Exact spin polarization in a boost-invariant Dirac fluid

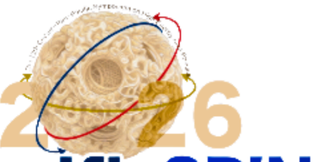
Based on

JHEP 07 (2026) 180

In collaboration with A. Palermo IDP, Tours

Daniele Roselli





The 13th Circum-Pan-Pacific Symposium on High Energy Spin Physics



UNIVERSITÀ DEGLI STUDI FIRENZE

Motivations

From hadron structure to spinful matter

One QFT language, two systems

Matrix elements in a Hadron

$$T^{\mu\nu}(x) = \langle P,S \left| \hat{T}^{\mu\nu}_{g,q}(x) \right| S,P \rangle$$

$A_{g,q}(0) \rightarrow$ momentum

$D(t) \rightarrow$ mechanical structure

$\frac{1}{2}(A+B) = J_{g,q}(0) \rightarrow$ angular momentum

X.Ji, *Phys.Rev.Lett.* 78 (1997) 610-613
Review: V.Burkert et all *Rev.Mod.Phys.* 95 (2023) 4, 041002

Statistical expectation value in a fluid

$$T^{\mu\nu}(x) = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{LE}}\hat{T}^{\mu\nu}(x)\right]$$

$\mathscr{E} \rightarrow$ Energy density $\quad \mathscr{P} \rightarrow$ pressure

$\Theta_{rs}(p) \rightarrow$ Spin density

$P^{\mu}(p) \rightarrow$ Particle polarization

F.Becattini, Lect.Notes Phys. 987 (2021) 15-52
F.Becattini, M.Buzzegoli, E.Grossi, *Particles* 2 (2019) 2, 197-207

Same QFT, different states, complementary questions

Local densities are not unique

Total angular momentum density: orbital part + spin part

$$\widehat{\mathcal{J}}^{\lambda,\mu\nu} = x^{\mu}\hat{T}^{\nu\lambda} - x^{\nu}\hat{T}^{\mu\lambda} + \widehat{\mathcal{S}}^{\lambda,\mu\nu}$$

$$\partial_{\lambda}\widehat{\mathcal{J}}^{\lambda,\mu\nu} = 0 \qquad \partial_{\lambda}\widehat{\mathcal{S}}^{\lambda,\mu\nu} = 2\hat{T}^{[\nu\mu]}$$

pseudo-gauge transformations:

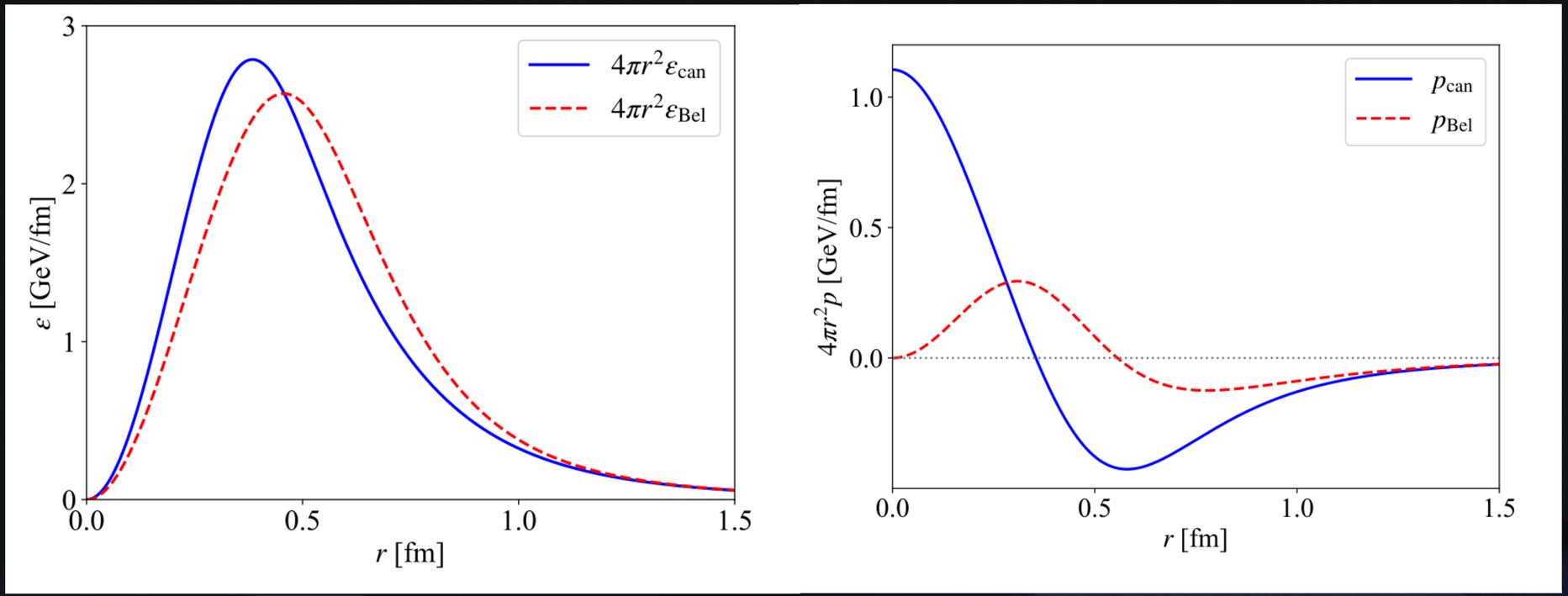
$$(\hat{T}, \widehat{\mathcal{S}}) \mapsto (\hat{T}_{\Phi}, \widehat{\mathcal{S}}_{\Phi}) \qquad \widehat{P}^{\nu} = \widehat{P}^{\nu}_{\Phi} \quad \hat{J}^{\mu\nu} = \hat{J}^{\mu\nu}_{\Phi}$$

Example: Belinfante Pseudo-gauge —> $\widehat{\Phi}\ \mathrm{t.c}\ \widehat{\mathcal{S}} = 0 \text{ and } \hat{T}^{[\mu\nu]} = 0$

Can local physics be sensitive to this decomposition?

Pseudogauge freedom in hadron structure

A recent hadronic example



K.Fukushima, T.Uji, *Phys. Rev. D 113, 016025 (2026)*

Canonical and Belinfante EMTs give same global charges but different energy density, pressure and shear distribution

Pseudogauge freedom at local equilibrium

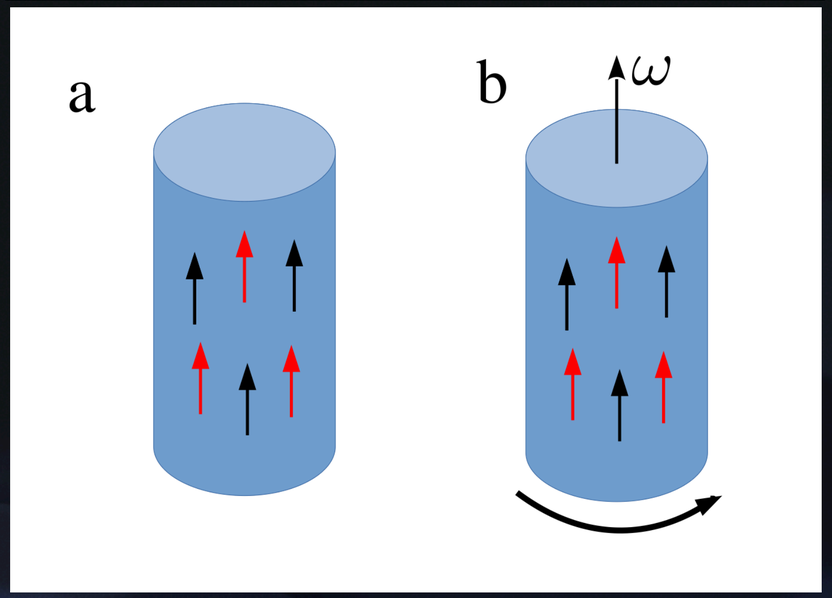
If spin is a slow variable

$$\mathfrak{S}^{\star}_{\mu\nu} \leftrightarrow \widehat{\mathcal{S}}^{\lambda,\mu\nu}$$

$$\widehat{\rho}_{\mathrm{LE}} \mapsto \widehat{\rho}_{\mathrm{LE}}[\widehat{\mathcal{S}}]$$

