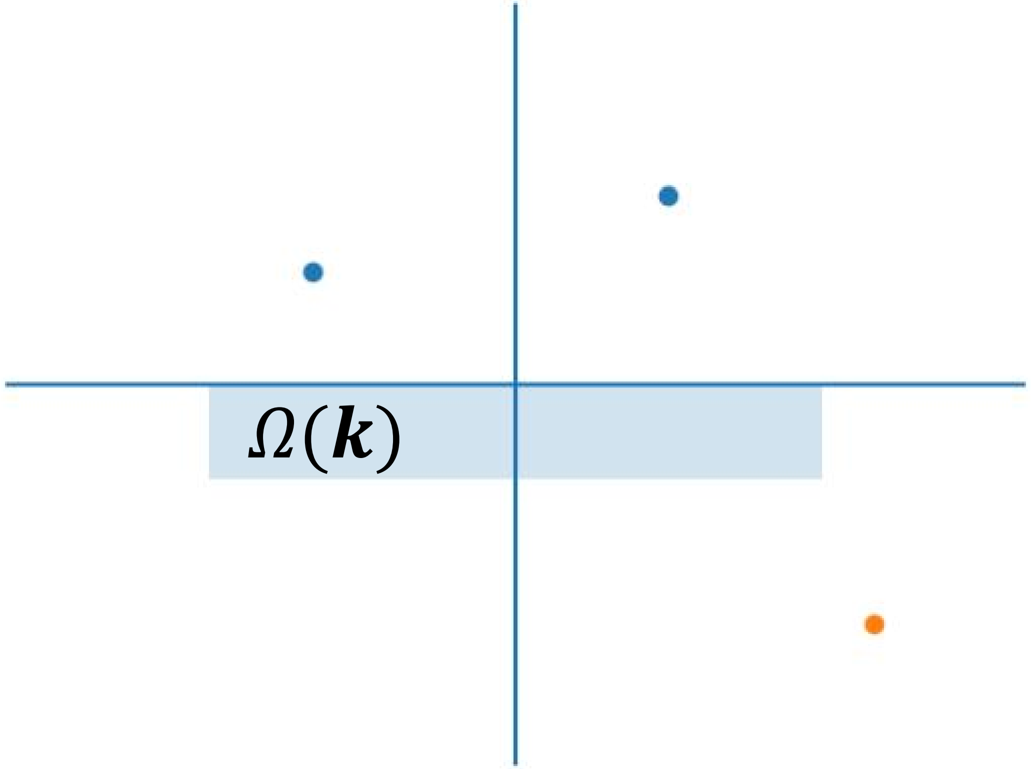
$\det \Pi^{\mu\nu}(k) = 0$ leads to the dispersion relation $\omega(\boldsymbol{k})$; $\omega > 0$ indicates instability

- Define the resonance set in the complex ω-plane: $\Omega(\boldsymbol{k}) = \{\boldsymbol{k}\cdot\boldsymbol{v} - i\Gamma(E,\boldsymbol{v}), |v| = 1, E\epsilon\mathbb{R}\}$
- $\int dP \frac{\Delta f(E,\boldsymbol{v}) v^\mu v^\nu}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Gamma(E)}$ defines a holomorphic function of $\omega(\boldsymbol{k})$ on $\mathbb{C} \setminus \Omega(\boldsymbol{k})$
- The solutions of $\omega(\boldsymbol{k})$ on $\mathbb{C} \setminus \Omega(\boldsymbol{k})$ are isolated in the ω plane

$$\Delta f \equiv f_{\nu_e} - f_{\nu_x}, \quad \Delta \bar{f} \equiv f_{\bar{\nu}_e} - f_{\bar{\nu}_x}, \quad \int dP = \int_0^\infty \frac{E^2\, dE}{2\pi^2} \int \frac{d\mathbf{v}}{4\pi}.$$

Analytic structure of $\boldsymbol{\Pi}$

"Spectrum" of ω at $\boldsymbol{k}$



Isolated eigenvalue ⇔ Collective eigenvectors i.e., coherence grows as $\delta S_j(E,\boldsymbol{v})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{v}}$

Continuun of eigenvalue ⇔ distributional-like eigenvectors i.e., like $\delta(E,\boldsymbol{v})$

People care about unstable modes, which are all isolated

All isolated solutions are determined by poles with induced pole densities

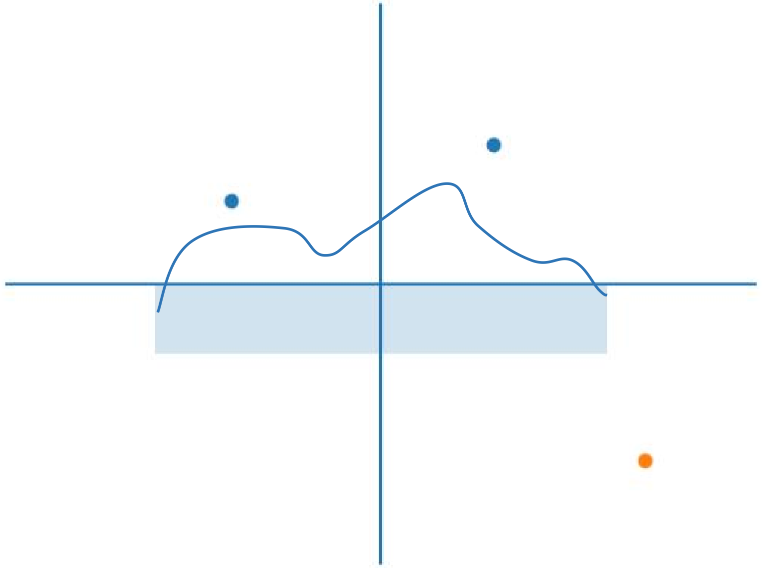
Eigenvectors: $\{\delta S_j(E,\boldsymbol{v})\}_{j\in J}$

Pole density formulation

- The key object is the nonlocal kernel $\int dP \frac{W(E,\boldsymbol{v})}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\Gamma(E,)}$
- Define the pole density induced by the push-forward

$$\sigma_{\pm}(\omega') \equiv \int dP\, W_{\pm}(E,\mathbf{v})\, \delta(\omega' - \mathbf{k}\cdot\mathbf{v} + \mathbf{i\Gamma(E)})$$

- The integrals become $\int_{\Omega'} d\omega' \frac{\sigma_{\pm}(\omega')}{\omega - \omega'}$



- Pole density generates a comlex potential
- Decays with inverse distance
- Try to approximate the potential for some ω

Geometry of an approximation …

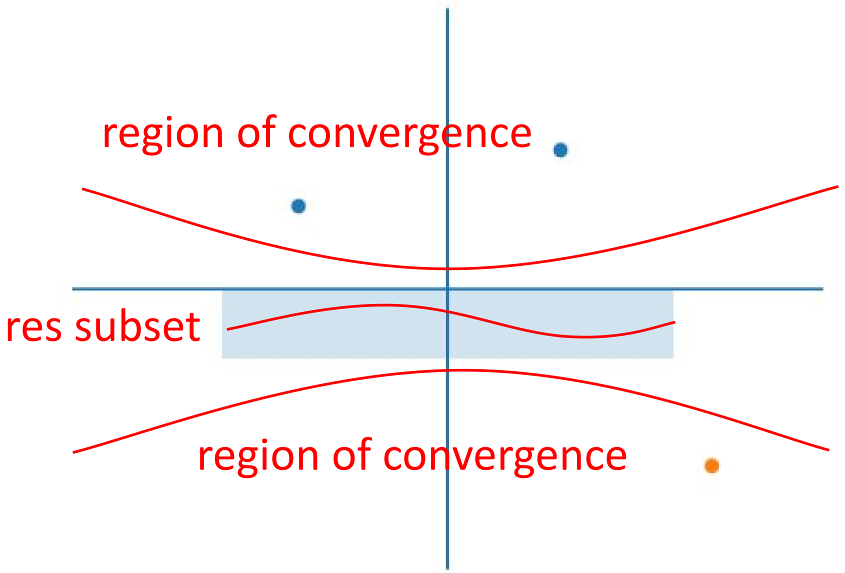
- Any faithful approximation is one that defines a resonance subset $\widetilde{\Omega}(\boldsymbol{k}) \subseteq \Omega(\boldsymbol{k})$ together with a reduced pole density $\widetilde{\sigma}_{\pm}(\omega)$ whose potential field "converges" onto the full potential field in the region of convergence
- Previous CFI approximations are reducing

a 1D line segment into 2 poles

- We may even reduce a 2D area into a

finite set of poles

- Practical efficiency, solvability



k=0 CFI on isotropic backgrounds

- With isotropic distribution and k=0, the $\boldsymbol{\Pi}$ matrix reduces to

$$1+\sqrt{2}G_F\int_0^\infty \frac{E^2\,dE}{2\pi^2}\left[\frac{\Delta f(E)}{\omega+i\Gamma(E)}-\frac{\Delta\bar{f}(E)}{\omega+i\bar{\Gamma}(E)}\right]=0$$

(00) component

$$-3+\sqrt{2}G_F\int_0^\infty \frac{E^2\,dE}{2\pi^2}\left[\frac{\Delta f(E)}{\omega+i\Gamma(E)}-\frac{\Delta\bar{f}(E)}{\omega+i\bar{\Gamma}(E)}\right]=0$$

degenerate (11,22,33) components

$$\Gamma^A_{\rm eff}=\frac{\int_0^\infty E^2\,dE\,\Delta f(E)\,\Gamma(E)}{\int_0^\infty E^2\,dE\,\Delta f(E)},$$

$$\bar{\Gamma}^A_{\rm eff}=\frac{\int_0^\infty E^2\,dE\,\Delta\bar{f}(E)\,\bar{\Gamma}(E)}{\int_0^\infty E^2\,dE\,\Delta\bar{f}(E)}.$$

Method A by Z. Xiong and collaborators

$$\Gamma_{{\rm eff},\nu_\alpha}=\frac{\int_0^\infty E^2\,dE\,f_{\nu_\alpha}(E)\,\Gamma_{\nu_\alpha}(E)}{\int_0^\infty E^2\,dE\,f_{\nu_\alpha}(E)},$$

$$\Gamma^B_{\rm eff}=\frac{\Gamma_{{\rm eff},\nu_e}+\Gamma_{{\rm eff},\nu_x}}{2},\qquad \bar{\Gamma}^B_{\rm eff}=\frac{\Gamma_{{\rm eff},\bar{\nu}_e}+\Gamma_{{\rm eff},\bar{\nu}_x}}{2}.$$

Method B by J. Liu and collaborators

The naming of “A” and “B” was first given by T. Wang + in arXiv:2507.01100

k=0 CFI on isotropic backgrounds

- With isotropic distribution and k

$$1 + \sqrt{2}G_F \int_0^\infty \frac{E^2\,dE}{2\pi^2}\left[\frac{\Delta f(E)}{\omega + i\Gamma(E)} - \frac{\Delta \bar{f}(E)}{\omega + i\bar{\Gamma}(E)}\right] =$$

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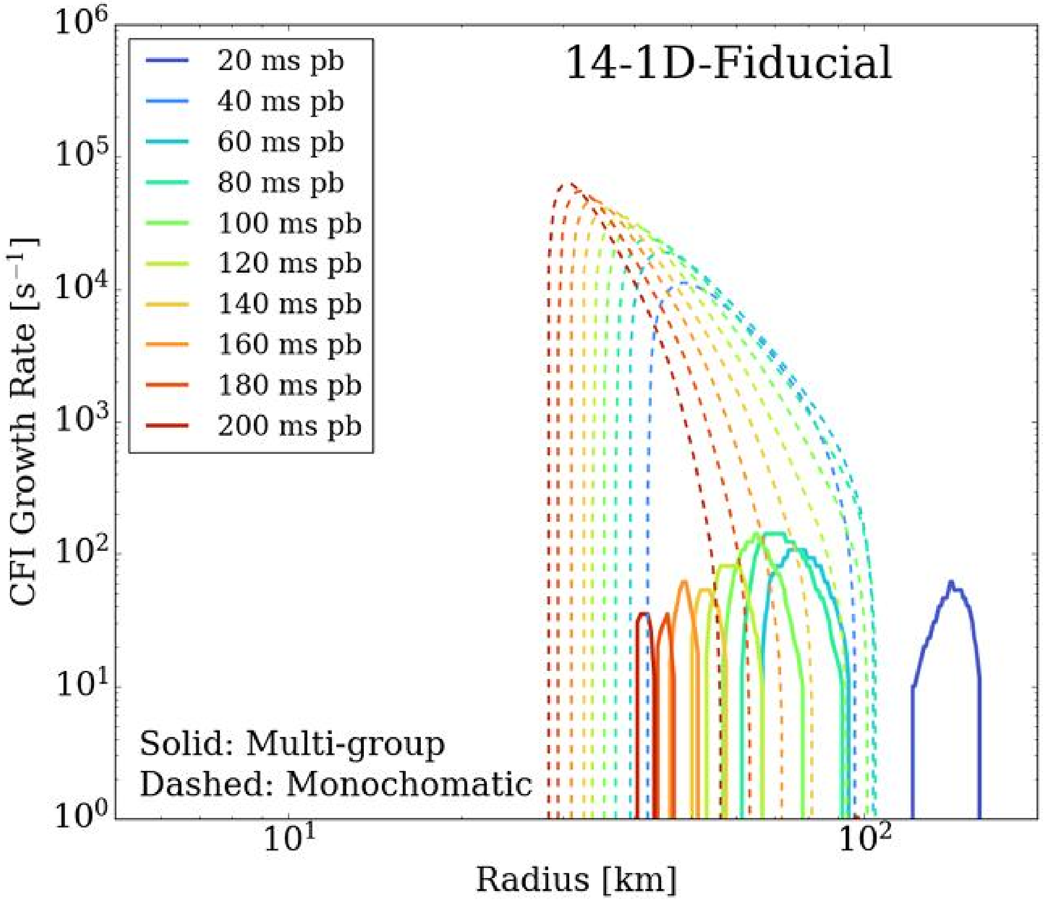
$$\Gamma_{\rm eff}^A = \frac{\int_0^\infty E^2\,dE\,\Delta f(E)\,\Gamma(E)}{\int_0^\infty E^2\,dE\,\Delta f(E)},$$

$$\bar{\Gamma}_{\rm eff}^A = \frac{\int_0^\infty E^2\,dE\,\Delta \bar{f}(E)\,\bar{\Gamma}(E)}{\int_0^\infty E^2\,dE\,\Delta \bar{f}(E)}.$$

T. Wang+ arXiv:2507.01100



Method A by Z. Xiong and collaborators

Method B by J. Liu and collaborators

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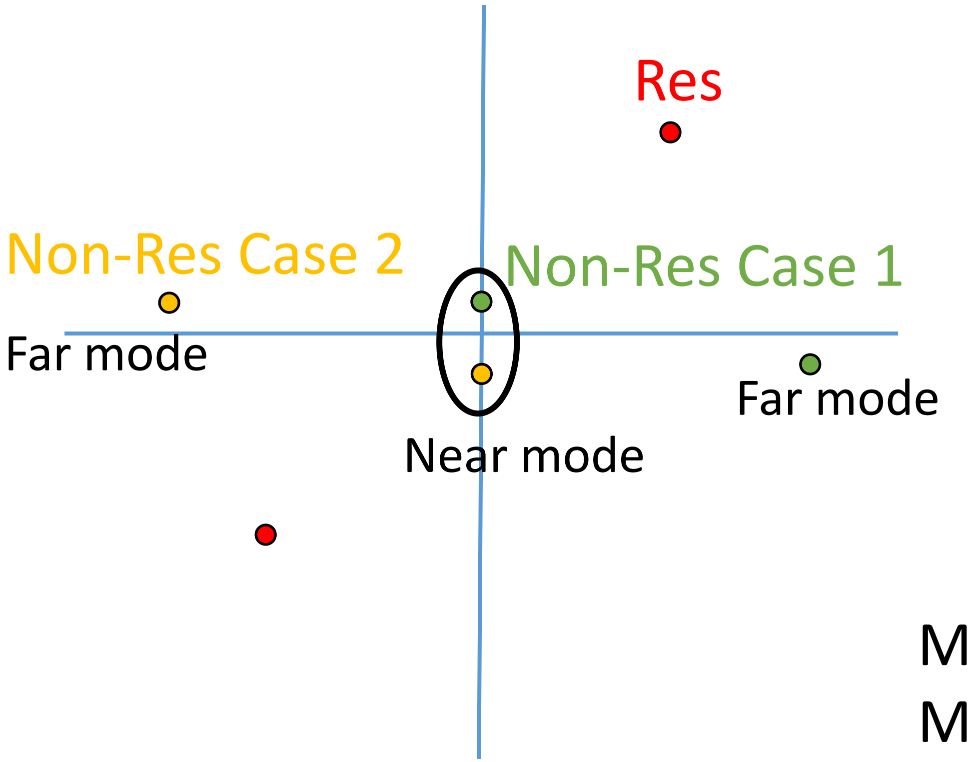
When do the previous methods fail



$$G \equiv \frac{g+\bar{g}}{2}, \qquad A \equiv \frac{g-\bar{g}}{2}, \qquad \gamma^X \equiv \frac{\Gamma^X_{\rm eff}+\bar{\Gamma}^X_{\rm eff}}{2}, \qquad \alpha^X \equiv \frac{\Gamma^X_{\rm eff}-\bar{\Gamma}^X_{\rm eff}}{2}.$$

$$\omega^X_\pm = -A - i\gamma^X \pm \sqrt{A^2 - (\alpha^X)^2 + i\,2G\alpha^X},$$

$$\omega^X_\pm \approx \begin{cases} -A - i\gamma^X \pm \left(|A| + i\,\dfrac{G\alpha^X}{|A|}\right), & A^2 \gg |G\alpha^X|, \text{ Non-Res} \\ -A - i\gamma^X \pm \sqrt{i\,2G\alpha^X}, & A^2 \ll |G\alpha^X|, \text{ Res} \end{cases}$$

