Monodromy Approach to Quasi-Normal Modes of Black Holes

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Question

What is monodromy?

What is Quasi-Normal Modes?

Question

What is monodromy?

This is a **math** question.

What is Quasi-Normal Modes?

Question

What is monodromy?

What is Quasi-Normal Modes?

This is a **physical** question.

It **motivated** us to learn the monodromy.

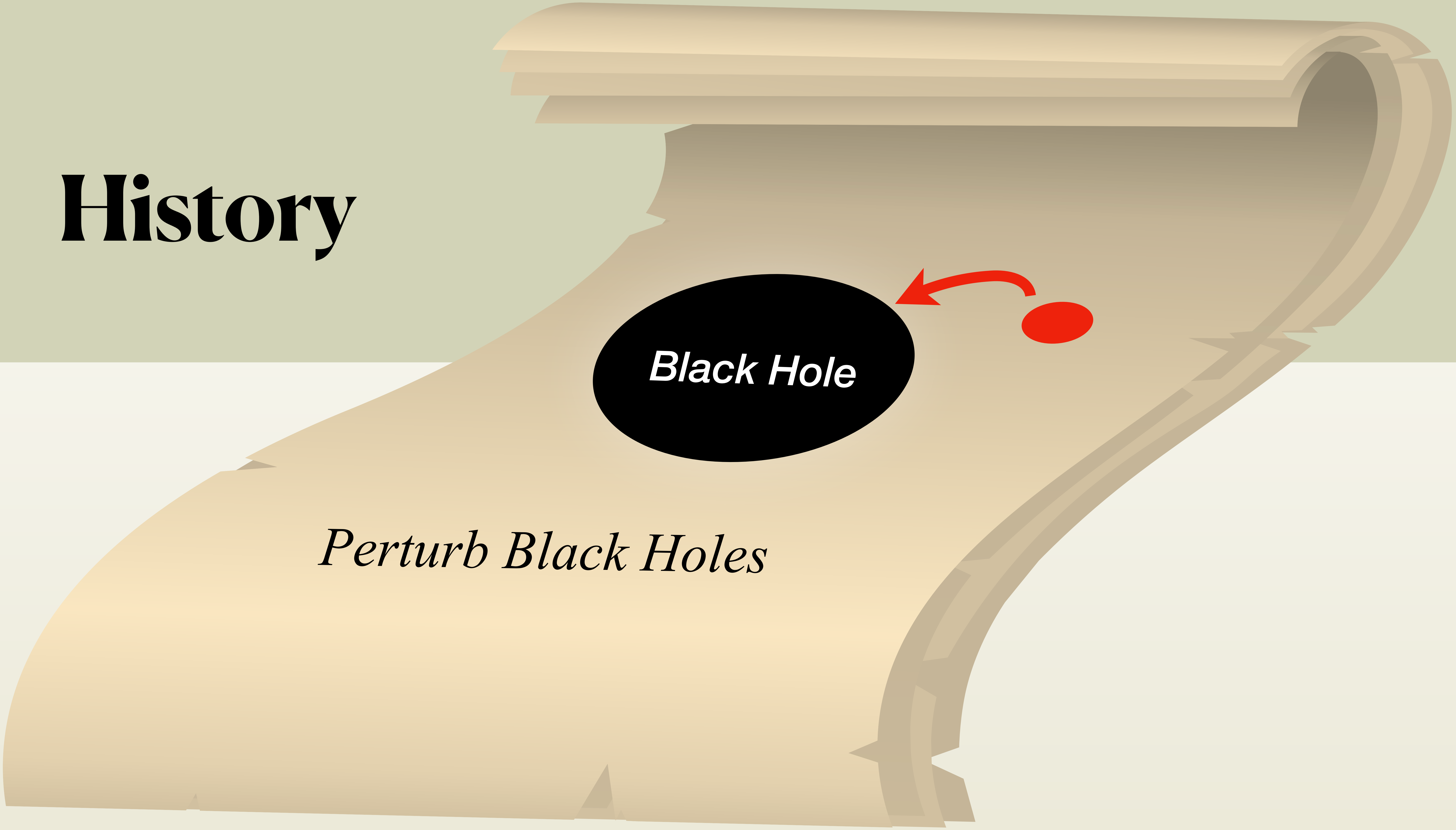
Outline

History —QNMs

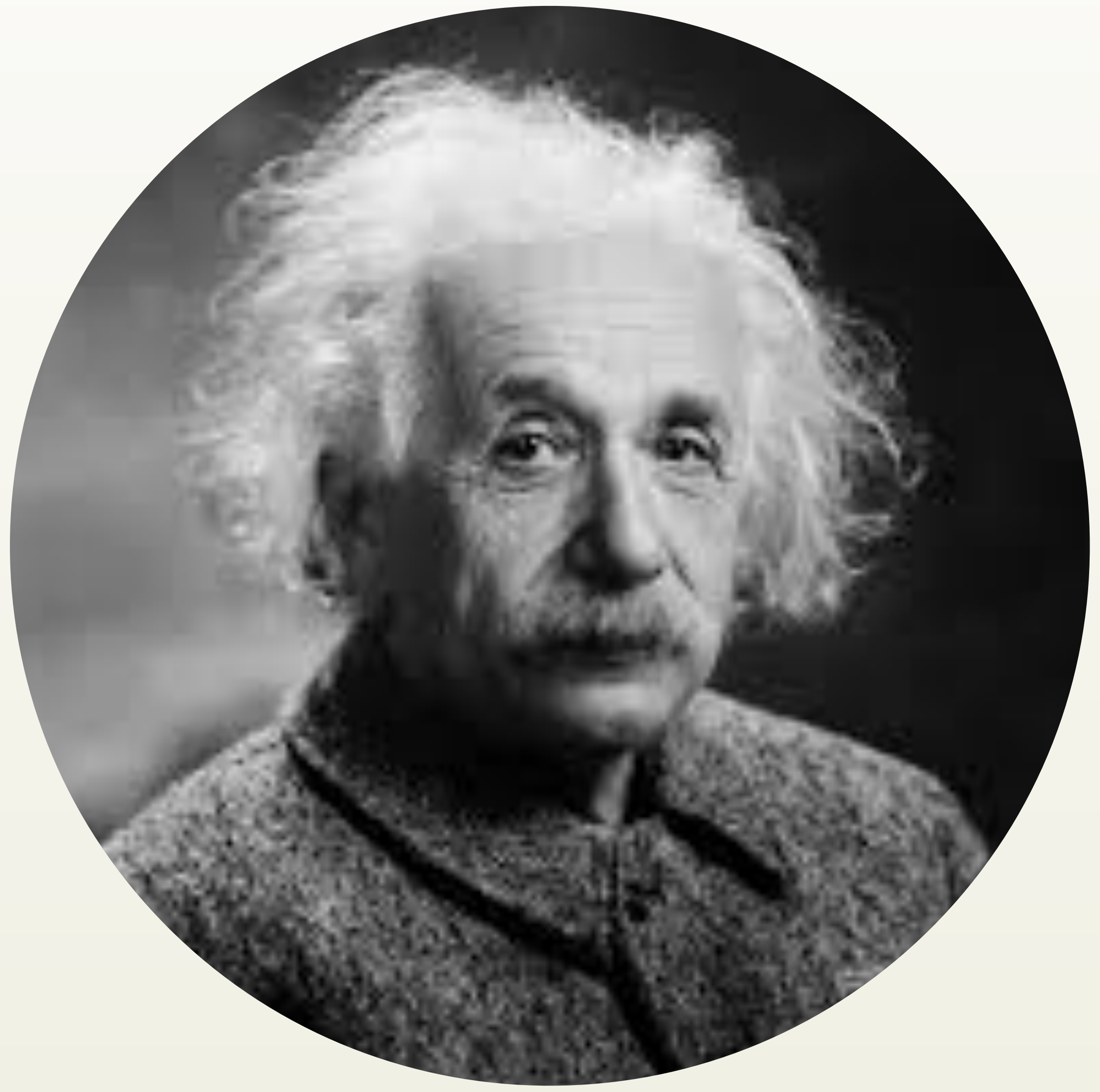
Monodromy

Summary

History



History



Einstein

developed General Relativity around 1907.