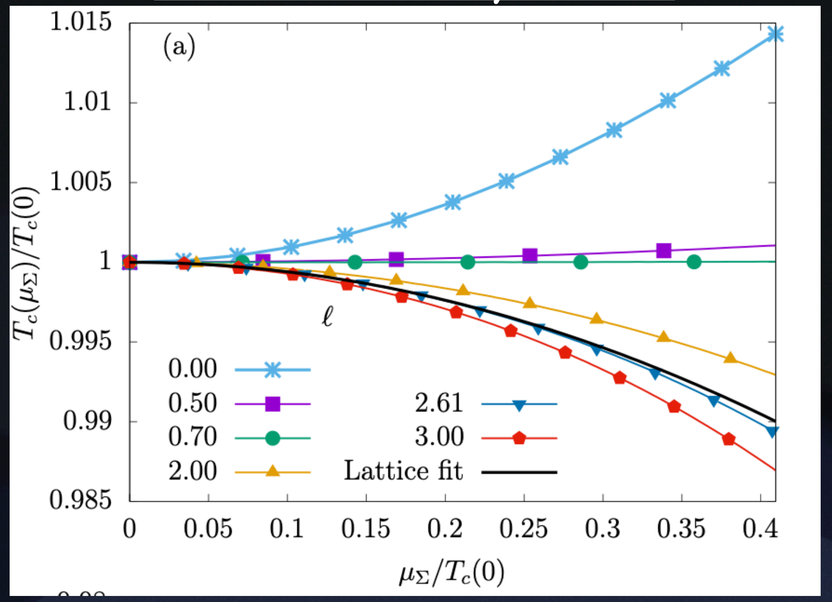
Spin potential = conjugate Lagrange multiplier

Polarization without thermal flow



F.Becattini, W.Flowrkowski, E.Speranza, *Phys.Lett.B* 789 (2019), 419.425

QCD thermodynamics



P.Singha, et.Al *Phys.Rev.D* 113 (2026) 11, 114029
V.Braguta, M.Chernodub, A.Roenko *Phys.Rev.D* 111 (2025) 11, 114508

Local equilibrium depends on how spin is constrained

Use an exact solvable QFT as a benchmark

Does thermodynamics remain consistent at finite spin density?

What is the exact polarization beyond linear response?

Quantum statistical field theory

Free Dirac field at Local thermodynamic equilibrium

Canonical $\quad \widehat{T}^{\mu\nu}_{C} = \frac{\mathrm{i}}{2}\overline{\psi}\gamma^{\mu}\partial^{\nu}\psi - \frac{\mathrm{i}}{2}\partial^{\nu}\overline{\psi}\gamma^{\mu}\psi \qquad \widehat{\mathcal{S}}^{\lambda,\mu\nu}_{C} = \frac{1}{2}\varepsilon^{\sigma\lambda\mu\nu}\overline{\psi}\gamma_{\sigma}\gamma_{5}\psi$

Belinfante $\quad \widehat{T}^{\mu\nu}_{B} = \frac{\mathrm{i}}{4}\overline{\psi}\gamma^{\mu}\partial^{\nu}\psi - \frac{\mathrm{i}}{4}\partial^{\nu}\overline{\psi}\gamma^{\mu}\psi + \left(\mu \leftrightarrow \nu\right) \qquad \widehat{\mathcal{S}}^{\lambda,\mu\nu}_{B} = 0$