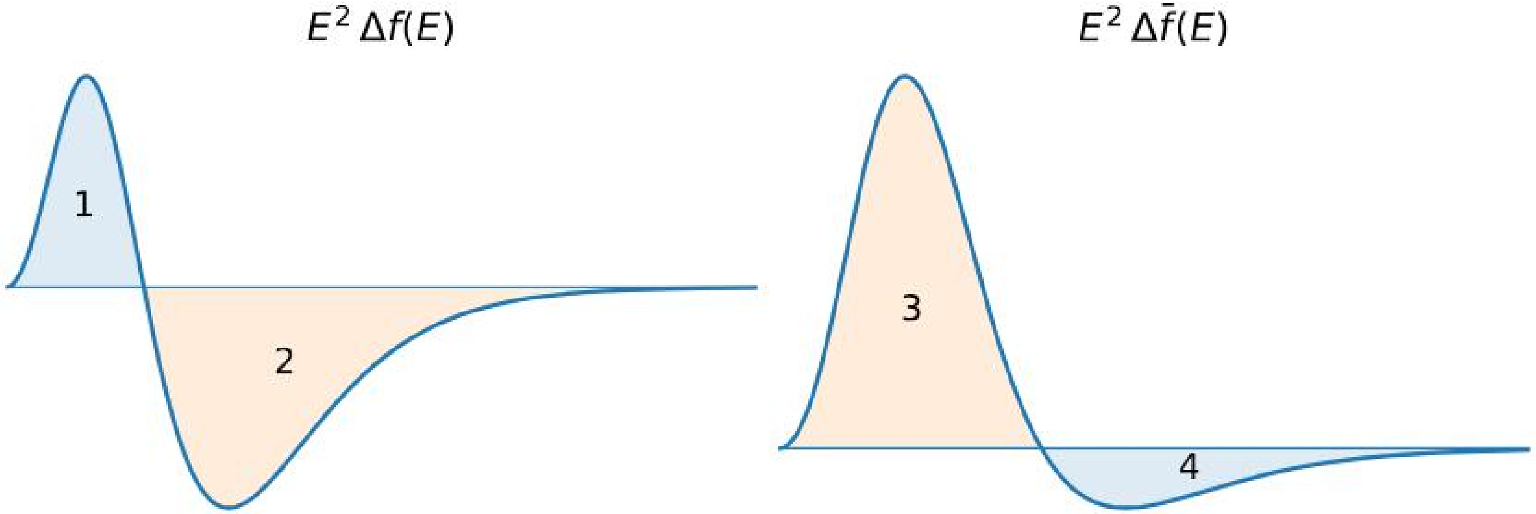
Method A fails when $\Gamma^A_{\rm eff}$ diverges; growth rate diverges
Method B fails when heavy-lepton flavors contribute a lot; not diverging, but with big "standard deviation"

A better approximation that...

- Completely avoids divergence and Reduces systematic bias

$$1+\sqrt{2}G_F\int_0^\infty \frac{E^2\,dE}{2\pi^2}\left[\frac{\Delta f(E)}{\omega+i\Gamma(E)}-\frac{\Delta\bar{f}(E)}{\omega+i\bar{\Gamma}(E)}\right]=0$$

We can group $\widetilde{\Omega_1}(\boldsymbol{k}), \widetilde{\Omega_2}(\boldsymbol{k}) \subseteq \Omega(\boldsymbol{k})$ such that the pole density is sign-definite



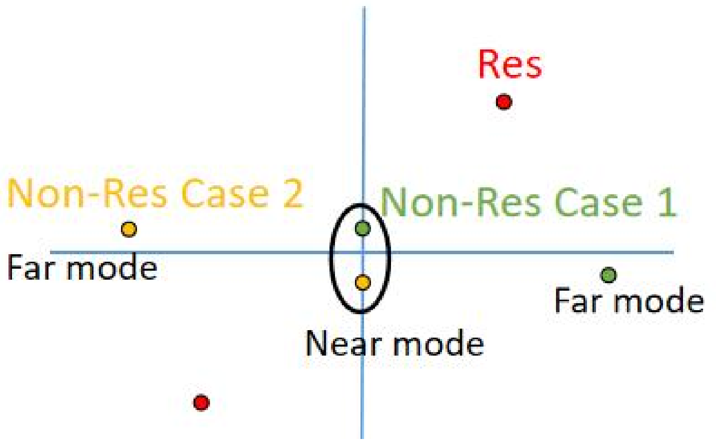
A better approximation…

$$\int_0^\infty \frac{E^2\,dE}{2\pi^2}\left[\frac{\Delta f(E)}{\omega + i\Gamma(E)} - \frac{\Delta \bar{f}(E)}{\omega + i\bar{\Gamma}(E)}\right] \approx \frac{\mathcal{N}_+}{\omega + i\Gamma_+} - \frac{\mathcal{N}_-}{\omega + i\Gamma_-}.$$

$$\mathcal{N}_+ = \int_0^\infty \frac{E^2\,dE}{2\pi^2}\Big([\Delta f(E)]_+ + [\Delta \bar{f}(E)]_-\Big), \qquad \mathcal{G}_+ = \int_0^\infty \frac{E^2\,dE}{2\pi^2}\Big([\Delta f(E)]_+\Gamma(E) + [\Delta \bar{f}(E)]_-\bar{\Gamma}(E)\Big),$$

$$\mathcal{N}_- = \int_0^\infty \frac{E^2\,dE}{2\pi^2}\Big([\Delta f(E)]_- + [\Delta \bar{f}(E)]_+\Big), \qquad \mathcal{G}_- = \int_0^\infty \frac{E^2\,dE}{2\pi^2}\Big([\Delta f(E)]_-\Gamma(E) + [\Delta \bar{f}(E)]_+\bar{\Gamma}(E)\Big).$$

$$\Gamma_+ \equiv \frac{\mathcal{G}_+}{\mathcal{N}_+}, \qquad \Gamma_- \equiv \frac{\mathcal{G}_-}{\mathcal{N}_-},$$



- is never divergent and faithful
- region of convergence is roughly $|\omega| \geq \Gamma_\pm$
- only near mode approximant may be refined

A better approximation…

$$\int_0^\infty \frac{E^2\,dE}{2\pi^2}\left[\frac{\Delta f(E)}{\omega + i\Gamma(E)} - \frac{\Delta \bar{f}(E)}{\omega + i\bar{\Gamma}(E)}\right] \approx \frac{\mathcal{N}_+}{\omega + i\Gamma_+} - \frac{\mathcal{N}_-}{\omega + i\Gamma_-}.$$

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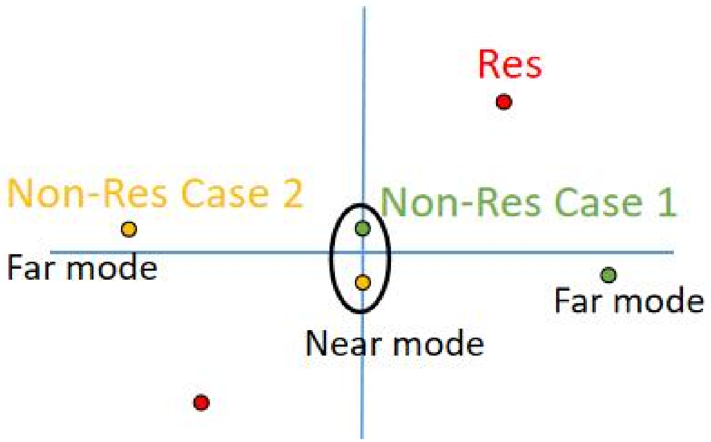
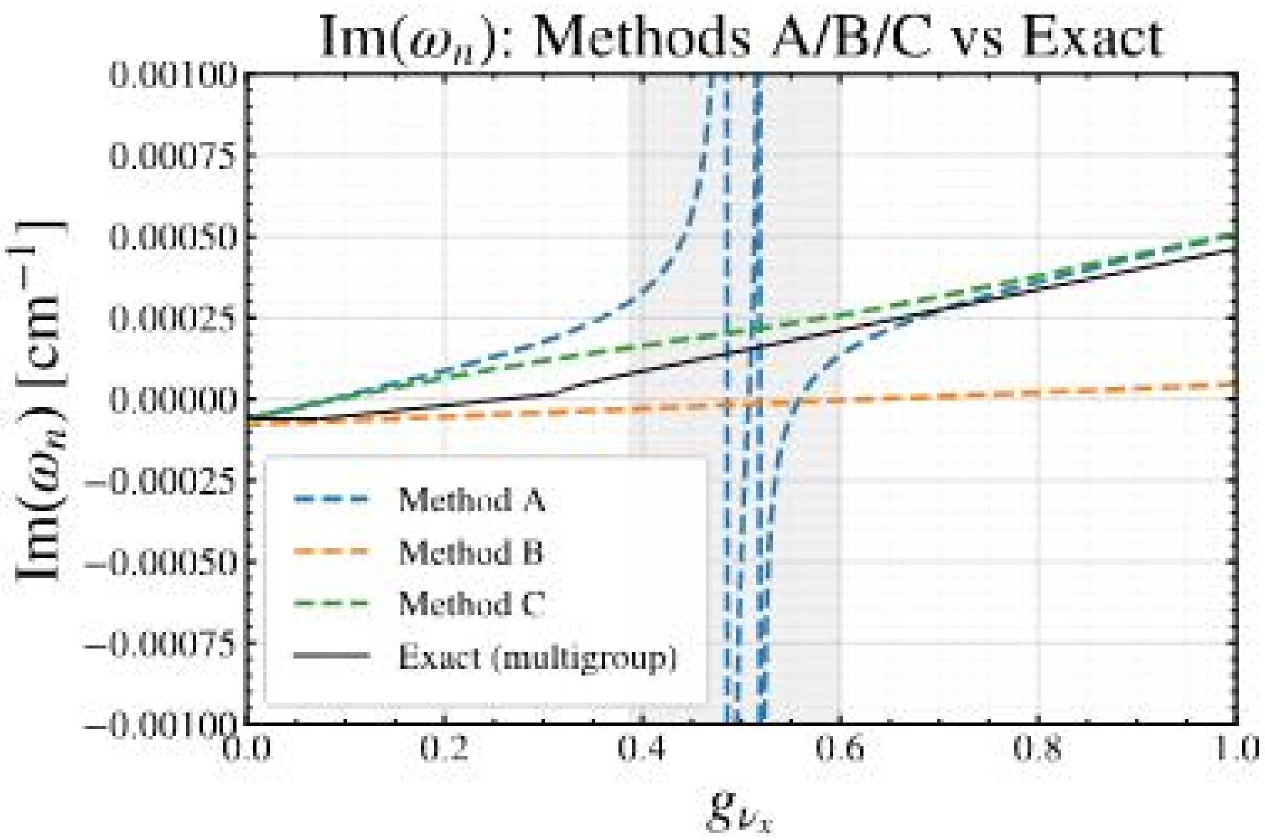
$$\mathcal{N}_- = \int_0^\infty \frac{E^2\,dE}{2\pi^2}\Big([\Delta f(E)]_- + [\Delta \bar{f}(E)]_+\Big), \qquad \mathcal{G}_- = \int_0^\infty \frac{E^2\,dE}{2\pi^2}\Big([\Delta f(E)]_-\Gamma(E) + [\Delta \bar{f}(E)]_+\bar{\Gamma}(E)\Big).$$

$$\Gamma_+ \equiv \frac{\mathcal{G}_+}{\mathcal{N}_+}, \qquad \Gamma_- \equiv \frac{\mathcal{G}_-}{\mathcal{N}_-},$$

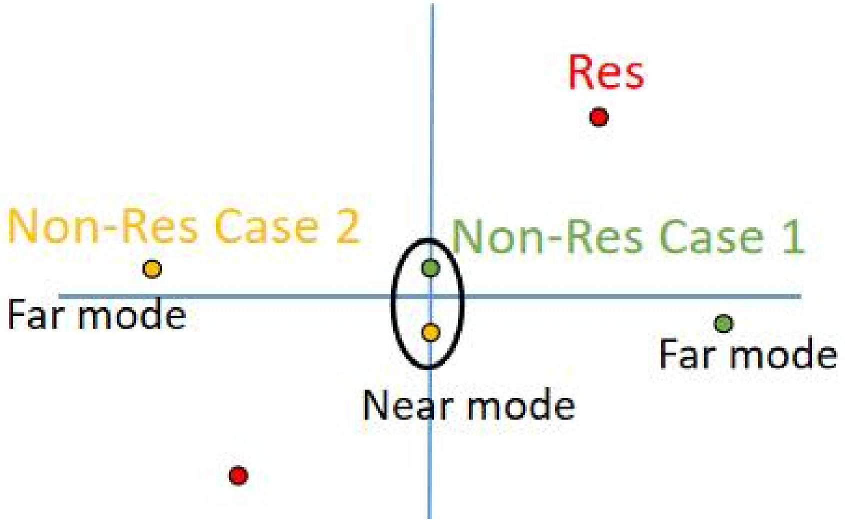
- is never divergent and faithful
- region of convergence is roughly $|\omega| \geq \Gamma_\pm$
- only near mode approximant may be refined



A better approximation



Method C for the near mode may need an efficient refinement method



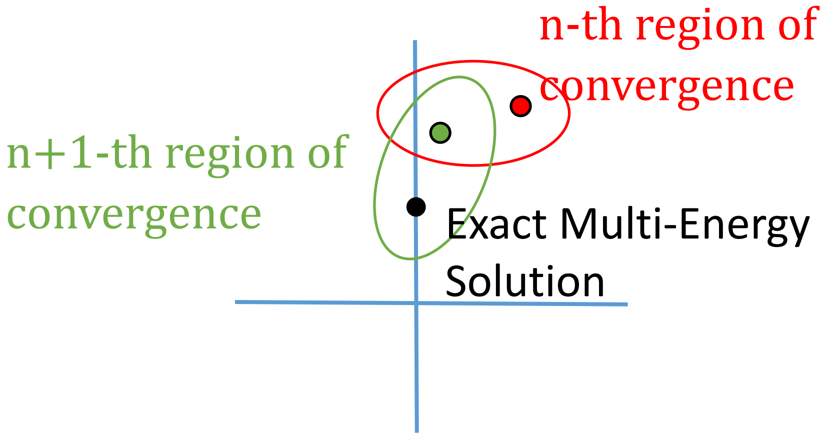
Convergence in operator space

We expand the operator $D(\omega) = 1 + R_+(\omega) - R_-(\omega)$, $R_+(\omega) = \sum_{j\in+} \frac{A_j^{(+)}}{\omega + i\gamma_j}$, $R_-(\omega) = \sum_{j\in-} \frac{A_j^{(-)}}{\omega + i\gamma_j}$.