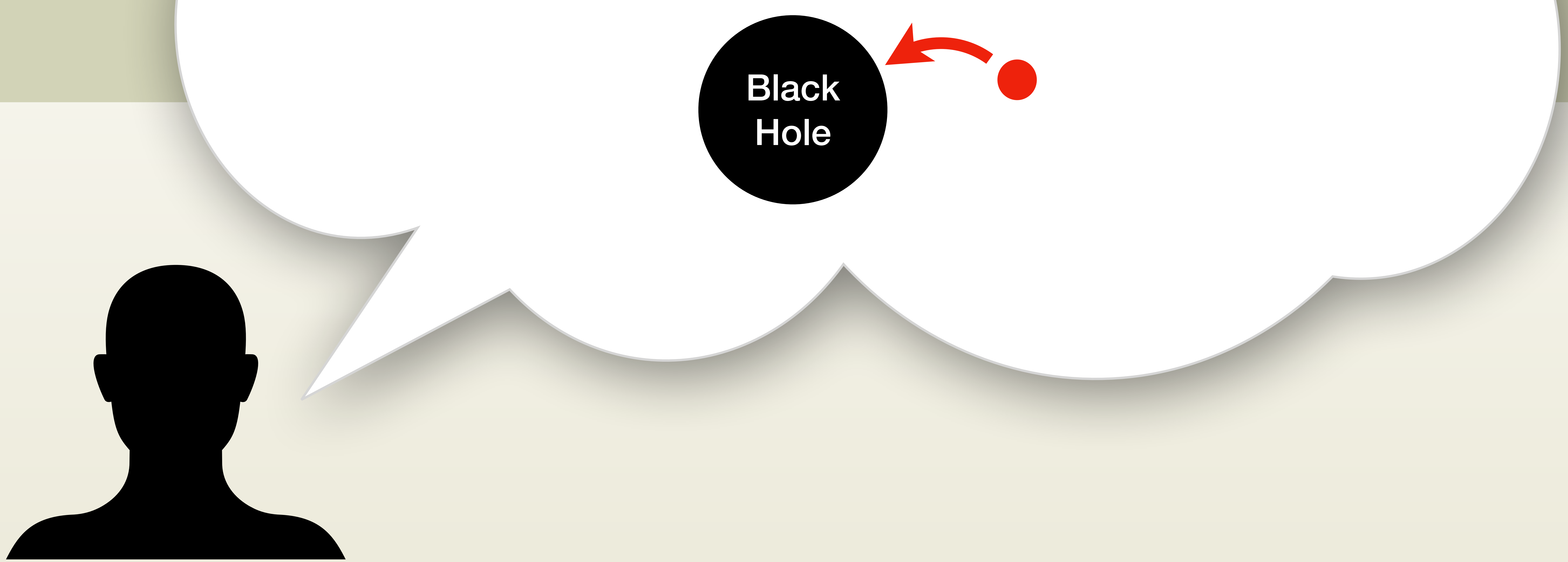
Schwarzschild

obtained the solution to Einstein equation in 1916.

Wheeler

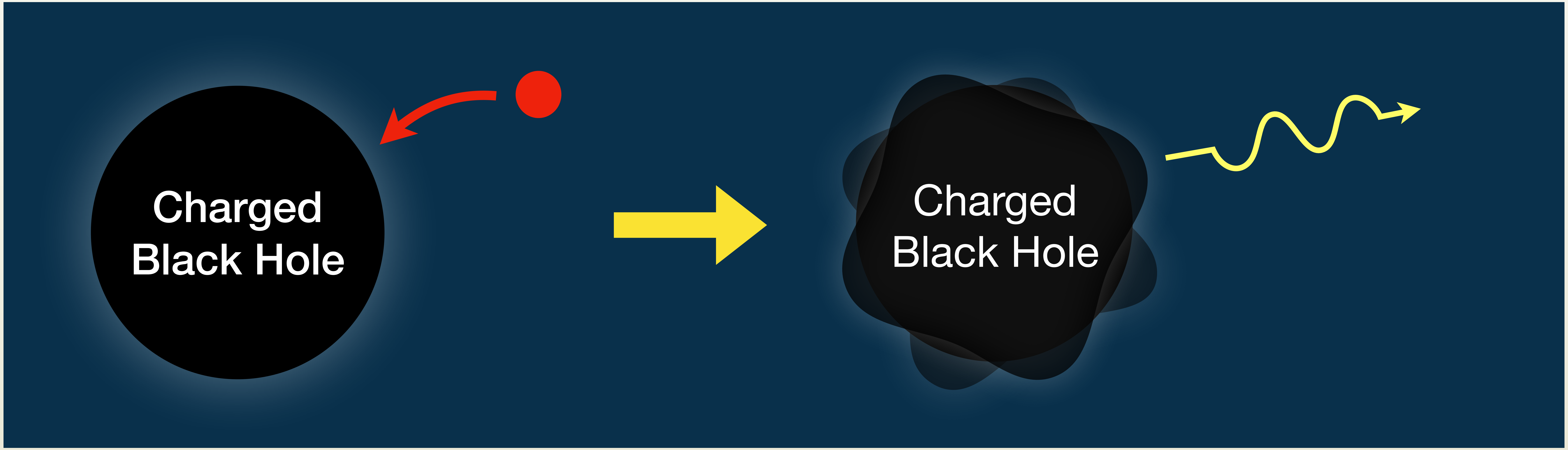
coined the term "**black hole**" in 1967.

Thought Experiment



Thought Experiment

Classic	Nothing escapes.
Semi-Classic	Some information will radiate out.



Thought Experiment

$$\mathrm{d}^2 s = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega^2$$

Schwarzschild BHs: $f(r) = 1 - \dfrac{2M}{r}$

Charged BHs: $f(r) = 1 - \dfrac{2M}{r} + \dfrac{Q^2}{r^2}$

+

Klein-Gordon equation

$$\left(D_\nu D^\nu - m^2\right)\Phi = 0$$

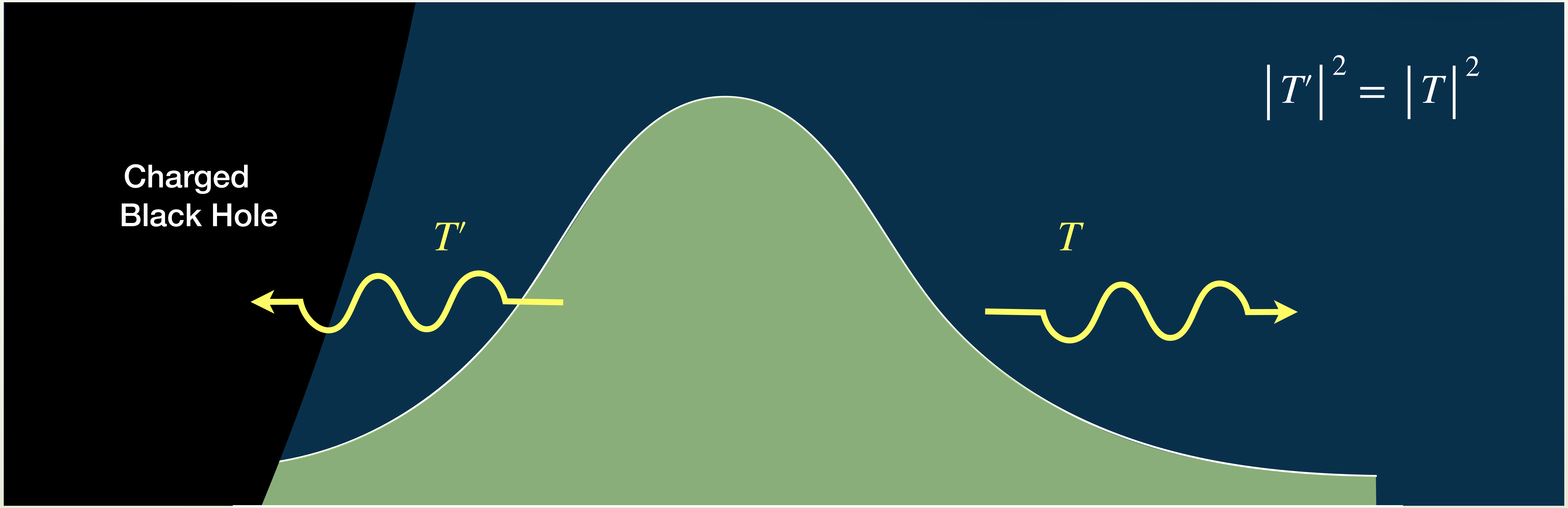
Thought Experiment

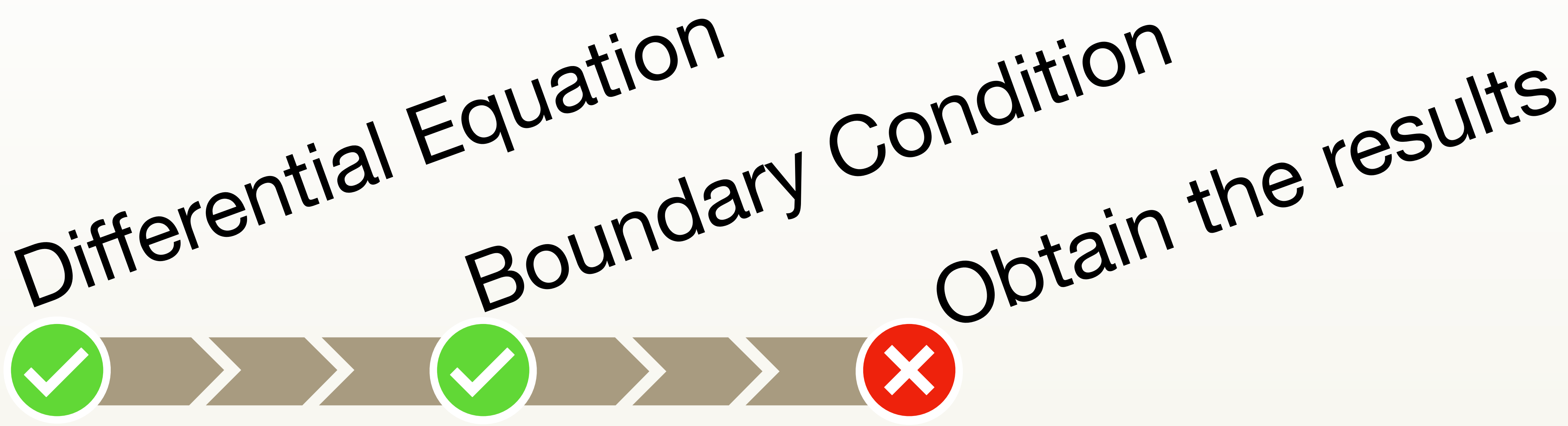
$$0 = \partial_r^2 R + \left(\frac{1}{r - r_+} + \frac{1}{r - r_-}\right)\partial_r R$$
$$+\frac{1}{r^4 f^2}\left[\left(\omega^2 - m^2\right) r^4 + 2\left(Mm^2 - qQ\omega\right) r^3 + Q^2\left(q^2 - m^2\right) r^2 - r^2 f \lambda_l\right] R$$