Quantum local thermodynamic equilibrium state:

$$\widehat{\rho}_{C} = \frac{1}{Z}\exp\left[-\int_{\Sigma}\mathrm{d}\Sigma_{\mu}\left(\widehat{T}^{\mu\nu}_{C}\beta_{\nu} - \varpi_{\lambda\nu}\,\widehat{\mathcal{S}}^{\mu\nu\lambda}_{C}\right)\right]$$

β_{ν} Four-Temperature $\qquad$ $\varpi_{\lambda\nu}$ Spin chemical potential

Boost invariant local equilibrium

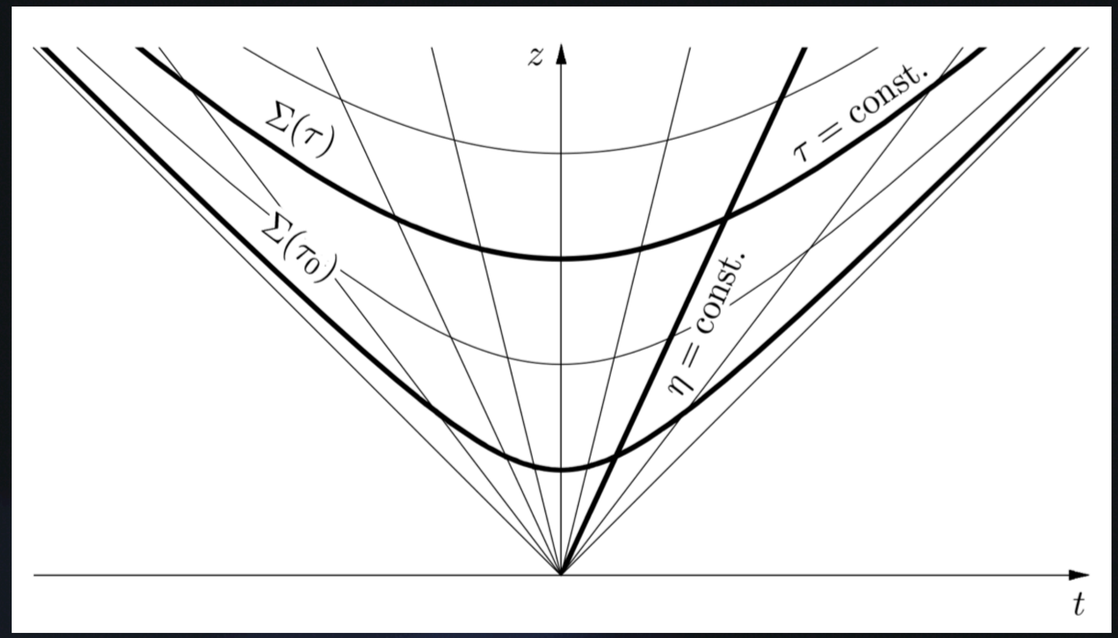
Milne metric

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \tau^2\mathrm{d}\eta^2$$

$$\Sigma : \tau = \sqrt{t^2 - z^2} = \text{const}\,.$$

$$\beta_\mu = \frac{u_\mu}{T(\tau)} \qquad u^\mu = \frac{1}{\tau}(t,0,0,z)$$

$$\mathfrak{S}_{\lambda\rho} = \frac{1}{2}\mathfrak{S}(\tau)\left(\delta^1_\lambda\delta^2_\rho - \delta^1_\rho\delta^2_\lambda\right)$$



$$\hat{\rho} = \frac{1}{Z}\exp\left[-\frac{1}{T(\tau)}\int \mathrm{d}^2x_\mathrm{T}\mathrm{d}\eta\left(\hat{T}^{\tau\tau} - \Omega(\tau)\,\hat{\mathcal{S}}^{\tau xy}\right)\right],\ \Omega = \mathfrak{S}/T$$

vanishing vorticity but **non-vanishing shear**

$$\varpi_{\mu\nu} = \partial_{[\mu}\beta_{\nu]} = 0 \qquad \xi_{\mu\nu} = \partial_{(\mu}\beta_{\nu)} \neq 0$$