around ω_n as $\widetilde{R}_{\pm,n}(\omega) = \frac{A_{\pm,n}}{\omega + i\Gamma_{\pm,n}}$, to iterate for ω_{n+1} as exact solutions to D_{n+1}

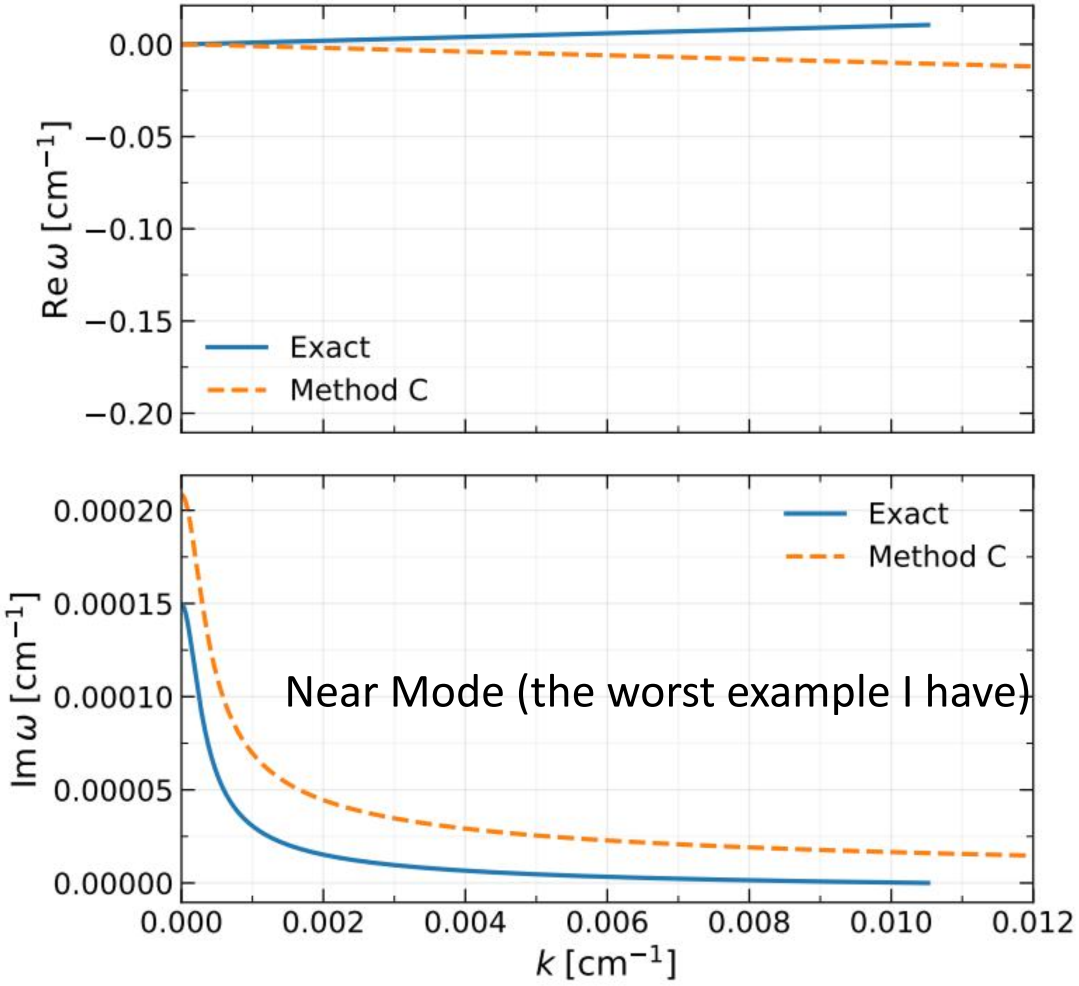
It never performs root-finding for the exact D operator
It iterates on a local low-dimensional expansion

iter	Re[w]	Im[w]
0	$-1.858661599898501\times10^{-7}$	0.000283213578923805 7
1	-1.01233×10^{-7}	0.000156813
2	-9.47236×10^{-8}	0.00014962
3	-9.46391×10^{-8}	0.000149561
4	-9.46391×10^{-8}	0.000149561



Not limited to k=0

With isotropic distributions



A minimal energy-reduction for axially symmetric distributions at k != 0

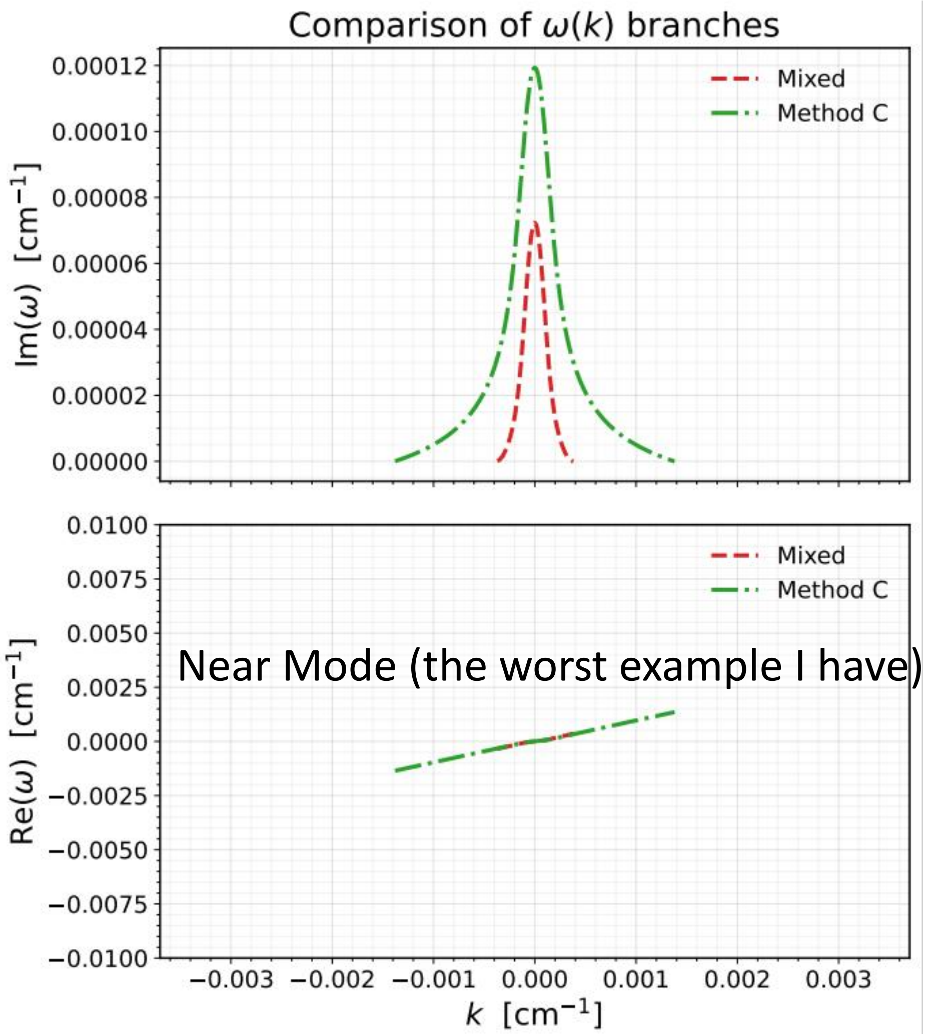
$$\int_{-1}^{1} \frac{v^n\,dv}{2} \int_0^\infty \frac{E^2\,dE}{2\pi^2} \frac{f(E,v)}{\omega - kv + i\Gamma(E,v)}$$
$$\approx \int_{-1}^{1} \frac{v^n\,dv}{2} \int_0^\infty \frac{E^2\,dE}{2\pi^2} \frac{f(E,v)}{\omega - kv + i\Gamma_{\rm eff}(v)},$$

$$\Gamma_{\rm eff}(v) \equiv \frac{\int_0^\infty E^2\,dE\,f(E,v)\,\Gamma(E,v)}{\int_0^\infty E^2\,dE\,f(E,v)}.$$

This example in fact shows an enhanced CFI peak at k>0

Not limited to isotropy and k=0

With strong axisymmetry



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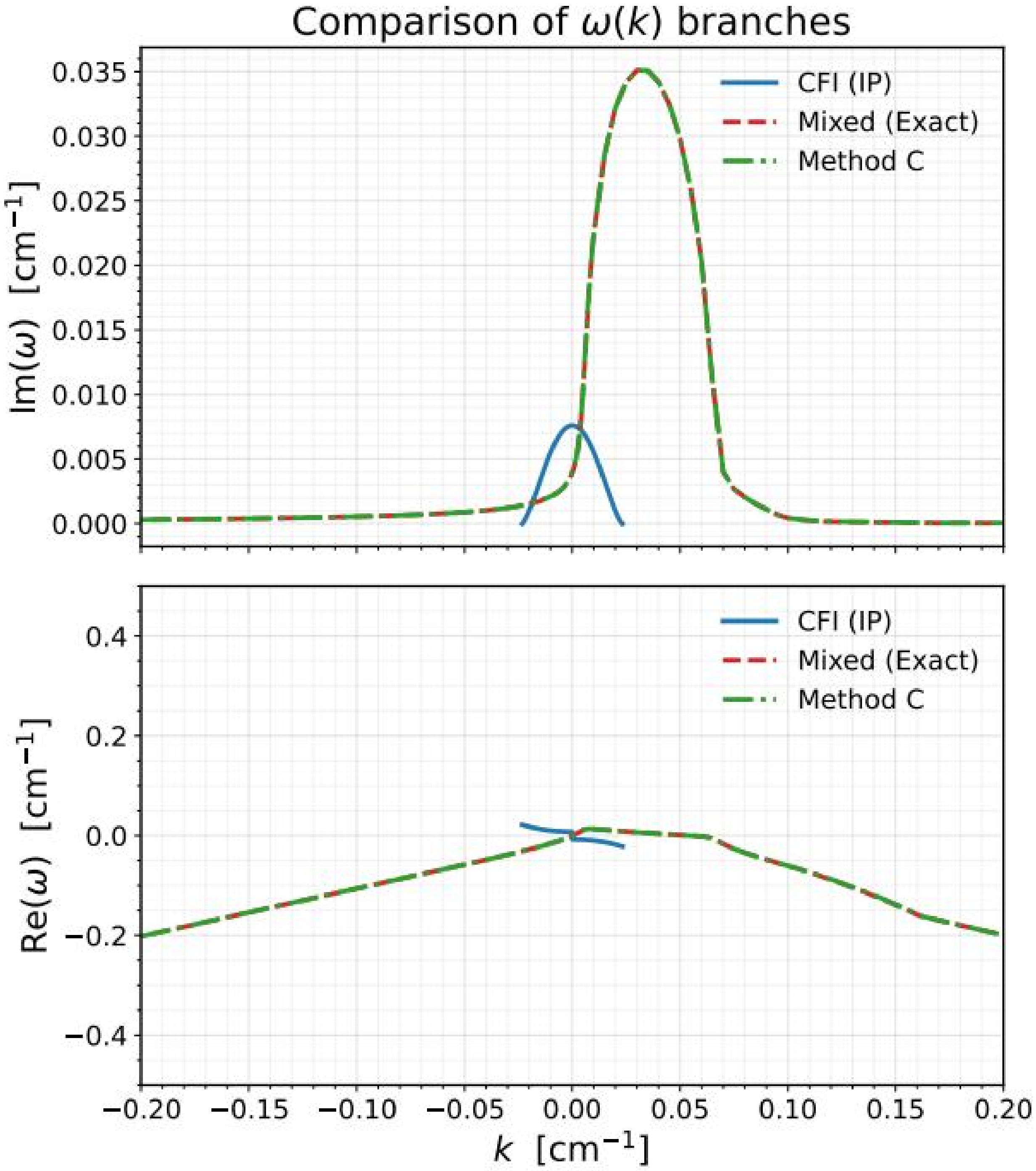


"Oh, CFI? No fast contributions here?"

"Oh, its peak is shifted from k=0?"

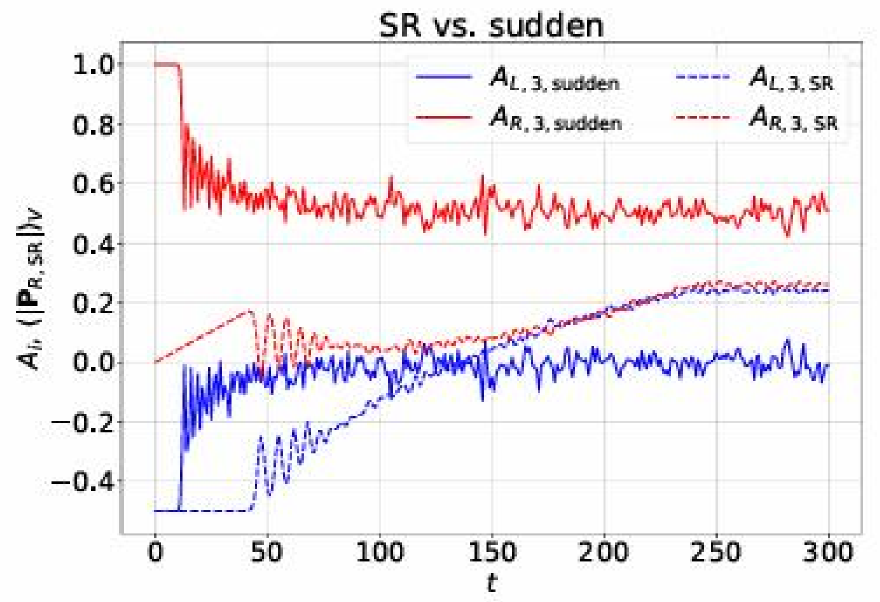
"Oh, it's enhanced by axisymmetry?"

"Oh, wait, the approximation agrees exactly at even k>0!"