These the radial part of Klein-Gordon equation is has two regular singular points at r_- and r_+ and one irregular singular point at infinity, which known as the kind of confluent Heun equation. But we don't completely know the confluent Heun function so far.

Quasi-Normal Mode

Thought Experiment





Thought Experiment

$$0 = \partial_r^2 R + \left(\frac{1}{r - r_+} + \frac{1}{r - r_-} \right) \partial_r R$$
$$+ \frac{1}{r^4 f^2} \left[\left(\omega^2 - m^2\right) r^4 + 2\left(Mm^2 - qQ\omega\right) r^3 + Q^2 \left(q^2 - m^2\right) r^2 - r^2 f \lambda_l \right] R$$

These the radial part of Klein-Gordon equation is has two regular singular points at r_- and r_+ and one irregular singular point at infinity, which known as the kind of confluent Heun equation. But we don't completely know the confluent Heun function so far.

History — Four Methods

Deweiler (1975)
computed
numerically QNMs.

Ferrari & Mashhoon (1983)
suggested to use WKB to compute
the frequencies of QNMs.

Leaver (1985)
provided recurrence
relation to solve QNMs.

Motl & Neitzke (2003)
used monodromy technique to
approach highly damped
frequencies of QNM.

?

History

Deweiler (1975)

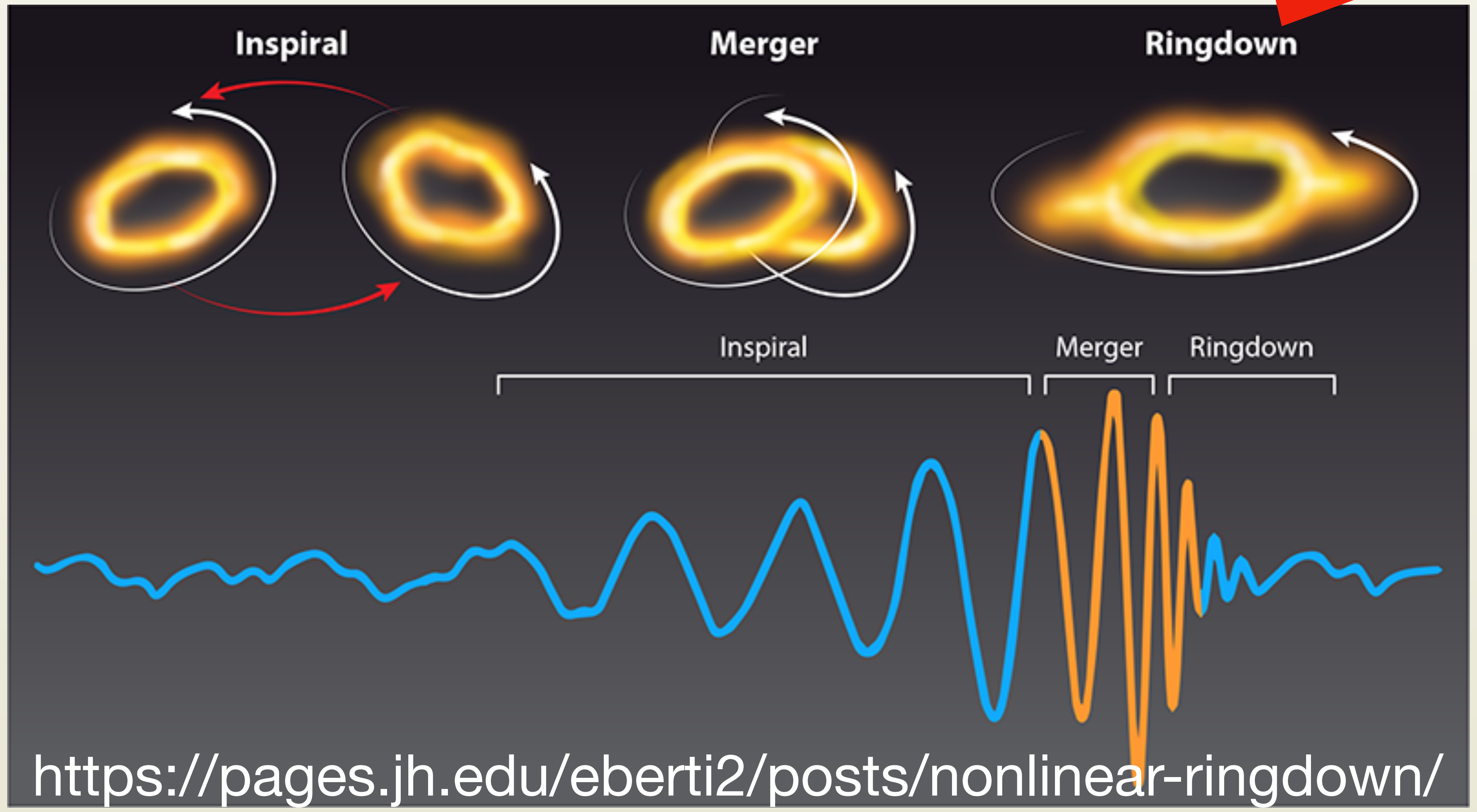
Ferrari & Mashhoon (1983)

Leaver (1985)

Hod (1998)
proposed discrete area
of black hole.

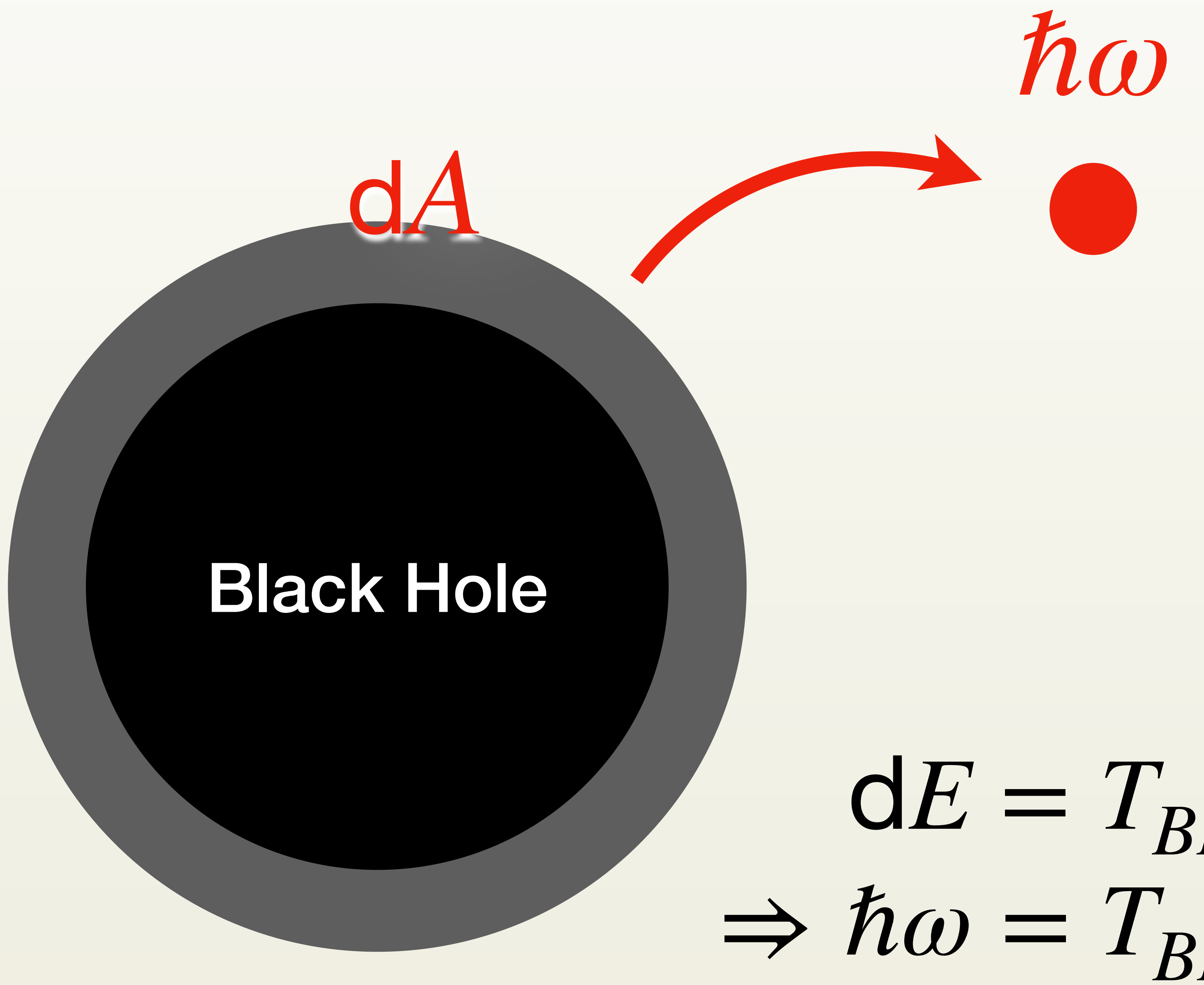
Thought Experiment

Motl & Neitzke (2003)