Thermodynamic observables

Direct QFT computation

$$\langle \hat{T}^{\mu\nu} \rangle \implies \mathscr{E}(\tau),\ \mathscr{P}_{\mathrm{T}}(\tau),\ \mathscr{P}_{\mathrm{L}}(\tau),\ \mathscr{T}(\tau)$$

$$\langle \widehat{\mathcal{S}}^{\lambda\mu\nu} \rangle \implies \mathcal{S}(\tau) \qquad \partial_\tau(\tau\mathcal{S}(\tau)) = \tau\mathscr{T}(\tau)$$

Thermodynamic relations

$$\mathscr{P}_{\mathrm{td}} = TV^{-1}\log Z \qquad \mathcal{S}_{\mathrm{td}} = \frac{\partial \mathscr{P}_{\mathrm{td}}}{\partial \Omega}$$

$$\mathscr{E}_{\mathrm{td}} = -\mathscr{P}_{\mathrm{td}} + \Omega\frac{\partial \mathscr{P}_{\mathrm{td}}}{\partial \Omega} + T\frac{\partial \mathscr{P}_{\mathrm{td}}}{\partial T}$$

Test thermodynamic relations at finite spin density

$$\mathscr{E} \stackrel{?}{=} \mathscr{E}_{\mathrm{td}} \qquad \mathscr{P} \stackrel{?}{=} \mathscr{P}_{\mathrm{td}} \qquad \mathcal{S} \stackrel{?}{=} \mathcal{S}_{\mathrm{td}}$$

Spin density matrix and spin polarization

Exact Dirac mode functions in Milne coordinates

$$\psi(\tau, x, y, \eta) = (2\pi)^{-3/2} \int \mathrm{d}^2 p_{\mathrm{T}} \mathrm{d}\mu \, \mathrm{e}^{\mathrm{i}(\mathbf{x}_{\mathrm{T}} \cdot \mathbf{p}_{\mathrm{T}} + \mu\eta)} \, \widehat{A}_r(\mathbf{p}_{\mathrm{T}}, \mu) U_r(\tau, \mathbf{p}_{\mathrm{T}}, \mu) + \mathrm{h.c}$$

Milne Creation/Annihilation operators

$$\mathrm{Tr}\left(\widehat{\rho}_{\mathrm{LE}} \, \widehat{A}^{\dagger}_r(\mathbf{p}_{\mathrm{T}}, \mu) \, \widehat{A}_r(\mathbf{p}'_{\mathrm{T}}, \mu') \right)$$

Spin density matrix and spin polarization vector:

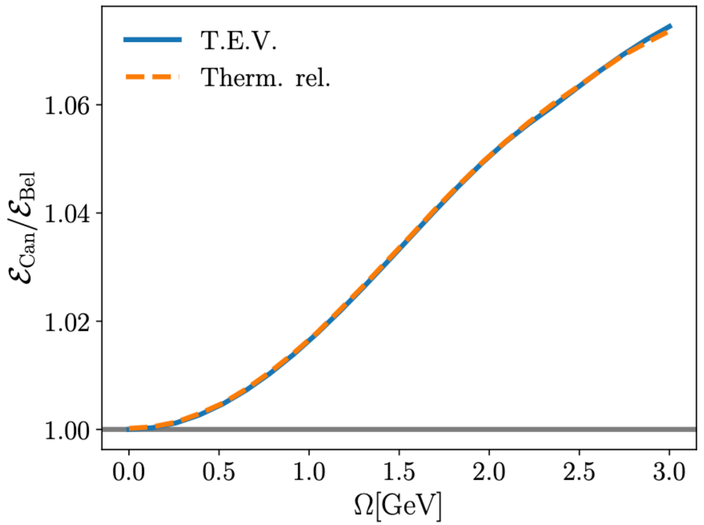
$$\Theta_{rs}(p) = \frac{\int_{-\infty}^{+\infty} \mathrm{d}\mu \langle \widehat{A}^{\dagger}_r(\mathbf{p}_{\mathrm{T}}) \, \widehat{A}_s(\mathbf{p}_{\mathrm{T}}, \mu) \rangle}{\sum_t \int_{-\infty}^{+\infty} \mathrm{d}\mu \langle \widehat{A}^{\dagger}_t(\mathbf{p}_{\mathrm{T}}) \, \widehat{A}_t(\mathbf{p}_{\mathrm{T}}, \mu) \rangle}, \quad P_i = \mathrm{tr}\left(\Theta_{rs} \sigma_i \right)$$