QuasiSteady Evolution of FFCs

J. Liu+ Phys. Rev. D 111, 023051

$$
\begin{aligned}
(\partial_t + \partial_r)P_{R,0} &= C_R(t), \\
(\partial_t - \partial_r)P_{L,0} &= C_L(t), \\
(\partial_t + \partial_r)\mathbf{P}_R &= 2\mathbf{P}_L \times \mathbf{P}_R + (0,0,C_R(t)), \\
(\partial_t - \partial_r)\mathbf{P}_L &= -2\mathbf{P}_L \times \mathbf{P}_R + (0,0,C_L(t)).
\end{aligned}
$$



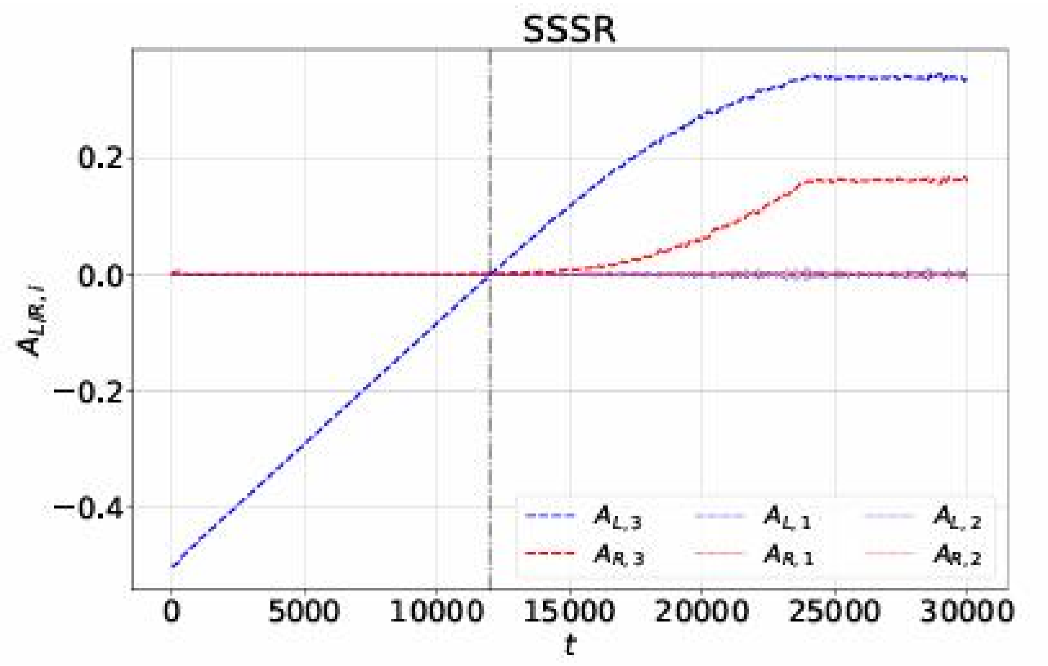
When there are no neutrino injections: erasing crossing

When there are slow neutrino injections: it seems to evolve along flavor equipartition

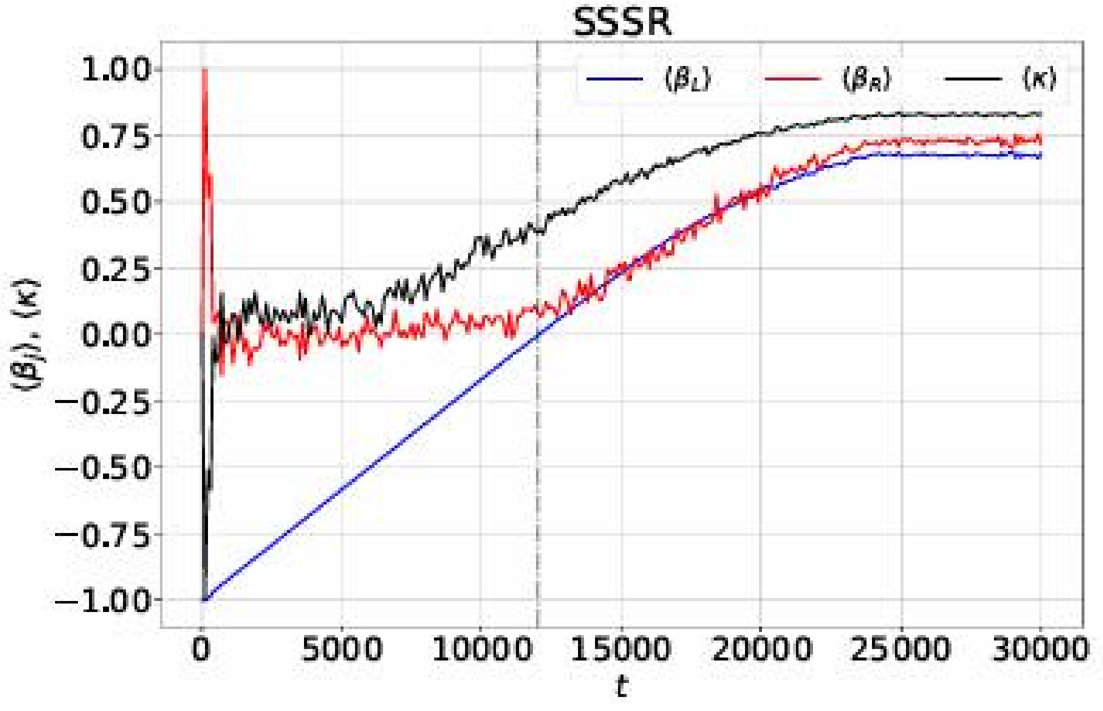
(Also see "Edge of Instability" Phys. Rev. Lett. 133, 221004 by D. F. G. Fiorillo & G. Raffelt

Ultra-Slow injection limit

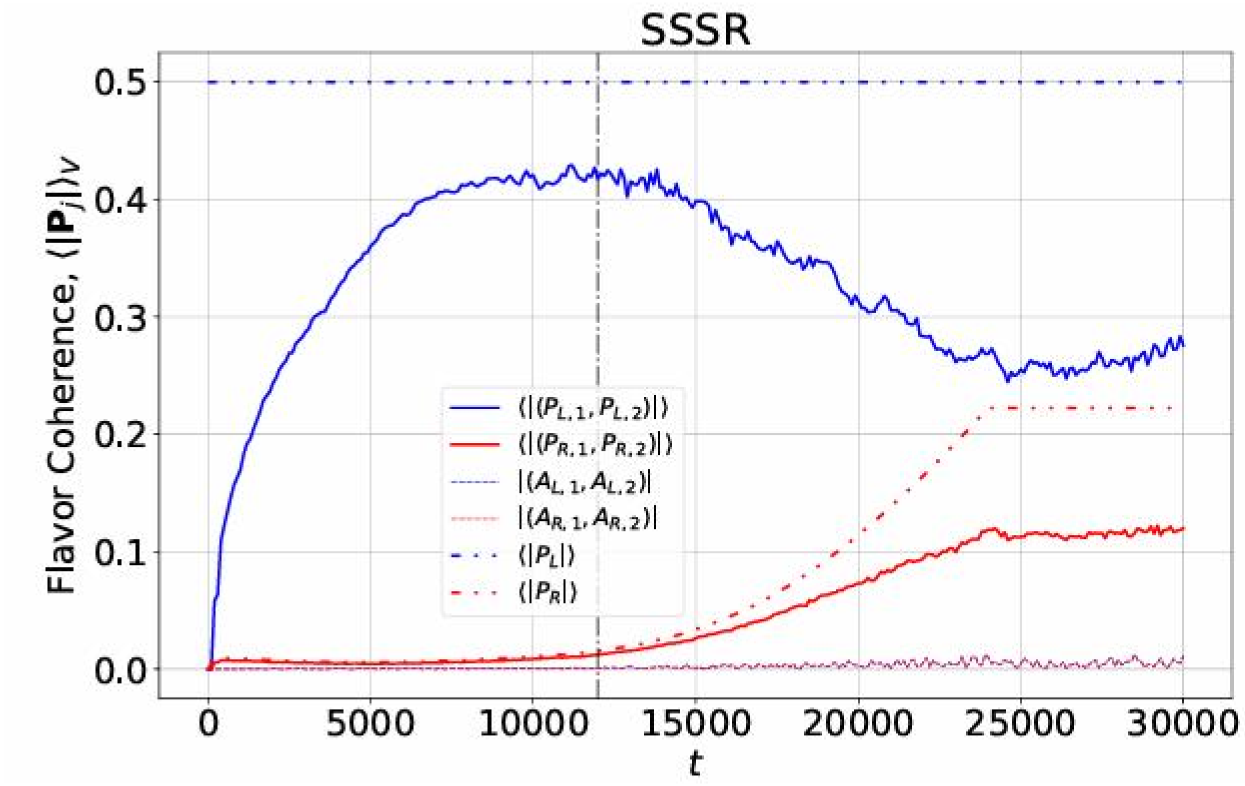
It is not tracing the line of equipartition, when the injection timescale is very long



Degree of polarization $\beta_{L/R} \equiv \frac{P_{L/R,3}}{|\mathbf{P}_{L/R}|}$.



Inhomogeneous information matters



The coarse-grained distribution is nearly coherence-free

At each spatial coordinate there is huge flavor coherence

Inhomogeneous spatial structure is driving continued collective flavor conversions

(Also see arXiv:2510.23917 by E. Urquilla & L. Johns)

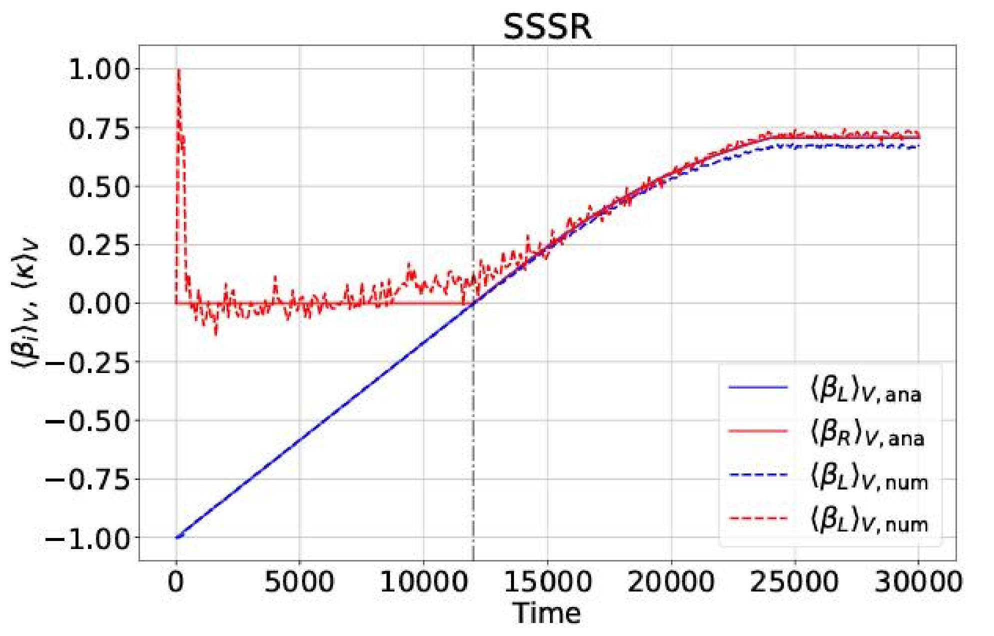
Degree of polarization as an asymptotic predictor?

By imposing an approximate sepraration $A_{L/R,3} = \langle|\mathbf{P}_{L/R}|\rangle_V \langle\beta_{L/R}\rangle_V$.

and the co-evolution of beta $\langle\beta\rangle_V \equiv \langle\beta_L\rangle_V = \langle\beta_R\rangle_V$.

We derive an analytical solution $\langle\beta\rangle_V = \dfrac{(t-t_m)C_R}{\langle|\mathbf{P}_L|\rangle_V + \langle|\mathbf{P}_R|\rangle_V}$.

$$\langle|\mathbf{P}_R|\rangle_V = -\langle|\mathbf{P}_L|\rangle_V + \sqrt{\langle|\mathbf{P}_L|\rangle_V^2 + (t^2 - 2t_m t)\, C_R^2 + 2F_{int}},$$



A more important question than merely the good agreement: can we construct BGK with co-evolutiong quantities effectively representing spatial structures?

Dynamical equilibrium of FFC

J. Liu, L. Johns, H. Nagakura, M. Zaizen, S. Yamada arXiv:2509.26418

- Why is there an asymptotic state?

- What determines the coarse-grained staticity?

Dynamical equilibrium of FFC

What are the most essential things to build a tower?

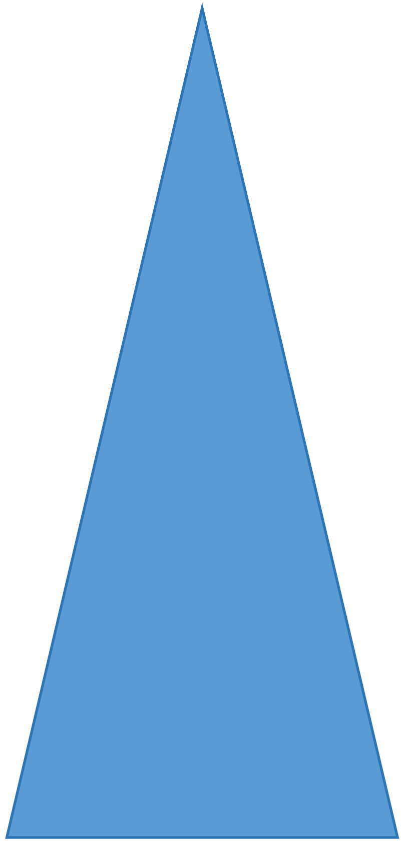
Tokyo Tower

Dynamical equilibrium of FFC

What are the most essential things to build a tower?

Tokyo Tower

First-principle analysis of Tokyo Tower



Helicity subspaces

Let (X, Y) be the space of complex-valued vector, then the set {(1,i), (1,-i)} forms a basis for the (X,Y) space

Call the subspace spanned by (1,i) the plus-polarization subspace; and the other is minus-polarization

(in QKE, they evolve independently!)

A closed subspace of QKE

Full QKE with only self-interaction Hamiltonian

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{P}(\mathbf{x},\mathbf{v}) = \sum_{\mathbf{v}'}(1-\mathbf{v}\cdot\mathbf{v}')\,\mathbf{P}(\mathbf{x},\mathbf{v}')\times\mathbf{P}(\mathbf{x},\mathbf{v})\,.$$

The Fourier transform of QKE

$$(\partial_t + i\,\mathbf{k}\cdot\mathbf{v}_a)\,\mathbf{P}_{\mathbf{k}}(\mathbf{v}_a) = \sum_{\mathbf{v}_b}(1-\mathbf{v}_a\cdot\mathbf{v}_b)\sum_{\mathbf{k}'}\mathbf{P}_{\mathbf{k}'}(\mathbf{v}_b)\times\mathbf{P}_{\mathbf{k}-\mathbf{k}'}(\mathbf{v}_a)\,.$$

is dynamically closed in the subspace with

$P_0(v_a) = (0,0,Z_a)$ and $P_k(v_a) = (X_a, Y_a, 0)$ for only one k, with $Z_a \in \mathbb{R}$ and all (X_a, Y_a) with the same helicity

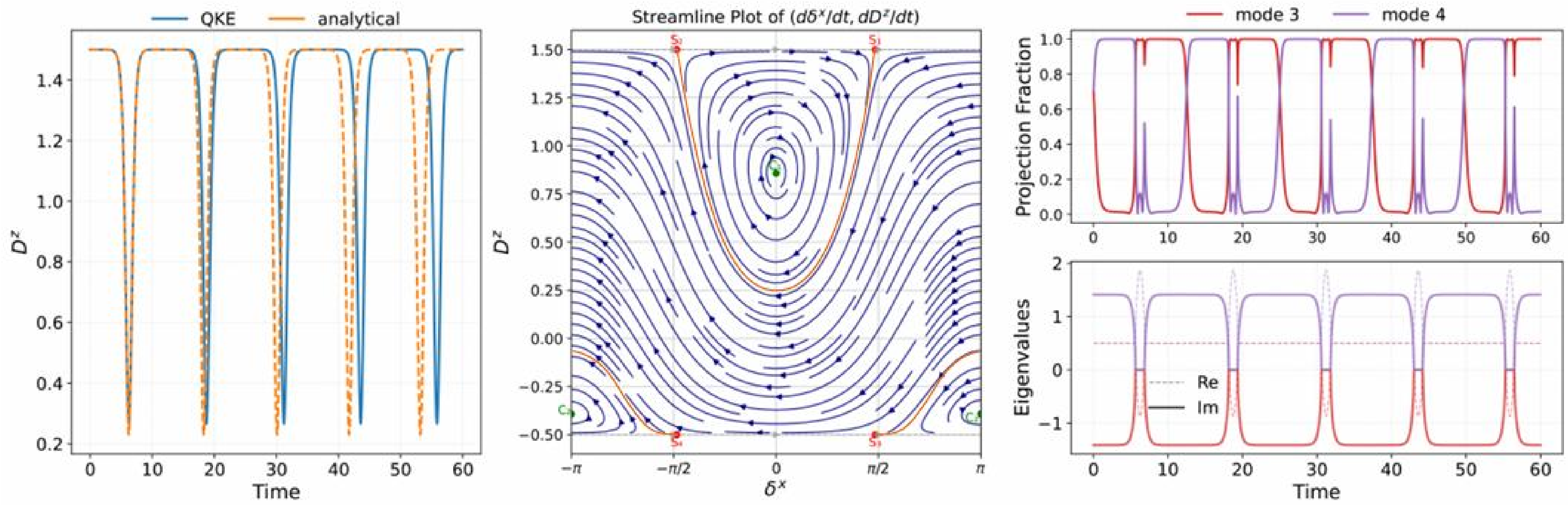
complex conjugation relation: $P_k^* = P_{-k}^*$

Two-beam realization and integrability

$$(\partial_t + \partial_r)\boldsymbol{P}_R = 2\boldsymbol{P}_L \times \boldsymbol{P}_R,$$
$$(\partial_t - \partial_r)\boldsymbol{P}_L = 2\boldsymbol{P}_R \times \boldsymbol{P}_L.$$

Initialize with a single Fourier mode (together with its conjugate) in the same helicity