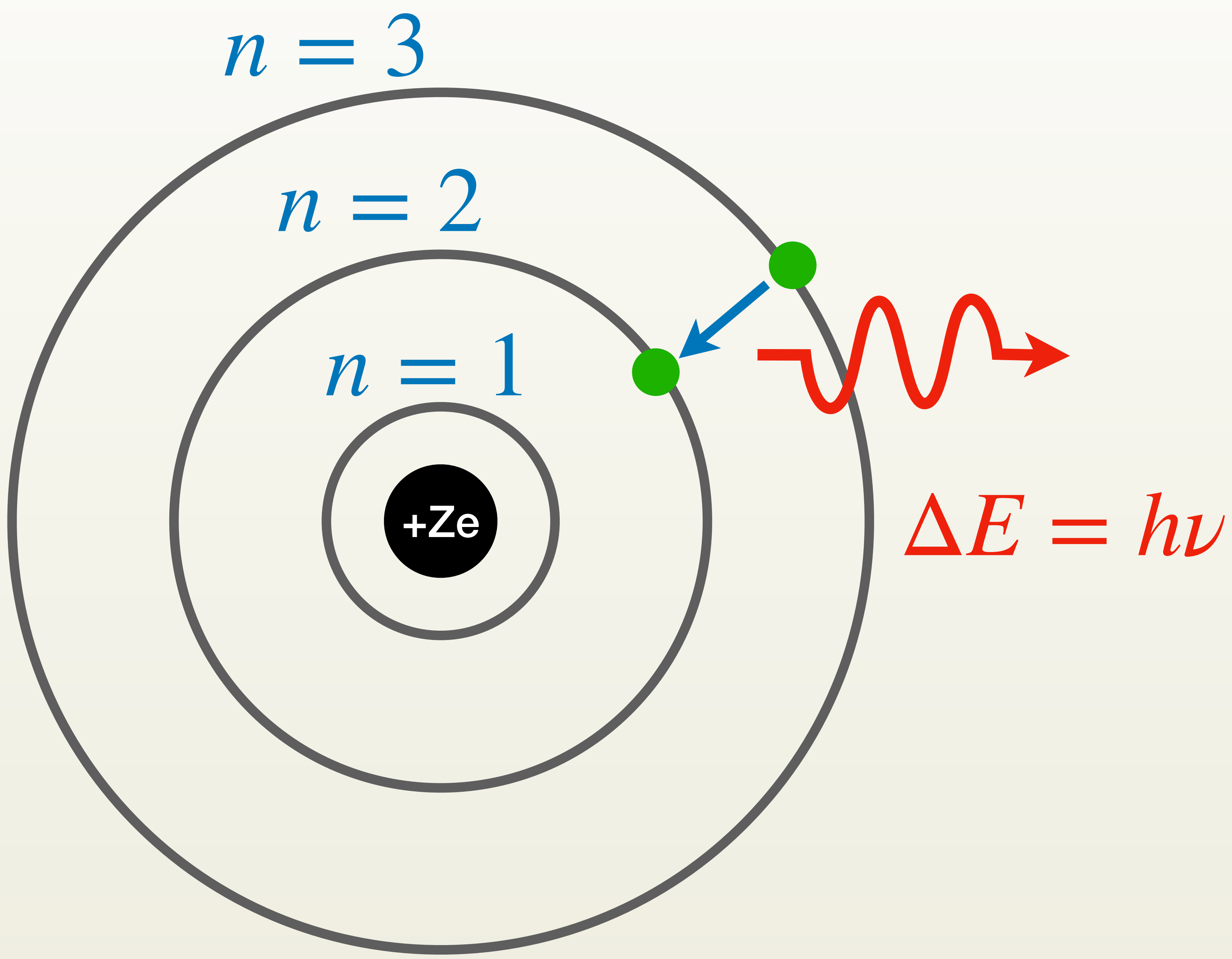


https://pages.jh.edu/eberti2/posts/nonlinear-ringdown/

History



$$
\begin{aligned}
& \mathrm{d}E = T_{BH}\mathrm{d}S \\
\Rightarrow\ & \hbar\omega = T_{BH}\mathrm{d}A \\
\Rightarrow\ & \quad \omega = T_{BH}\left(\mathrm{d}A/\hbar\right) \\
& \quad\ = T_{BH}\ln(3)
\end{aligned}
$$



Bohr's hydrogen atom model

Outline

History —QNMs

Monodromy

Summary

Question

What is monodromy?

This is a **math** question.

What is Quasi-Normal Modes?

Question

What is monodromy?

Monodromy is a math technique which is defined in **complex plane**. It can help us to analyze the differential equation with **many** regular singular points by **topology** and algebra property.

Monodromy

Usual Method

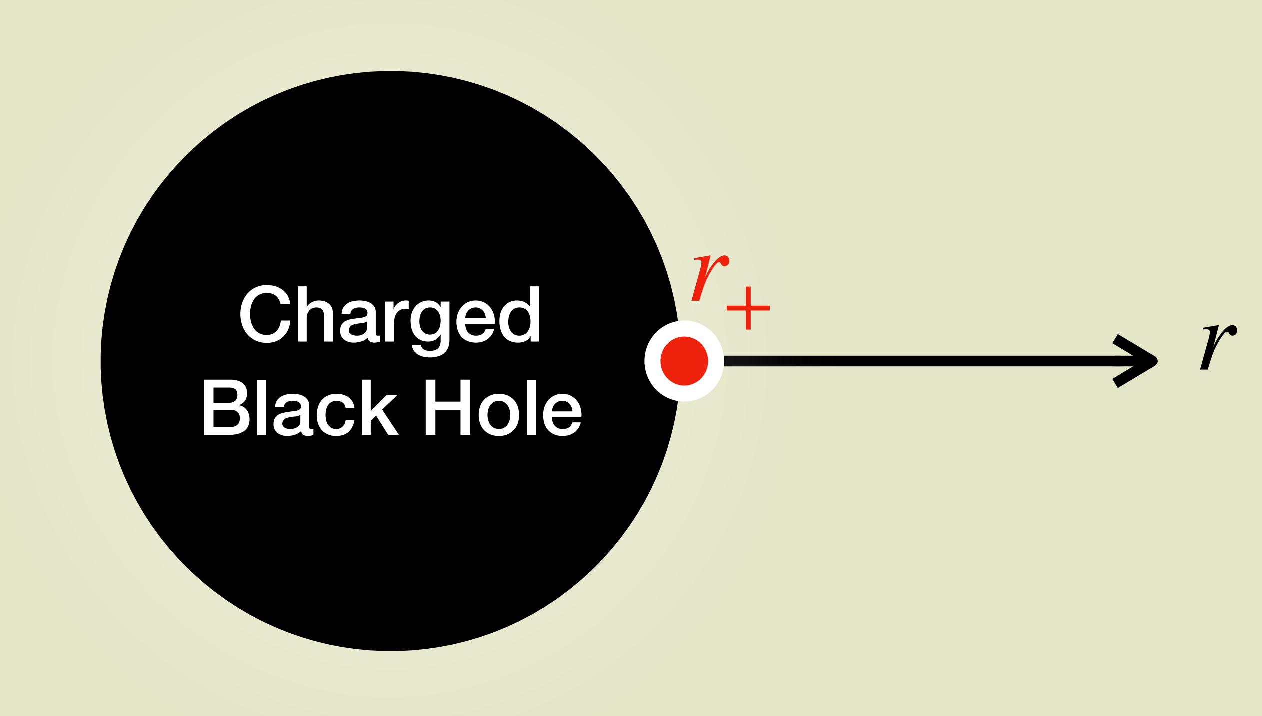
- Solve Differential Equation
- Impose Boundary Condition
- Obtain the results