Results

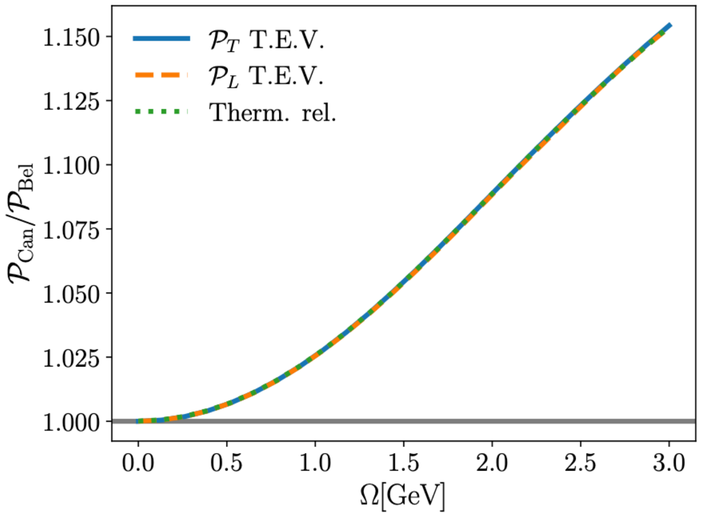
QFT and thermodynamic derivatives agree

$$\mathscr{E} = \mathscr{E}_{\rm td} \quad \mathscr{P} = \mathscr{P}_{\rm td} \quad \mathscr{S} = \mathscr{S}_{\rm td}$$

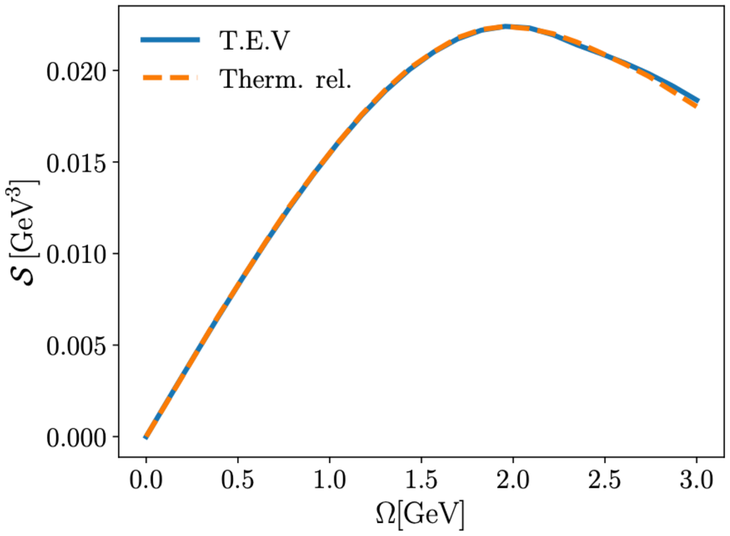
Energy density



Pressure



Spin density



Nontrivial: analogous relations need not hold at finite thermal vorticity

$$\frac{\partial \mathscr{P}}{\partial \varpi_{\mu\nu}} \neq u_\lambda \mathcal{S}^{\lambda,\mu\nu}$$