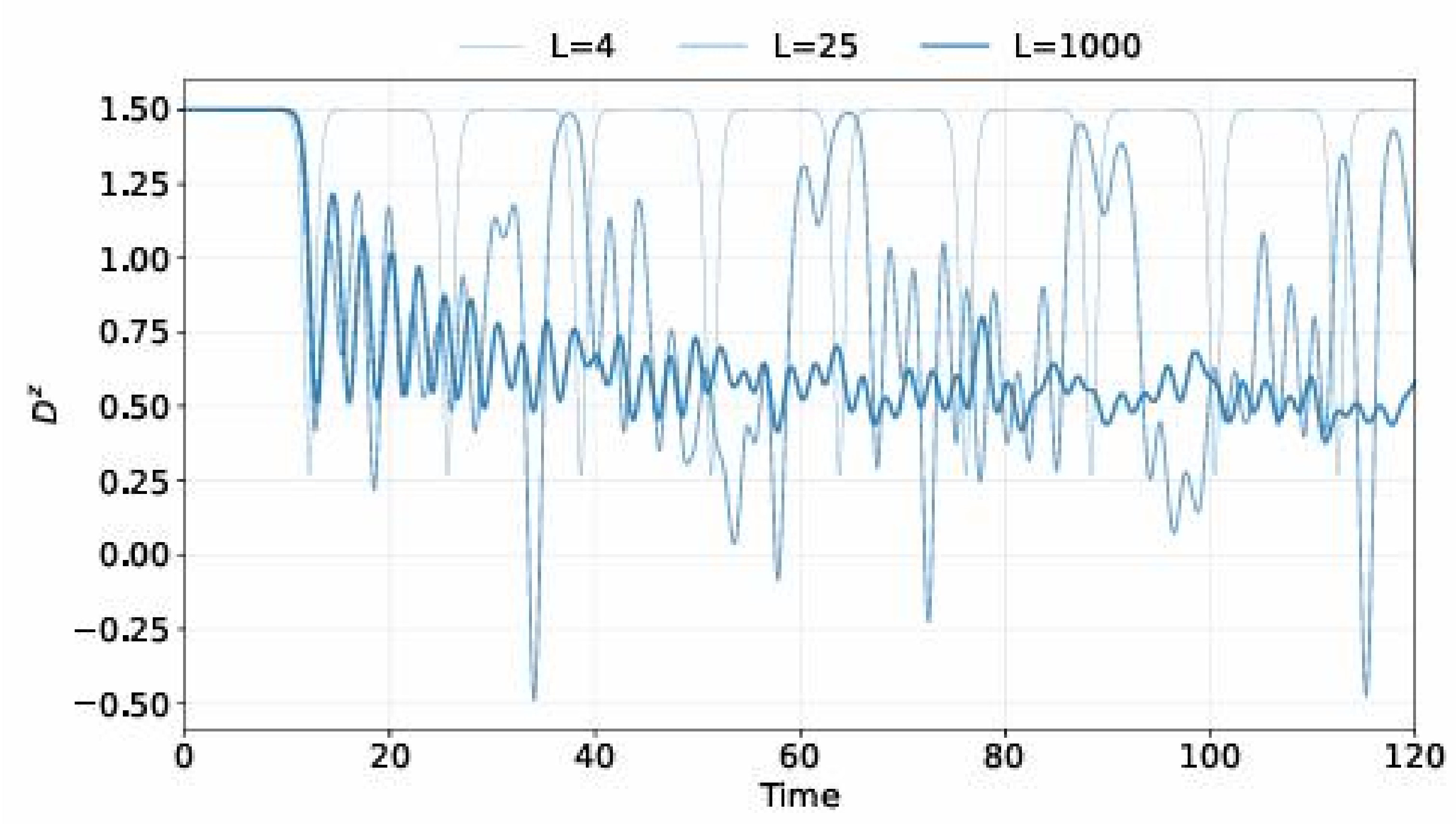
Integrable, repeated fast instability growing and decaying



The stable eigenvector is entering the dynamics

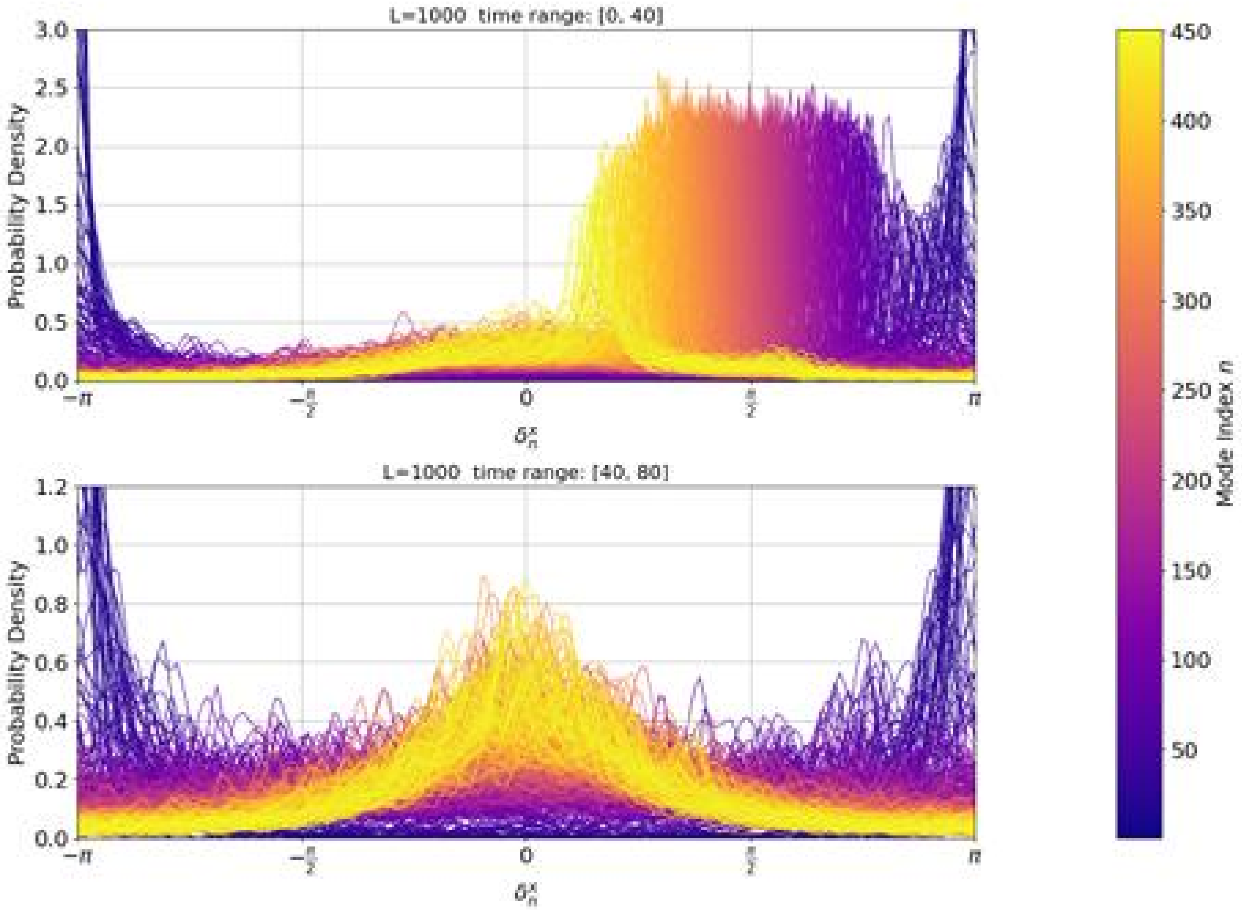
A neater proof of integrability is given in "Single-wave solutions ..." by D. F. G. Fiorillo and and G. Raffelt

Multi-wave inteference and relaxation



We lose recurrence to the flavor eigenstate if multiple Fourier modes admit FFIs

Phase is beign synchronized



Phase difference between $(X_{L,n}, Y_{L,n})$ and $(X_{R,n}, Y_{R,n})$ defined by

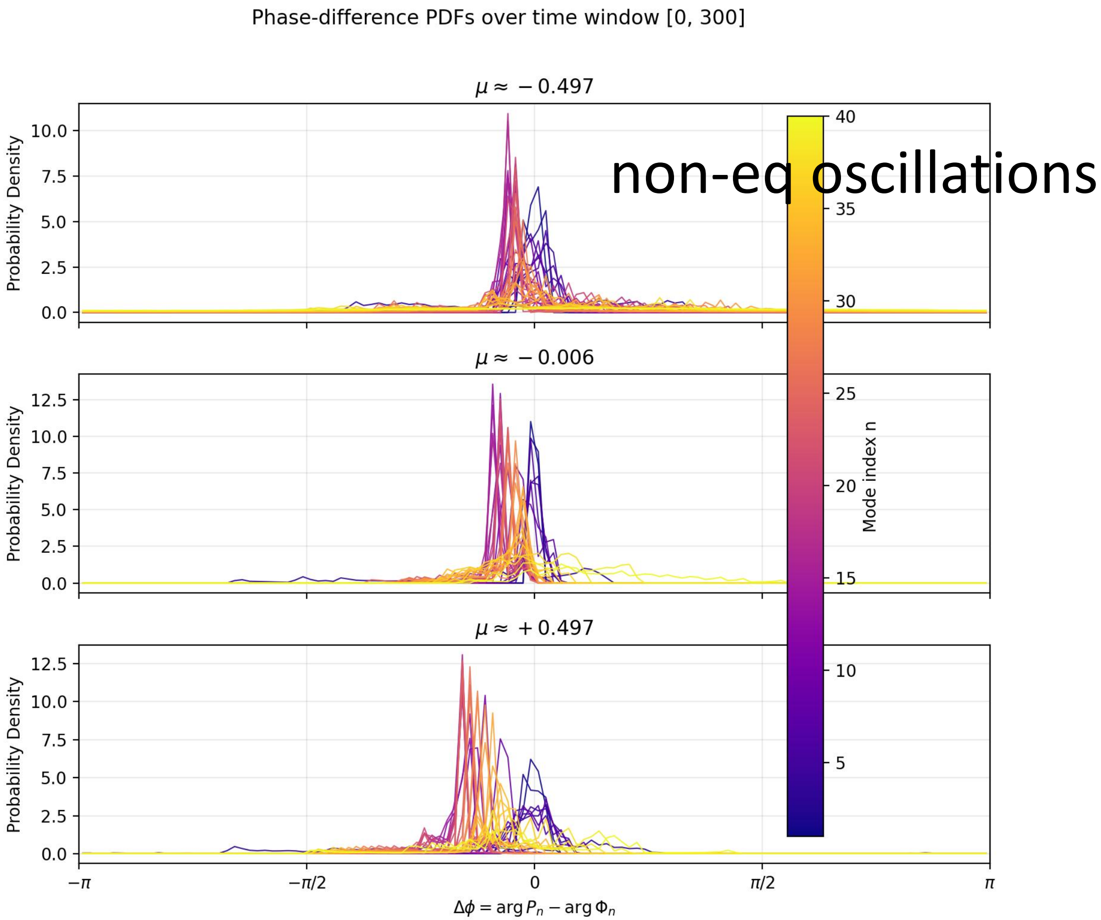
$$\delta_n = arg(X_{R,n}) - arg(X_{L,n})$$

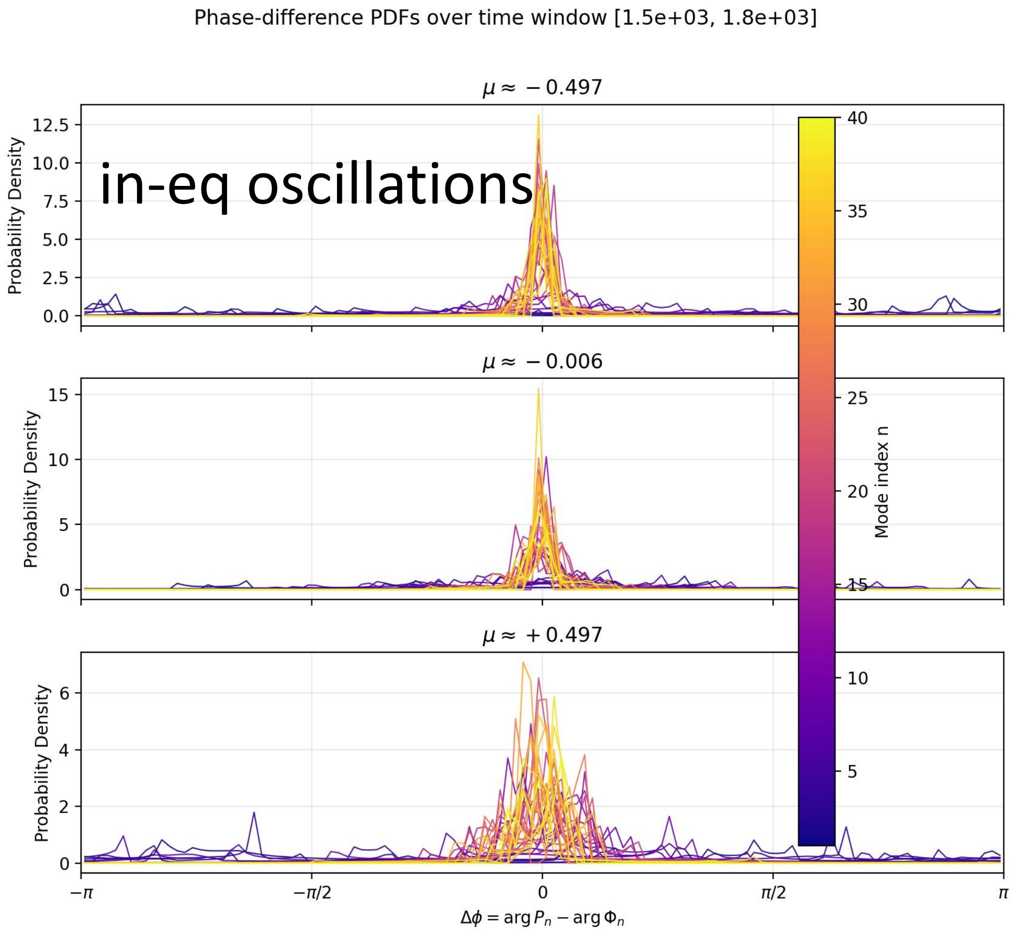
Statistics over some time window out of equilibrium and in equilibrium

“Phase synchronization” -> zero feedback to n=0

256-beam realization

The coupling $P_{L,n}^{*} \times P_{R,n}$ is extended to $(J_n^0 - v_a J_n^1)^* \times P_{a,n}$