Now

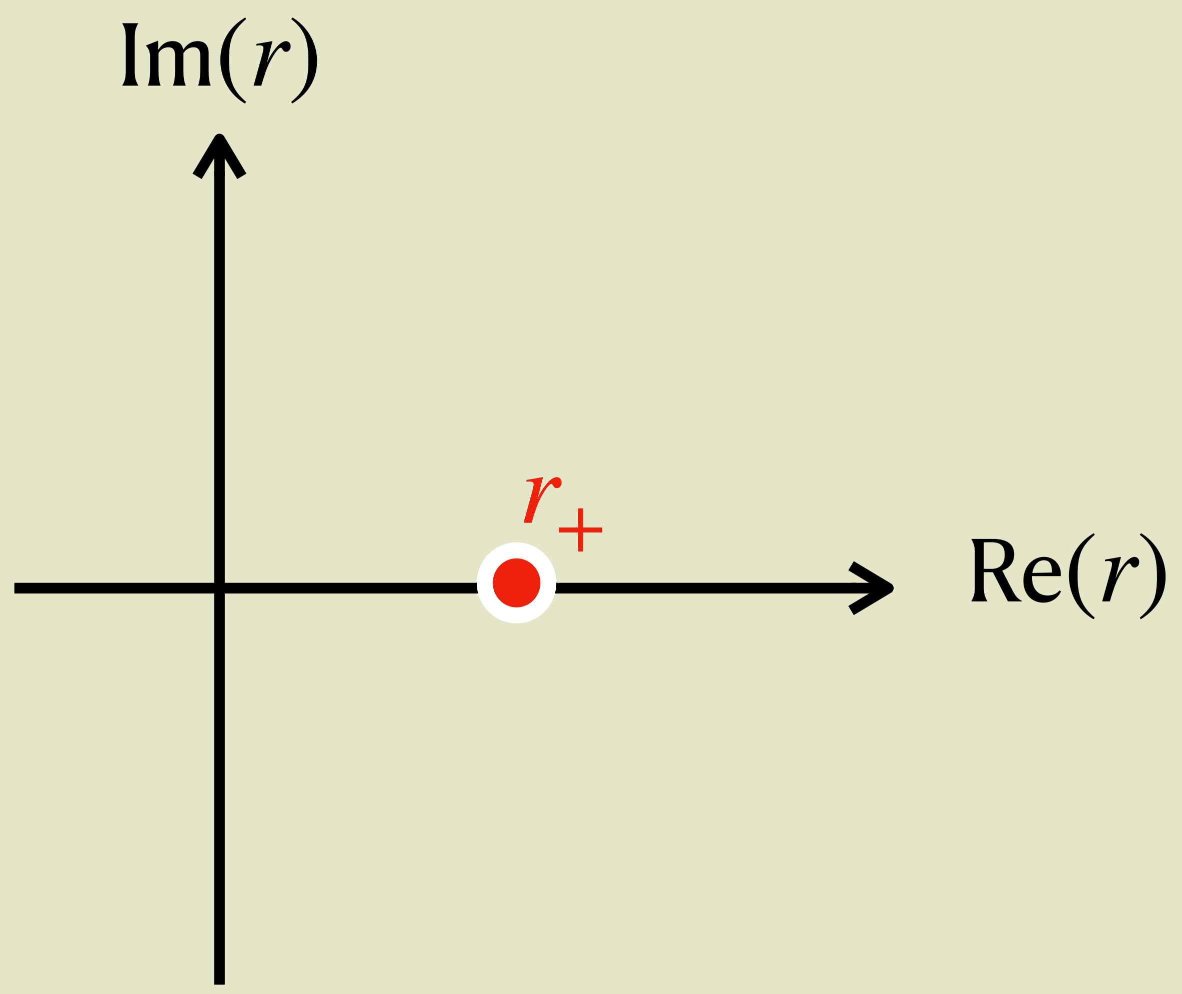
- Transform into Topological Space
- Monodromy Matrices
- Boundary condition of QNM
- Frequencies of QNM