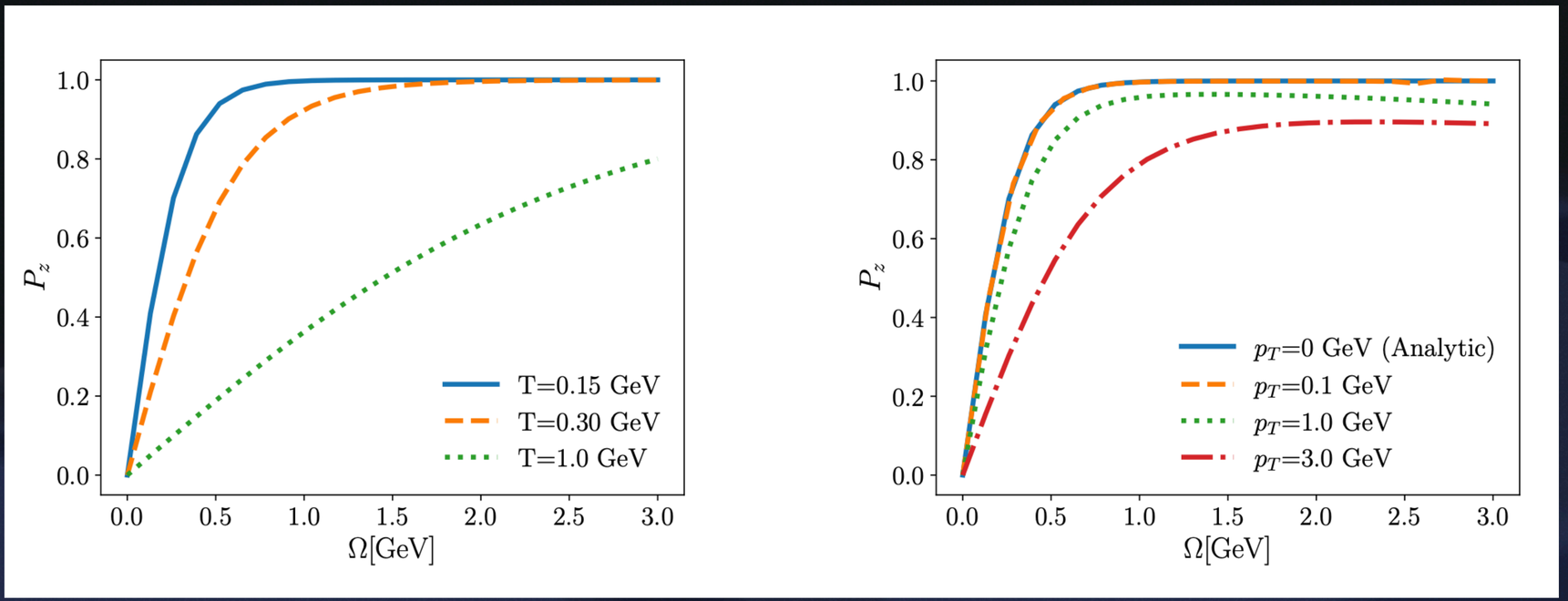
F.Becattini, R.Singh, *Eur.Phys.J.C* 85 (2025), 11 1338

Shear alone does not polarize the system

No-shear induced polarization in boost invariant system

$$\mathfrak{S} = 0, \ \xi_{\mu\nu} \neq 0 \implies P_i = 0$$

Exact formula for spin density induced polarization



(1) Saturates at $\Omega/T \gg 1$

(2) Linear at small Ω/T

(3) Suppressed at higher T

Conclusions

(1) Spin thermodynamics

QFT observables = thermodynamic relations

(2) Exact polarization: Belinfante

$\Omega = 0 \implies P_i = 0$ even though $\xi_{\mu\nu} \neq 0$

(3) Exact polarization: Canonical

$P_z(T, \Omega; \mathbf{p}_\mathrm{T})$ without any gradient expansion

An exactly solvable QFT benchmark for spin thermodynamics and polarization

Thank you for your attention

Backup slides

Global equilibrium

Local equilibrium operator:

$$\hat{\rho} = \frac{1}{Z}\exp\left[-\int_{\Sigma} \mathrm{d}\Sigma_{\mu}(y)\left(\hat{T}^{\mu\nu}(y)\beta_{\nu}(y) - \hat{j}^{\mu}(y)\zeta(y) - \varpi_{\lambda\sigma}(y)\,\hat{\mathcal{S}}^{\mu,\lambda\sigma}(y)\right)\right]$$