QuasiLinear Dynamics

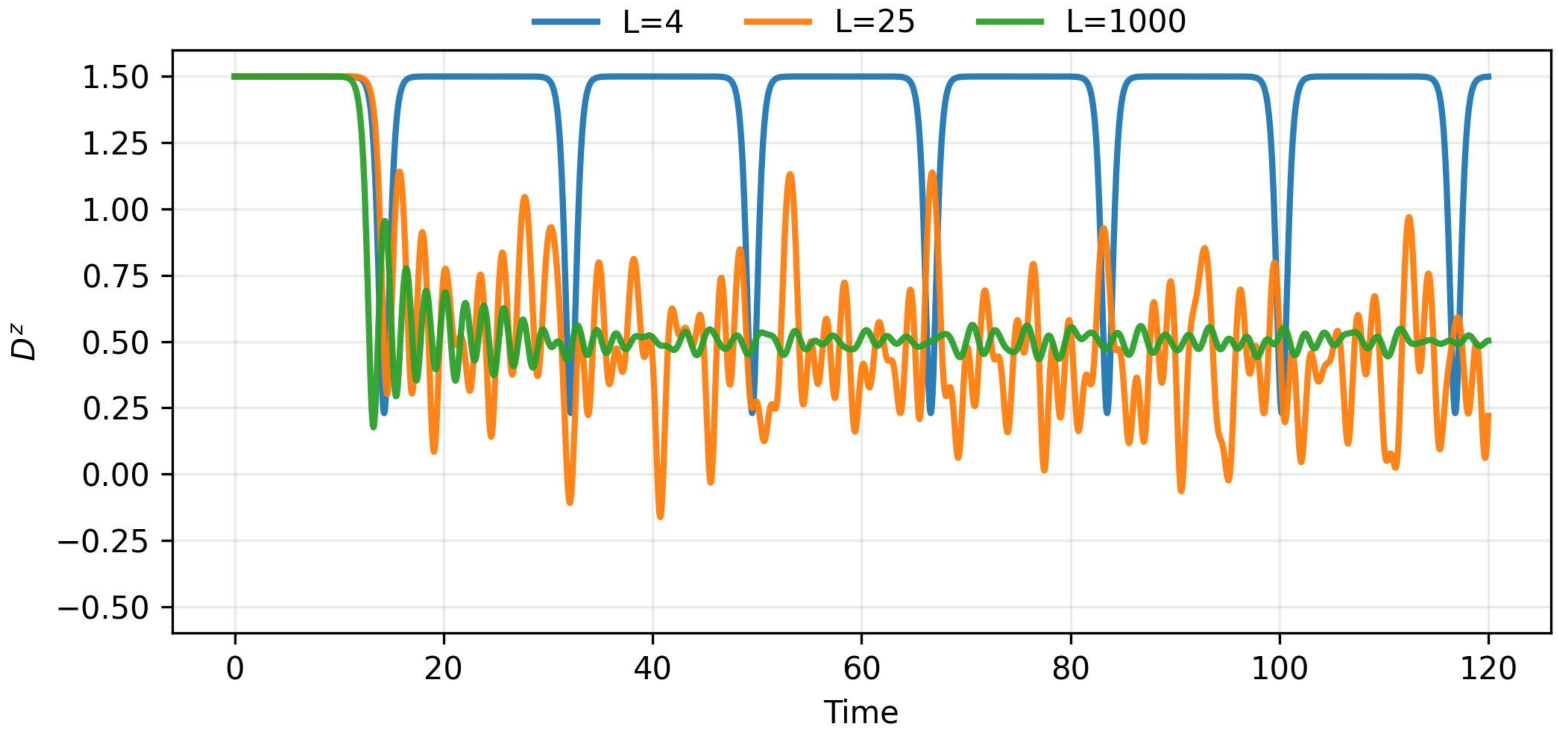
- The phase synchronization mechanism is a core feature of the QuasiLinear dynamics:

$$\dot{P}_{I,0} = 2\mathrm{Re}[\sum_n \sum_{I'} w_{I'}(1 - v_I v_{I'}) P^*_{I',n} \times P_{I,n}],$$

$$\dot{P}_{I,n} = -i v_I k_n P_{I,n} + \sum_{I'} w_{I'}(1 - v_I v_{I'})[P_{I',0} \times P_{I,n} + P_{I',n} \times P_{I,0}].$$

- The coupling (n,m) -> n+m is turned off
- The multi-wave single-helicty subspace is QL closed

Relaxation with two beams



The QL system converges onto the same asymptotic states with the QKE system over the same timescale

The phase synchronization mechanism

The full QL subspace can be separated into a collection of the n-th subspace

$$\dot{P}_{I,0} = 4Re[P^*_{I',n} \times P_{I,n}] + 4Re \sum_{n' \neq n} [P^*_{I',n'} \times P_{I,n'}],$$

$$\dot{P}_{I,n} = -iv_I k_n P_{I,n} + 2[P_{I',n} \times P_{I,0} + P_{I',0} \times P_{I,n}].$$

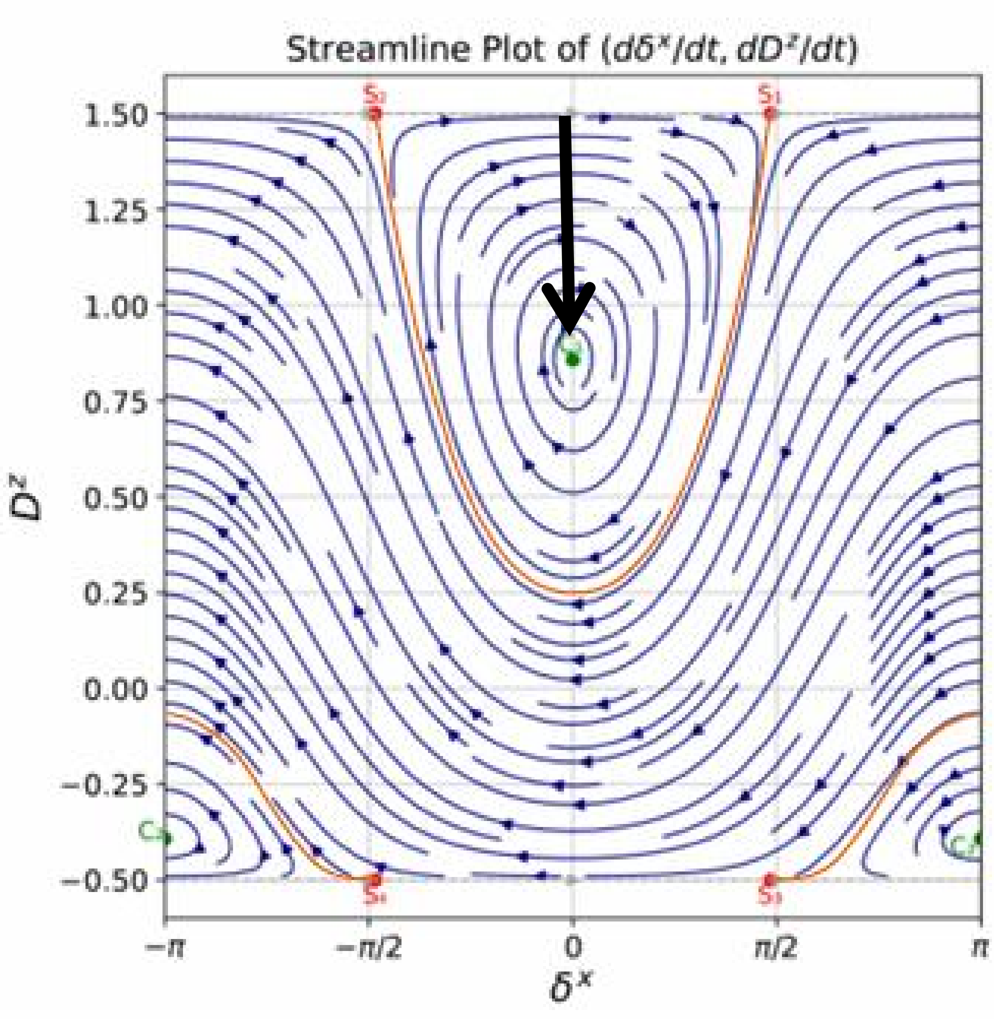
The collective feedback from the collection of $n' \neq n$ flavor waves is rendered as a noise in the n-th subspace

The phase synchronization mechanism

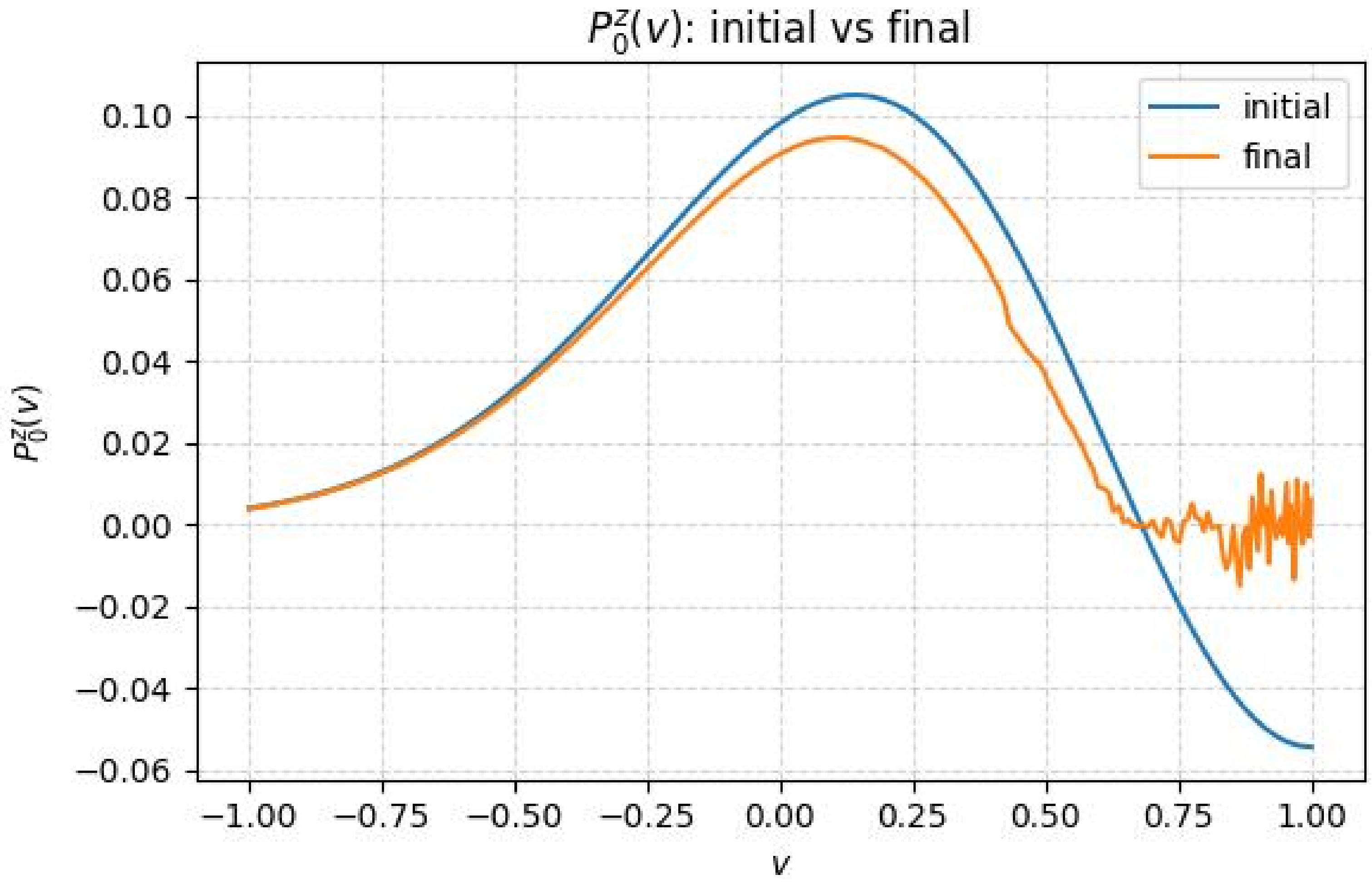
Noise collapse the periodic orbit to the center

A collection of n-th subspaces collectives lock onto a common center

The center is phase-synchronized



QL 256-beam relaxation

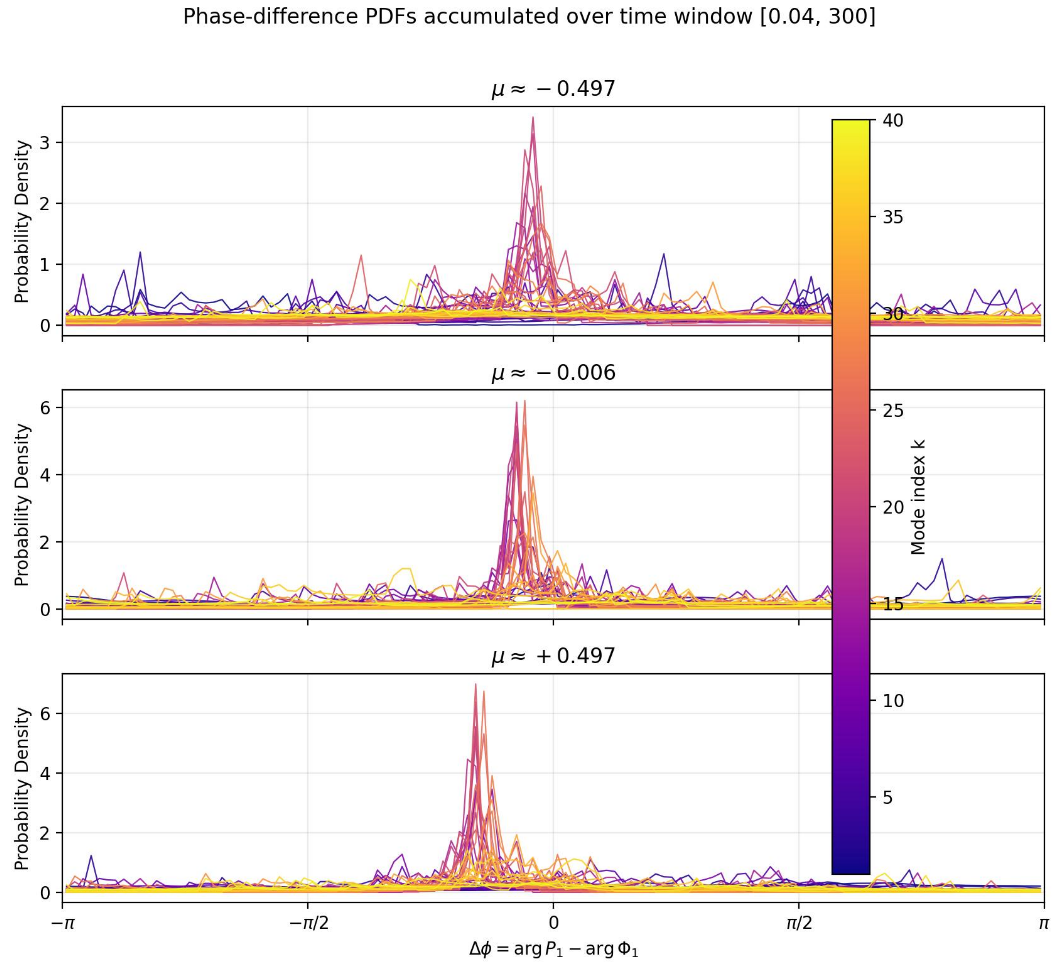


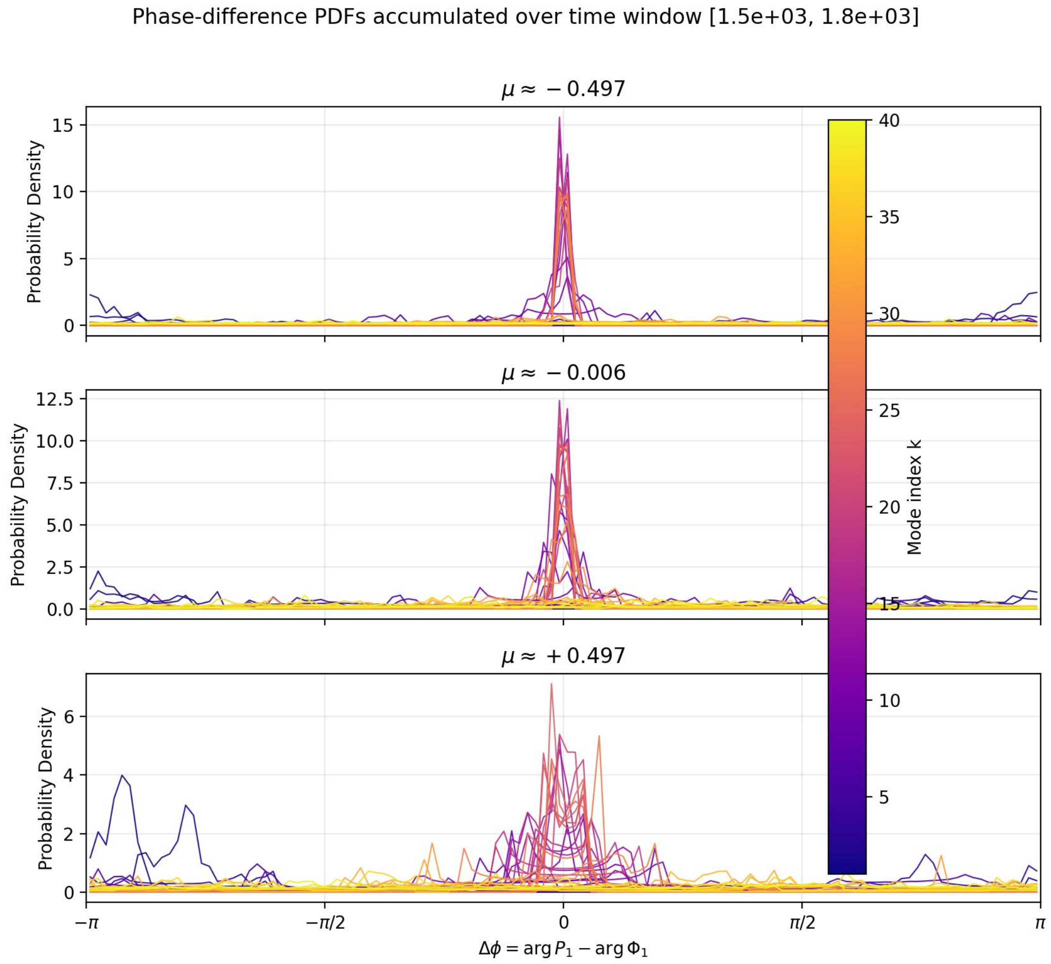
Convergence timescale: same as QKE

Asymptotic state: same as QKE

QL phase synchronization mechanism: the same argument as with 2 beams

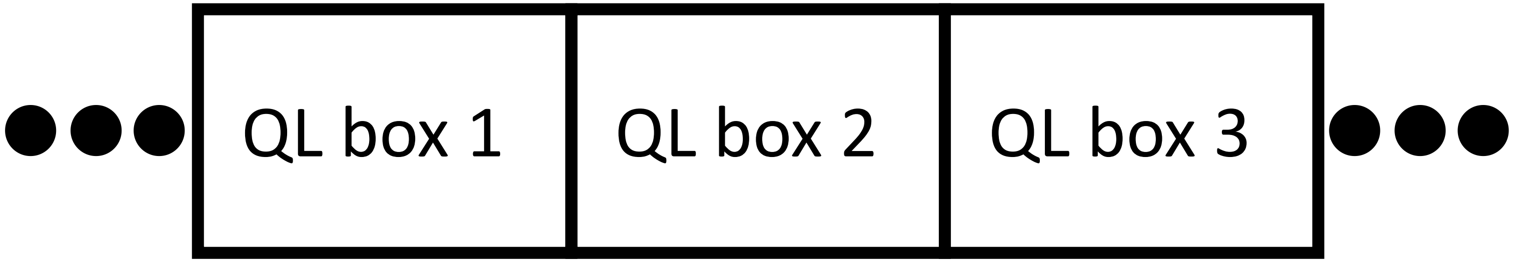
QL 256-beam phase synchronization





Ambitious proposals of QL dynamics

- We transport asymptotics states together with spatial structures represented by flavor waves across adjacent classical mesh points