Monodromy

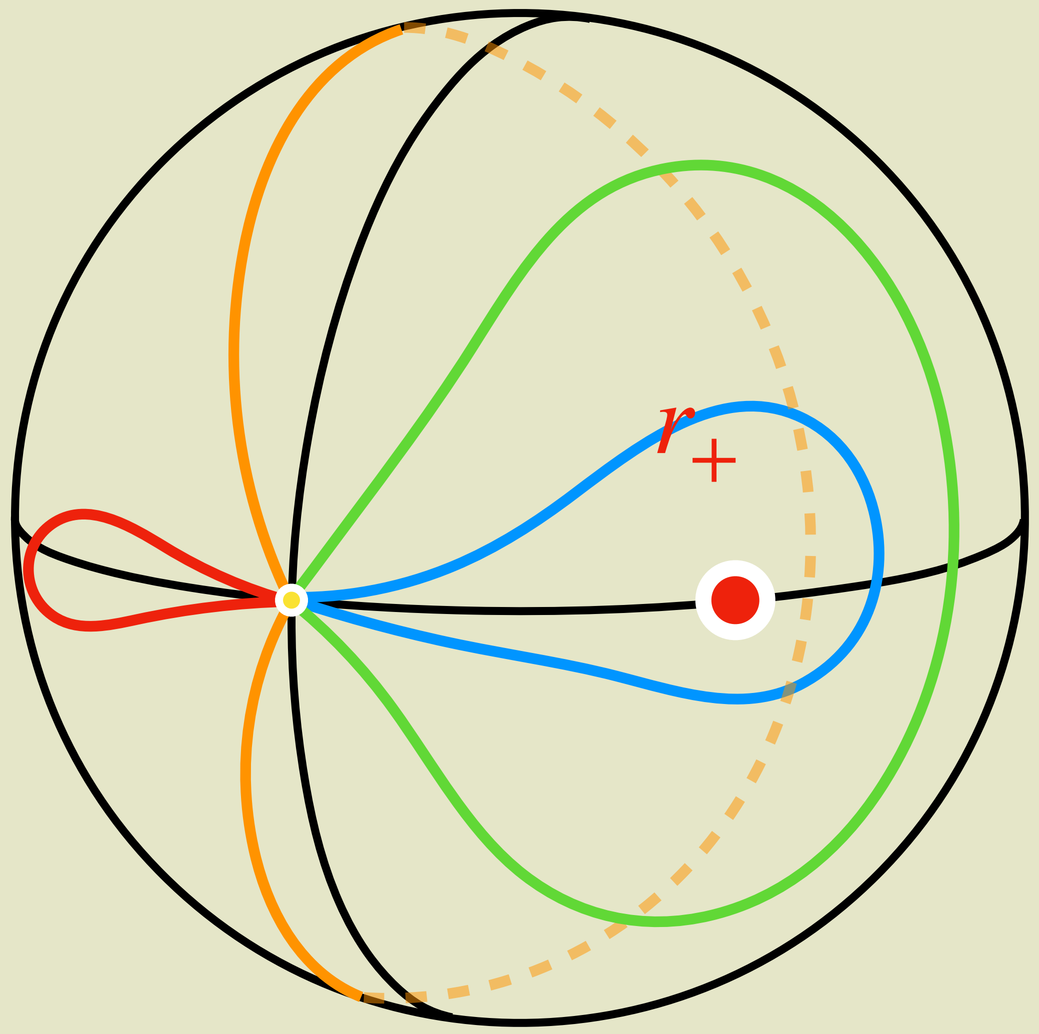
Real Space



Imaginary Space



Reimann Sphere



Monodromy

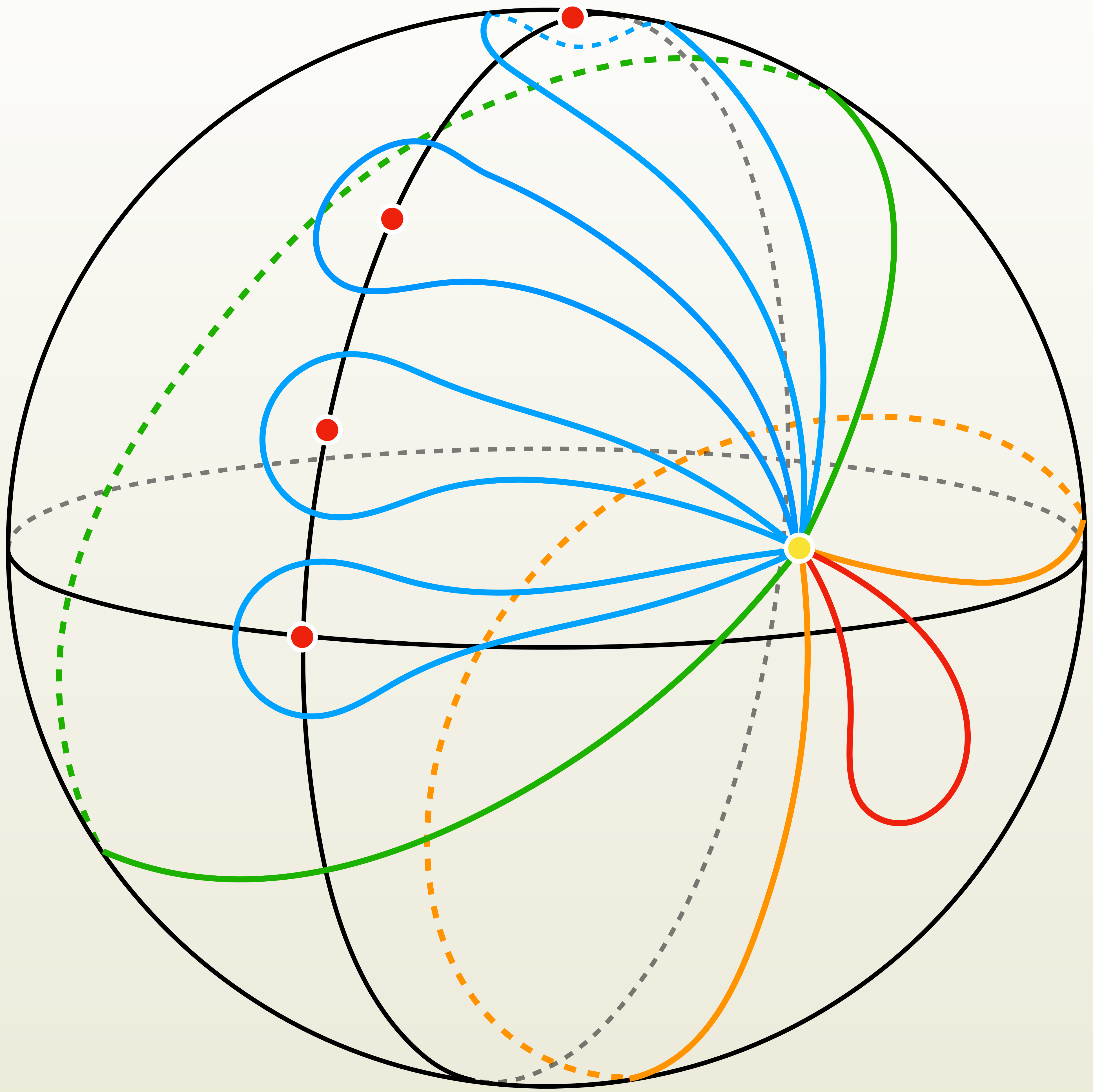
$$\boxed{\Gamma_i * \mathbf{F}_j \stackrel{def}{=} \mathbf{F}_j \mathbf{M}_{ij}}$$

Properties:

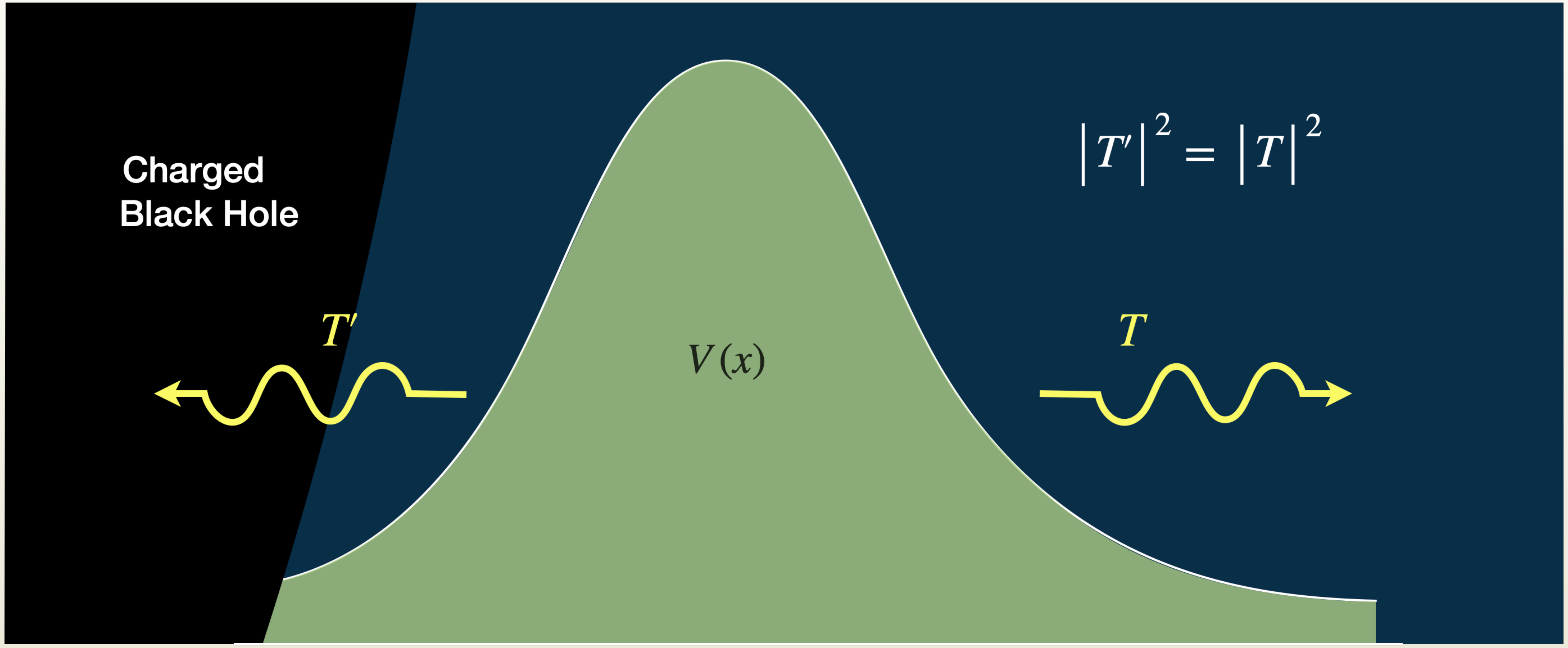
$$\det\left(\Gamma_i *\right) = \det\left(\mathbf{M}_{ij}\right)$$

$$\mathrm{tr}\left(\Gamma_i *\right) = \mathrm{tr}\left(\mathbf{M}_{ij}\right)$$

$$\mathbf{M}_{\infty j}\mathbf{M}_{aj}\mathbf{M}_{+j}\mathbf{M}_{-j} = \mathbf{1}$$



Monodromy (Quasi-Normal Modes)



Monodromy (QNM for Schwarzschild Black Holes)

Charged BHs?

Kerr BHs?

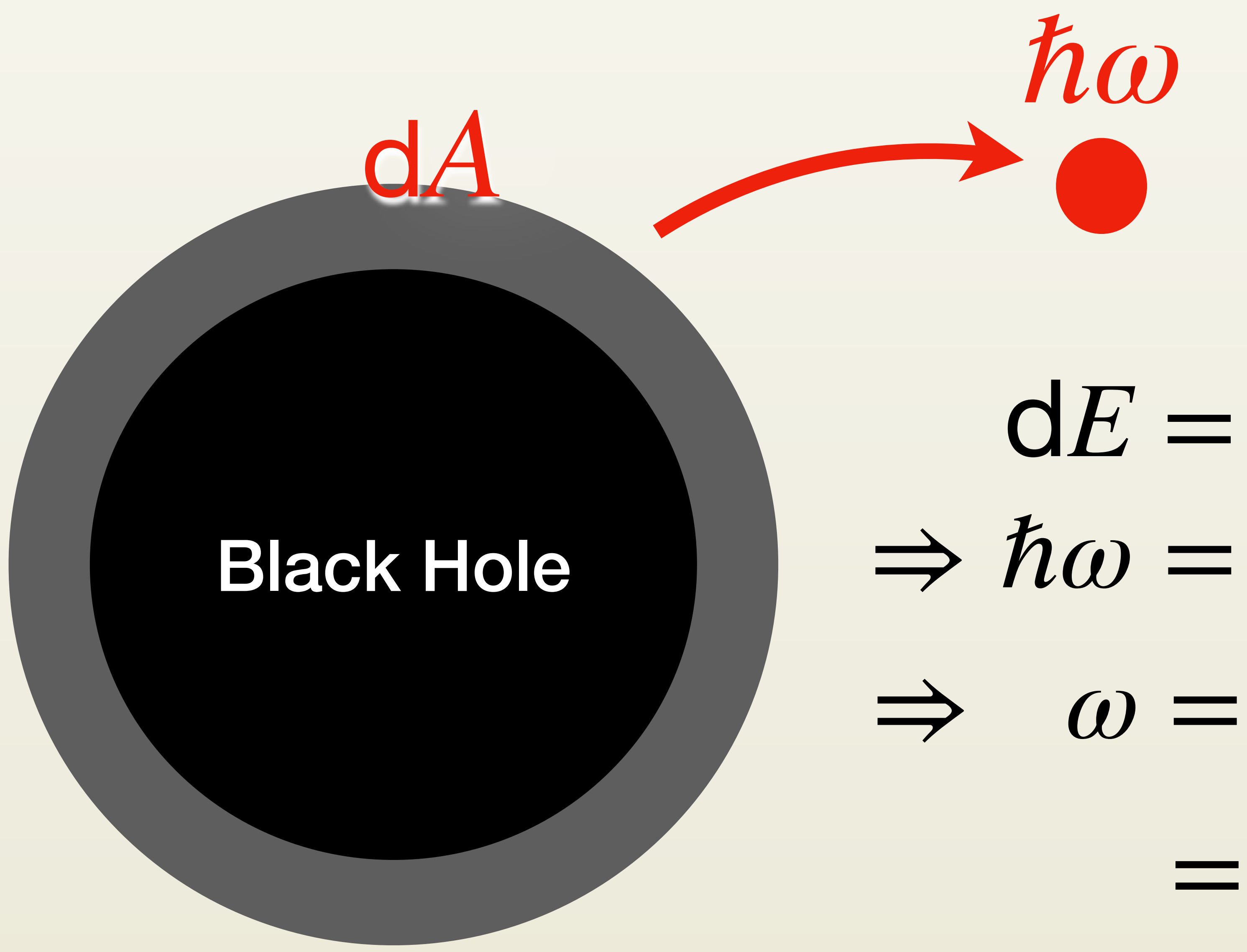
In 2003, Motl & Neitzke

$$\omega = T_{BH} \ln(3) + 2\pi \mathrm{i}\, T_{BH} \left(n - \frac{1}{2} \right)$$