- Adopt the QL asymptotic state as the QKE asymptotic state with arbitrary instabilities and collisions

Three-flavor BGK

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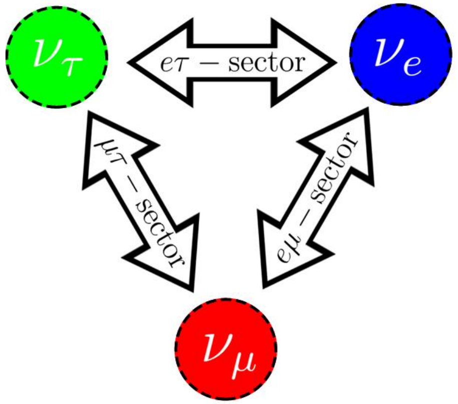
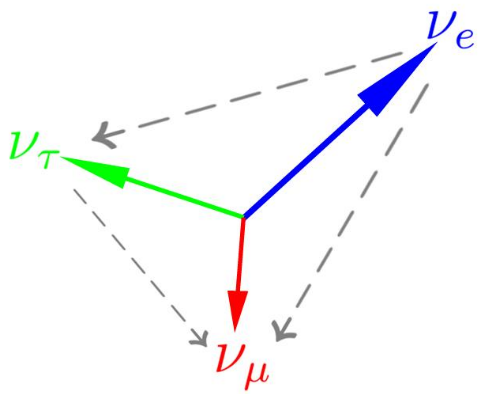
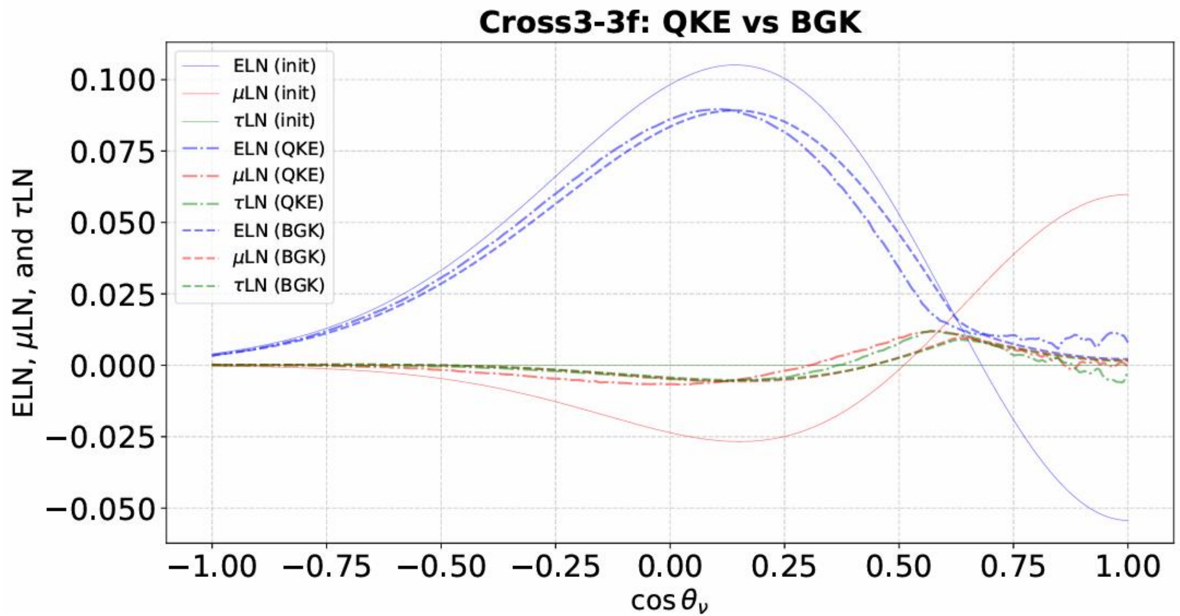
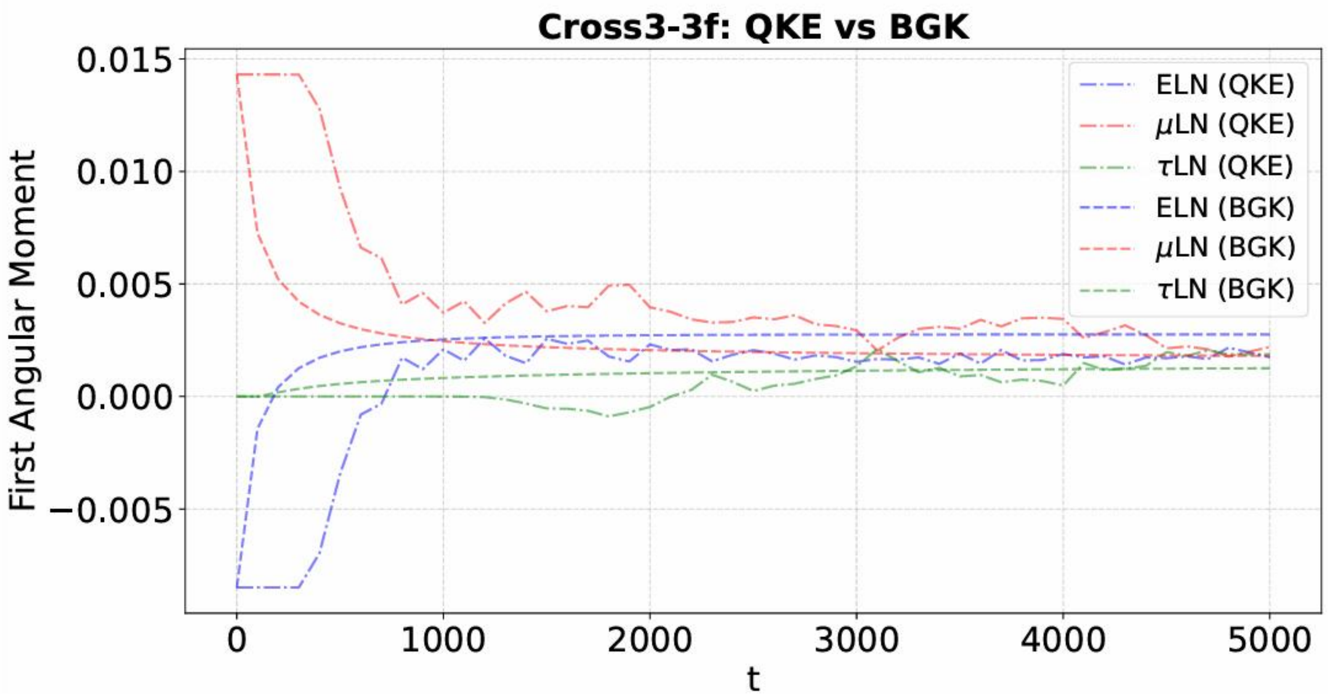
$$\frac{df_e}{dt} = -\frac{f_e - f_e^{a,e\mu}}{\tau^{e\mu}} - \frac{f_e - f_e^{a,e\tau}}{\tau^{e\tau}},$$
$$\frac{df_\mu}{dt} = -\frac{f_\mu - f_\mu^{a,e\mu}}{\tau^{e\mu}} - \frac{f_\mu - f_\mu^{a,\mu\tau}}{\tau^{\mu\tau}},$$
$$\frac{df_\tau}{dt} = -\frac{f_\tau - f_\tau^{a,e\tau}}{\tau^{e\tau}} - \frac{f_\tau - f_\tau^{a,\mu\tau}}{\tau^{\mu\tau}}.$$







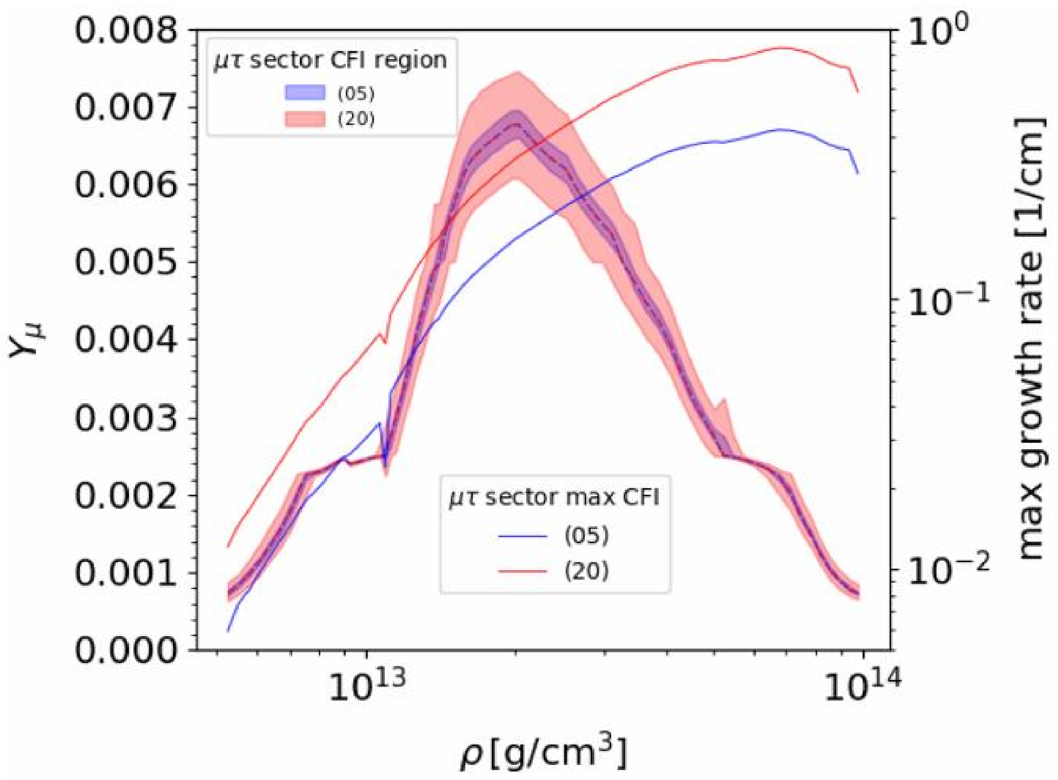


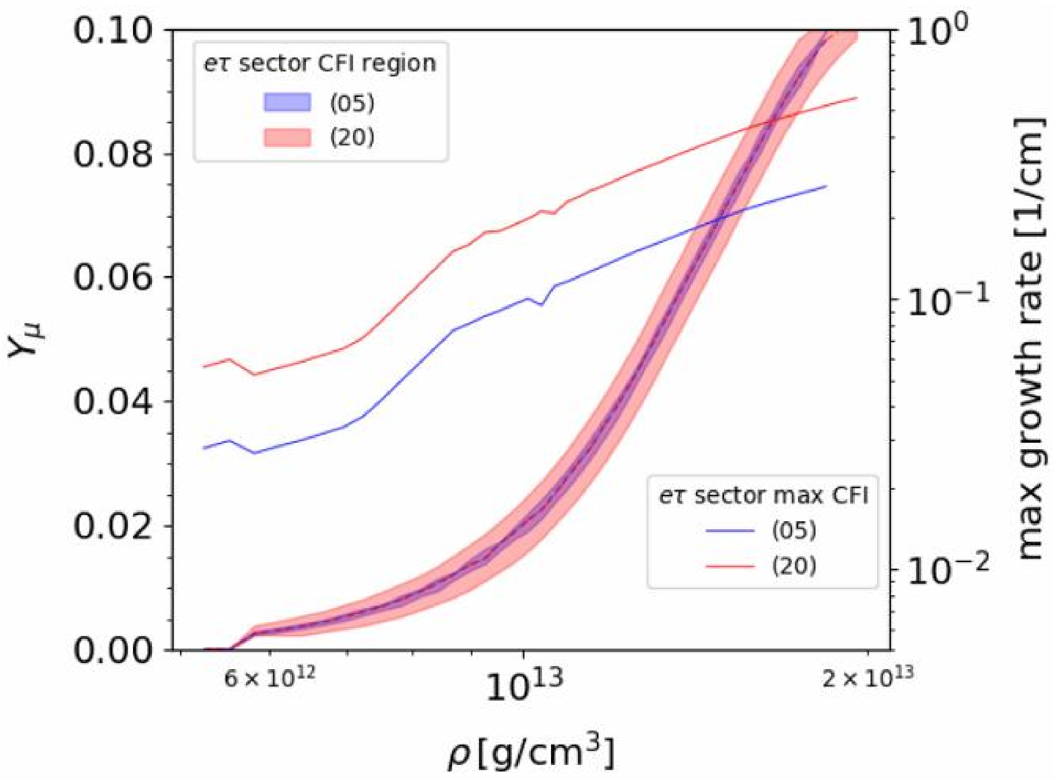
We can glue 2-flavor components together to make a 3-flavor recipe

Muonization boosts resonance CFI

J. Liu+ Phys. Rev. D 110, 043039

Muons inevitably exist near PNS and in mergers
They enhance asymmetry and induce resonacne CFI





Appendix

- Linear Stability Analysis (2 pages)
- Oh, a peak shift and big enhancemant in CFI
- Method C for mixed modes
- A P vector drags a P vector
- A nice visualization of nearly synchronized flavor waves
- Exercise 1: stochastic QKE

Linear stability analysis

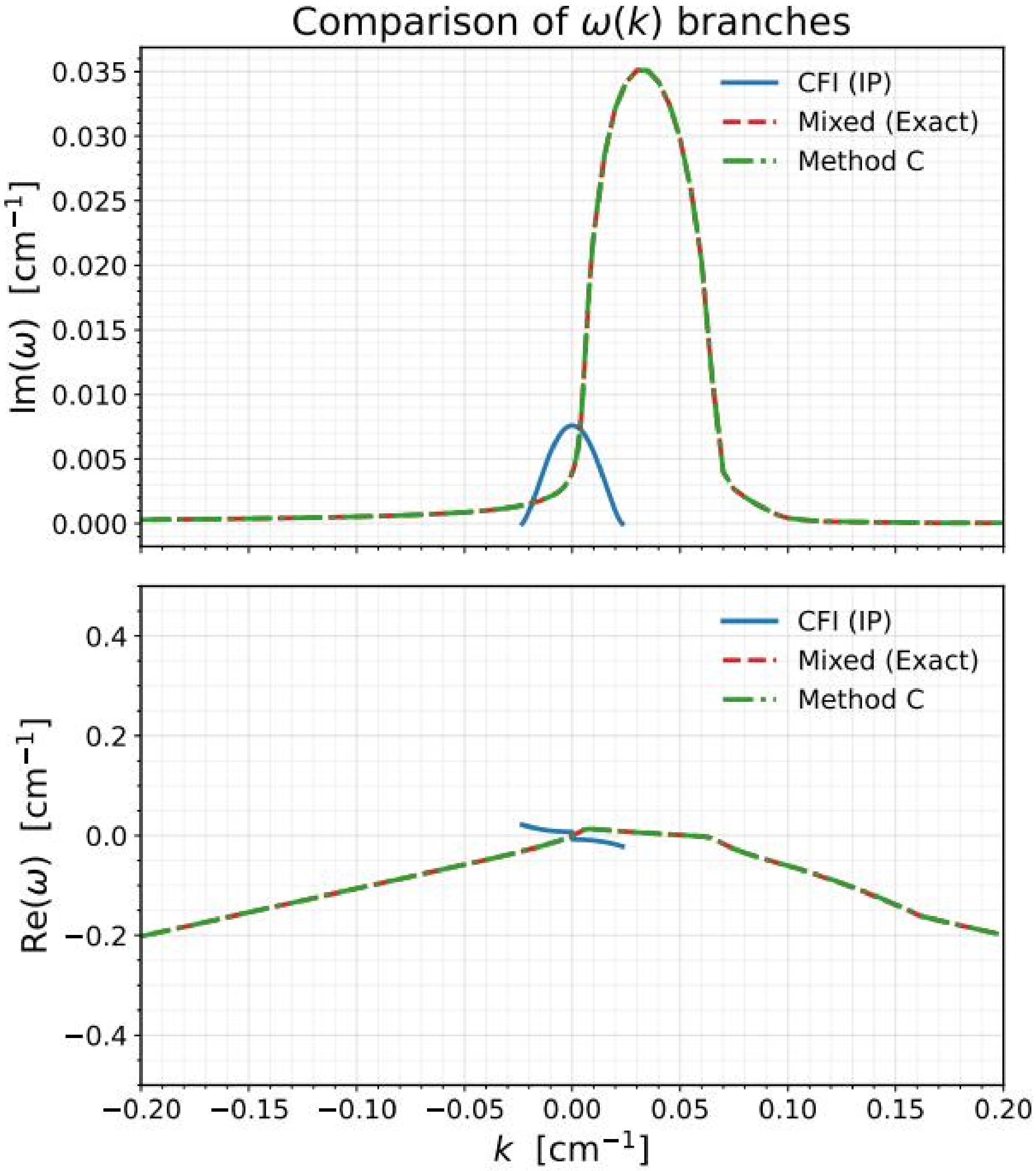
- **Linearize in S:** $\rho = \rho_0 + \delta\rho \quad \delta\rho = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$
- **Seek normal modes of the linearized transport operator:** $S(x,P) = S(k,P)e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}$
- **Instability corresponds to $\omega(\boldsymbol{k})$ with $\boldsymbol{Im}[\boldsymbol{\omega}] > \mathbf{0}$, i.e., exponential spectral growth**

Solving for normal modes

- **Linearized QKE can be recast into the form**

$\Pi^{\mu\nu} a_\nu = 0$, **with** $\Pi^{\mu\nu}(\omega, \boldsymbol{k}) = \eta^{\mu\nu} + \int dP \, \frac{\Delta f(E,\boldsymbol{v}) v^\mu v^\nu}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Gamma(E)}$**,**
and $\boldsymbol{a^\mu} \propto \int \boldsymbol{dP\, v^\mu\, S(E, v)}$

- **Nontrivial solutions requires** $\boldsymbol{det\, \Pi = 0}$
- **Instabilities correspond to roots in the upper half of the complex ω-plane**
- **Solutions are known as dispersion relations** $\omega(\boldsymbol{k})$

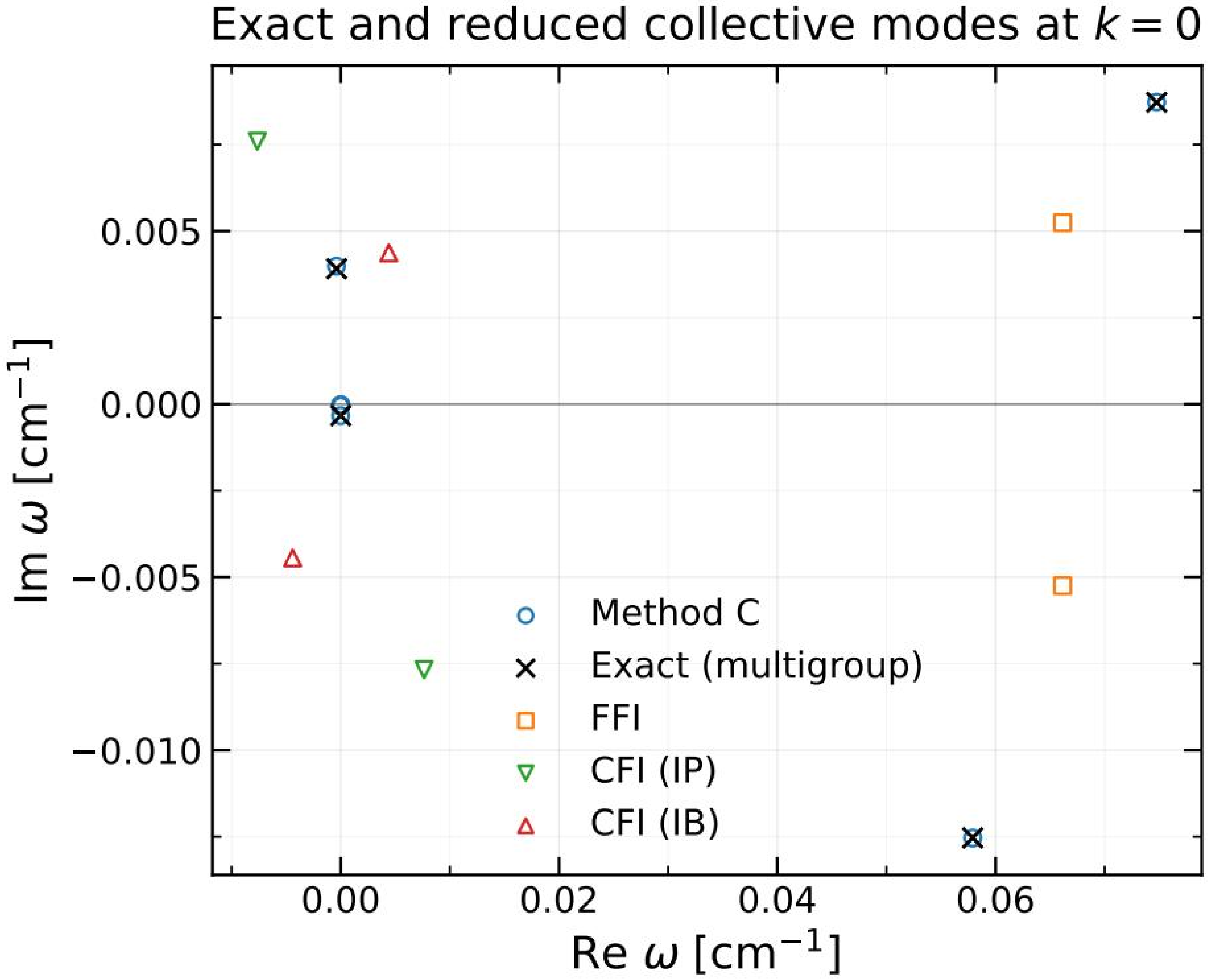


"Oh, CFI? No fast contributions here?"

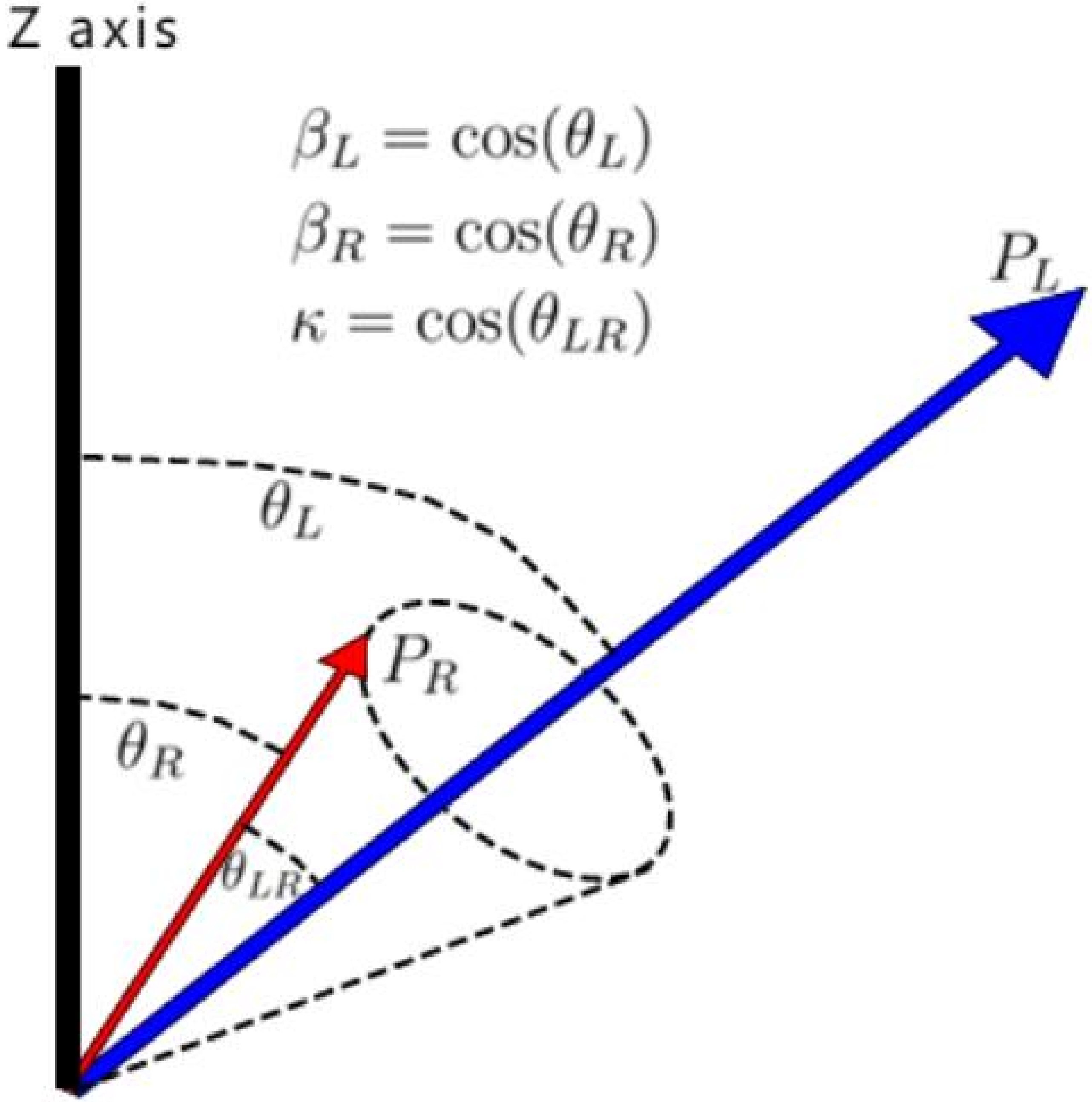
"Oh, its peak is shifted from k=0?"

"Oh, it's enhanced by axisymmetry?"

"Oh, wait, the approximation agrees exactly at even k>0!"

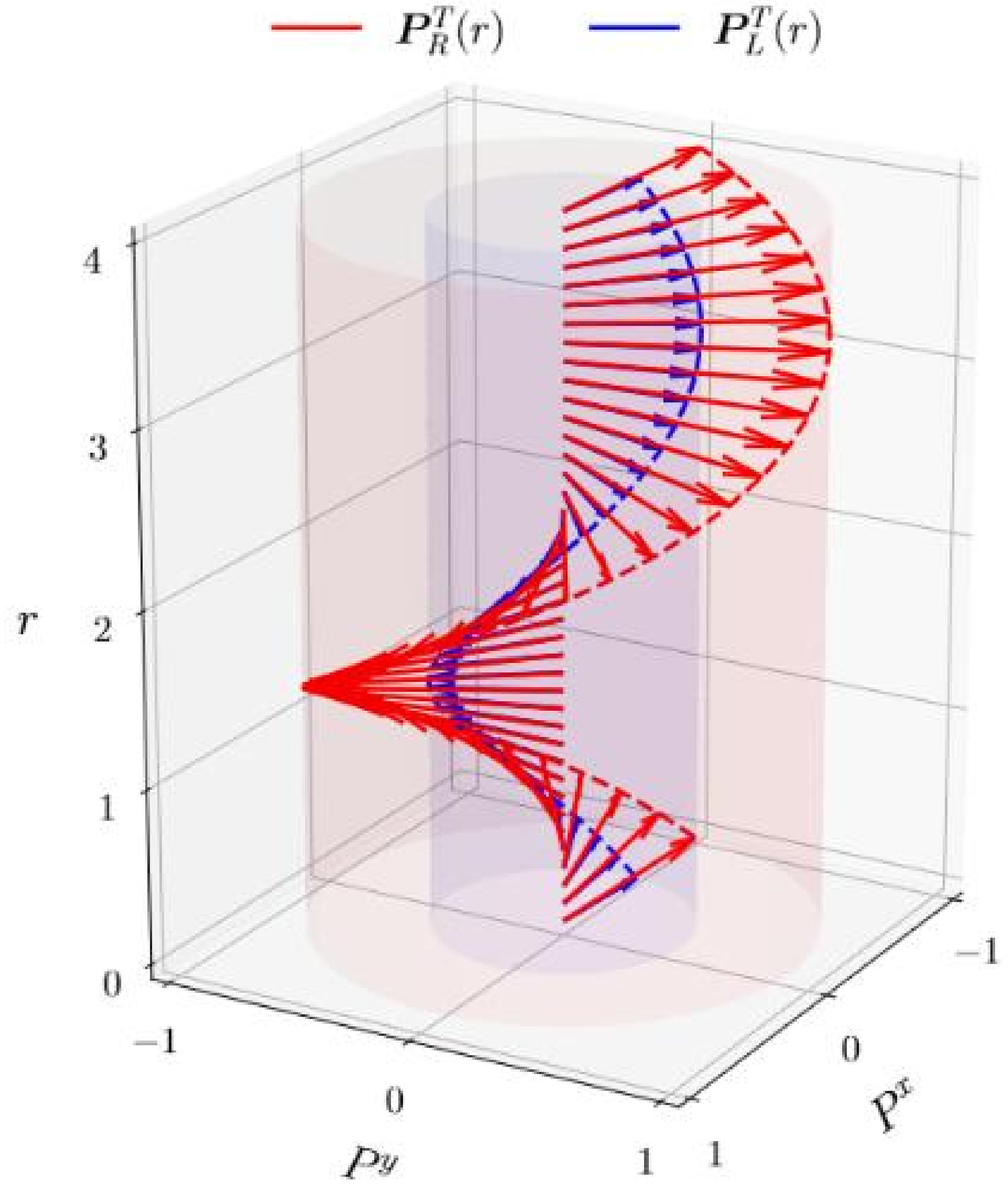


The most significant advantage of a first-principle based approximation guideline is it enables us to faithfully envision what should be seen



Question 1:
Can you interpret the evolution by treating the neutrino injection as a Coriolis force?

Question 2:
Can you derive the analytical solution to the full QKE evolution ignoring spatial advection?



Flavor waves are completely natural structure that people should consider in flavor conversions since they are selected by the polarization of the coarse-grained vector to be aligned along the Z axis!

The transverse space is natually decomposed into + and - helicities!

Exercise 1

- Show that the stochastic single-wave quantum kinetic ODE does collapse onto the center manifold. What types of noises lead to the collapse?