where $T_{BH} = \dfrac{1}{4\pi r_S} = \dfrac{1}{4\pi}$

and $r_S = 2M = 1$

In 1998, Hod



$$\begin{aligned} \mathrm{d}E &= T_{BH}\mathrm{d}S \\ \Rightarrow \hbar\omega &= T_{BH}\mathrm{d}A \\ \Rightarrow \omega &= T_{BH}\left(\mathrm{d}A/\hbar\right) \\ &= T_{BH}\ln(3) \end{aligned}$$

Outline

History —QNMs

Monodromy

Summary

Summary

- In the semiclassical conception, the quasi-normal modes (QNMs) are emitted out by perturbed black holes.

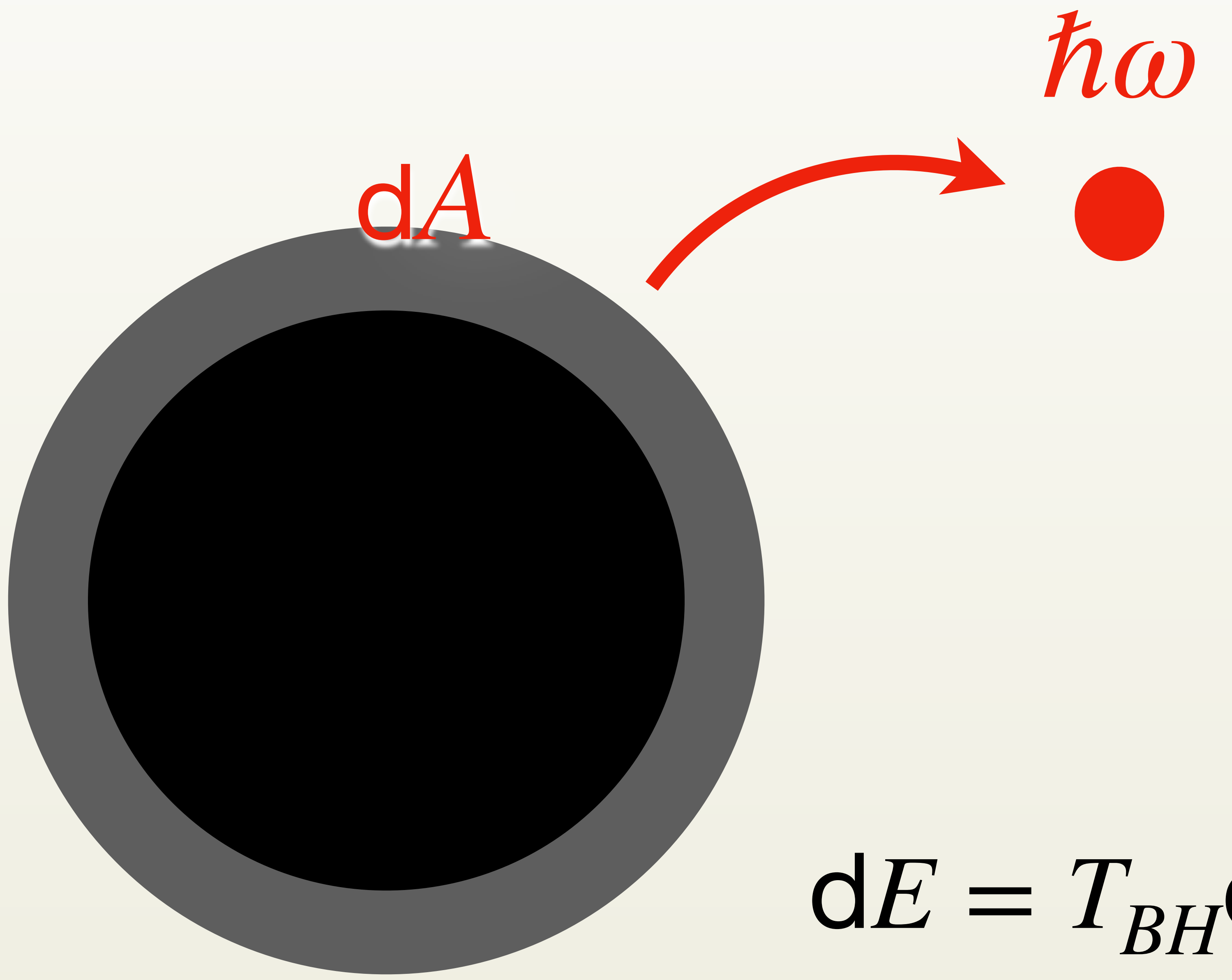
- In 1998, Hod predicted that each loss of energy form a black hole in the ringdown process is quantized.

- The Monodromy technique is one of math methods used to solve the QNM problem, and we expect it to provide more precise results.

Thank You!

Appendix

History



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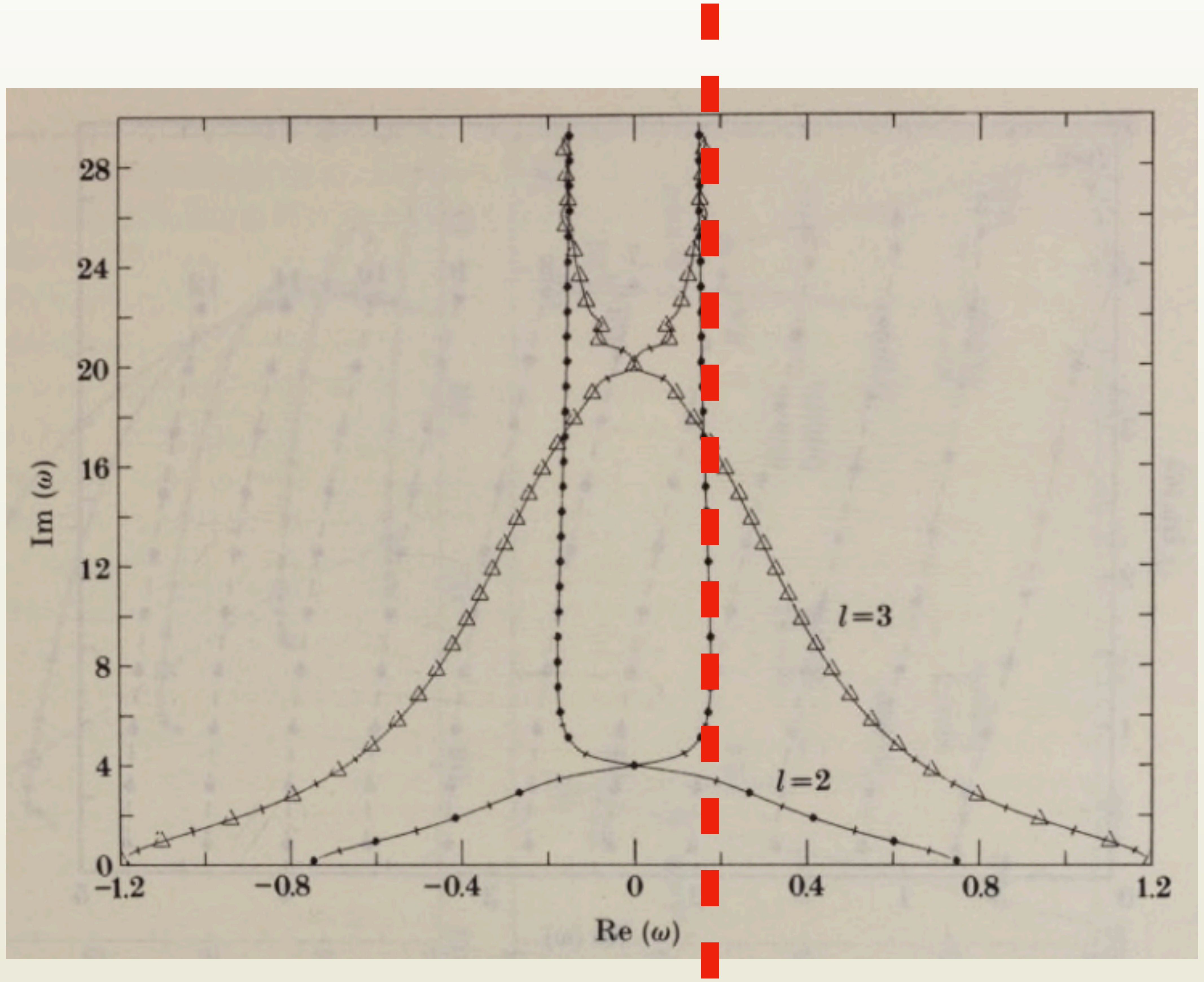
&= T_{BH}\ln(3)
\end{aligned}
$$

$$\mathrm{Re}\,(\omega) = T_{BH}\ln(3)$$

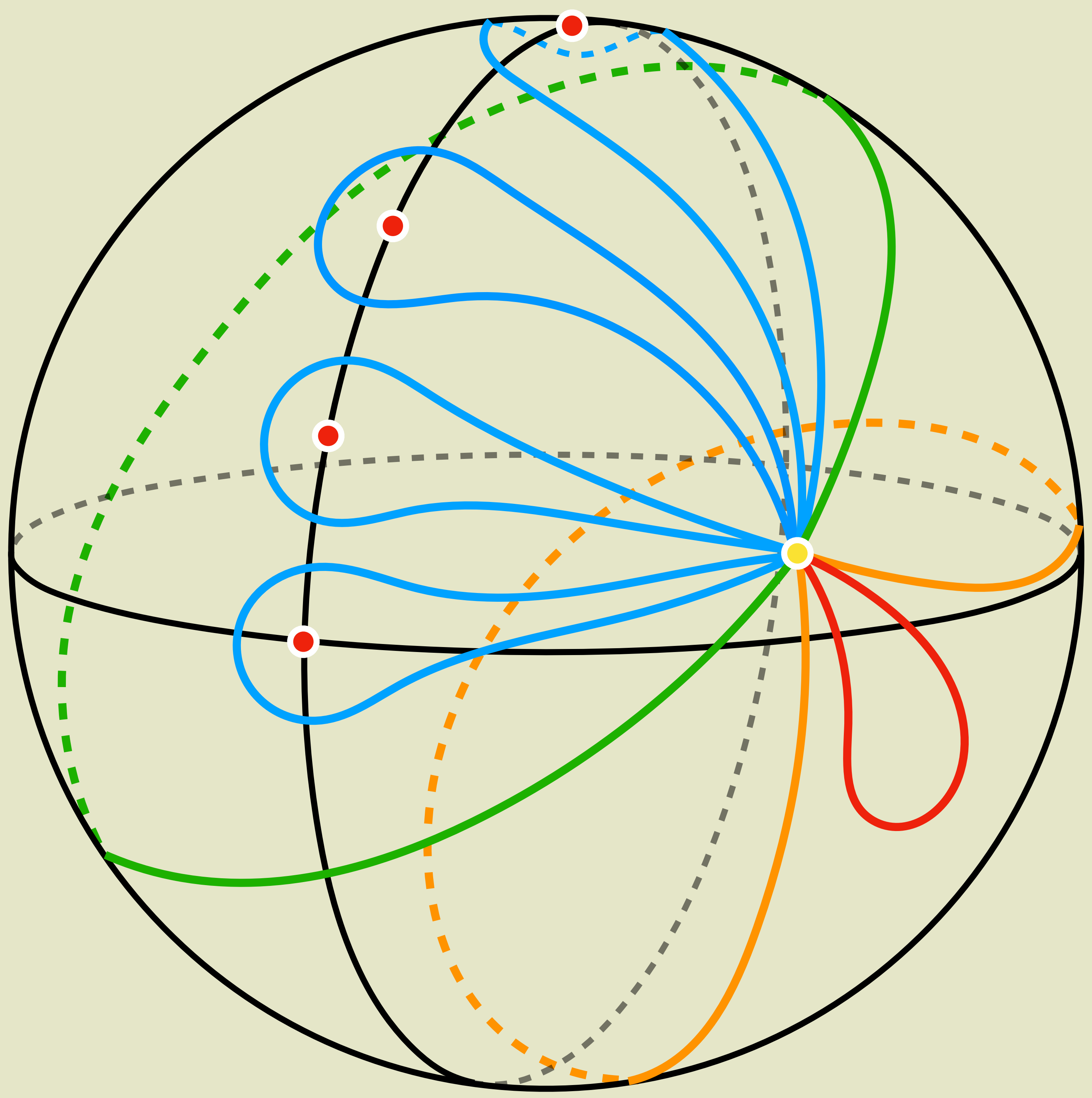


E. W. Leaver. An Analytic representation for the quasi normal modes of Kerr black holes. Proc. Roy. Soc. Lond., A402:285–298, 1985.

Monodromy

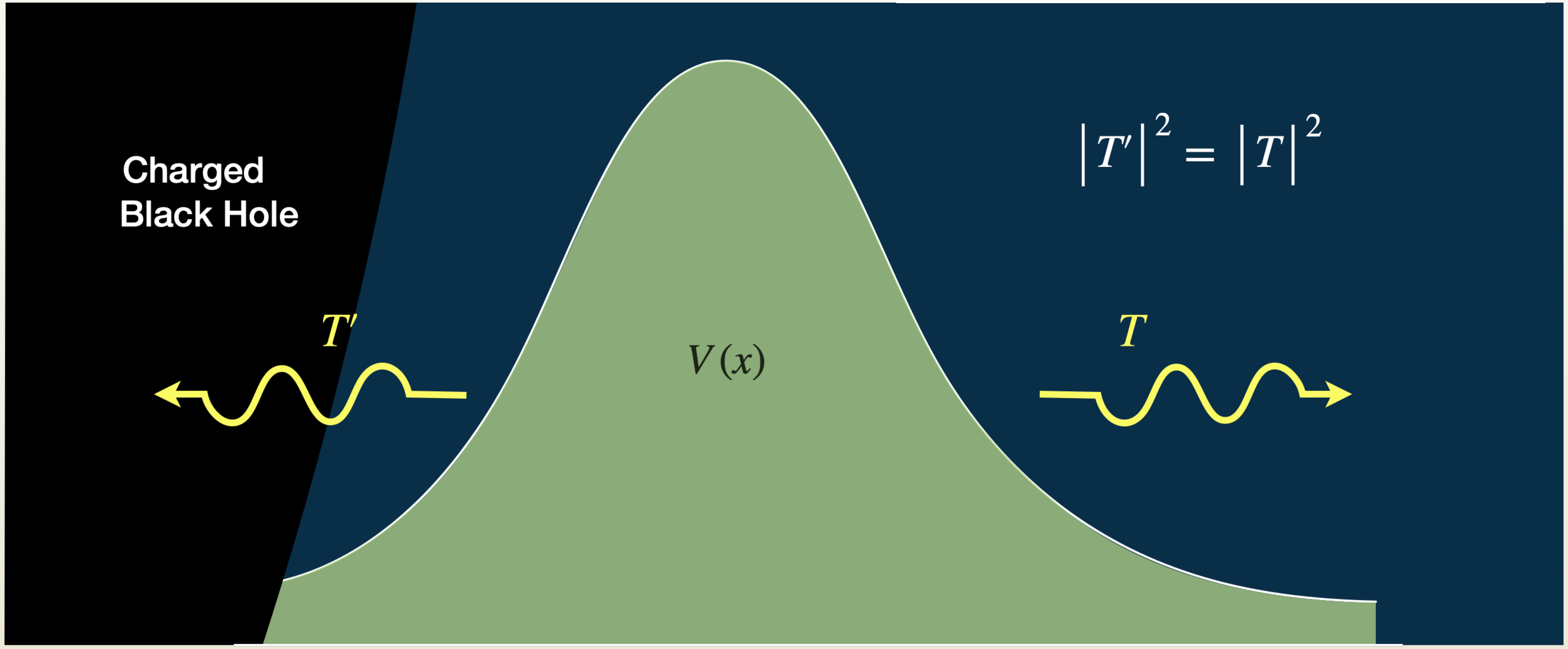
Singular Points of
Charged Black Hole in
Real Space

Singular Points of
Charged Black Hole in
Riemann Sphere



Quasi-Normal Mode

$$\mathbf{P}_{+\infty} = \begin{bmatrix} 0 & \left(\frac{\mathcal{T}'}{\mathcal{T}}\right)^* \\ \frac{\mathcal{T}'}{\mathcal{T}} & 0 \end{bmatrix}$$



Abstract

The quasi-normal modes (QNMs) are the oscillation modes of black holes. This kind of mode will be interesting when we study the ringdown of gravitational waves. In theoretical conception, the highly damped frequencies of QNMs would show quantum phenomena such as Bohr's hydrogen atom model. One of the analyzing methods is called "Monodromy" which is more precise to solve the highly damped frequencies of QNMs.