Global equilibrium state only depends on the global charges → It must be independent on the hypersurface and the pseudo-gauge

Equilibrium Condition:

$$\partial_{\mu}\beta_{\nu} + \partial_{\nu}\beta_{\mu} = 0, \quad \partial_{\mu}\zeta = 0, \quad \varpi_{\mu\nu} = \text{const}$$

Four-temperature is a Killing vector

Global thermodynamic equilibrium state

$$\hat{\rho}_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}}\exp\left(-\beta(x)\cdot\hat{P} + \zeta(x)\hat{Q}\right)$$

$$\beta_{\mu} = b_{\mu} + \varpi_{\mu\nu}x^{\nu} \qquad \varpi_{\mu\nu} = \text{const}. \quad b_{\mu} = \text{const}.$$

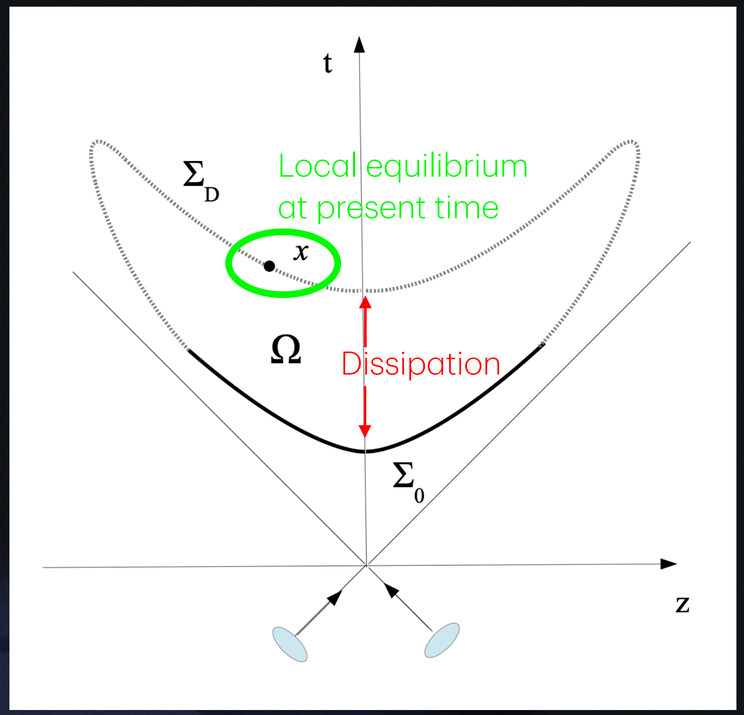
Local equilibrium and dissipation

System reaches local equilibrium, at some time, on the initial surface Σ_0

$$\hat{\rho} = \frac{1}{Z} \exp\left(- \int_{\Sigma_0} d\Sigma_\mu(y)\, \hat{T}^{\mu\nu}(y) \beta_\nu(y) \right)$$

Gauss theorem from Σ_0 to Σ_D

$$\hat{\rho} = \frac{1}{Z} \exp\left(- \int_{\Sigma_D} d\Sigma_\mu(y)\, \hat{T}^{\mu\nu}(y) \beta_\nu(y) + \int_{\Omega} d^4y\, \hat{T}^{\mu\nu}(y) \partial_\mu \beta_\nu(y) \right)$$



Entropy production rate

$$\partial_\mu s^\mu(x) = \left(\langle \hat{T}^{\mu\nu}(x) \rangle - \langle \hat{T}^{\mu\nu}(x) \rangle_{\rm LE} \right) \partial_\mu \beta_\nu(x)$$

$$\langle \hat{T}^{\mu\nu} \rangle_{\rm LE} = \frac{1}{Z_{\rm LE}} {\rm Tr} \left[\exp\left(- \int_{\Sigma_D} d\Sigma_\mu(y)\, \hat{T}^{\mu\nu}(y) \beta_\nu(y) \right) \hat{T}^{\mu\nu}(x) \right]$$

F.Becattini, M.Buzzegoli, E.Grossi, *Particles* 2 (2019) 2, 197-207

Vorticity induced polarization

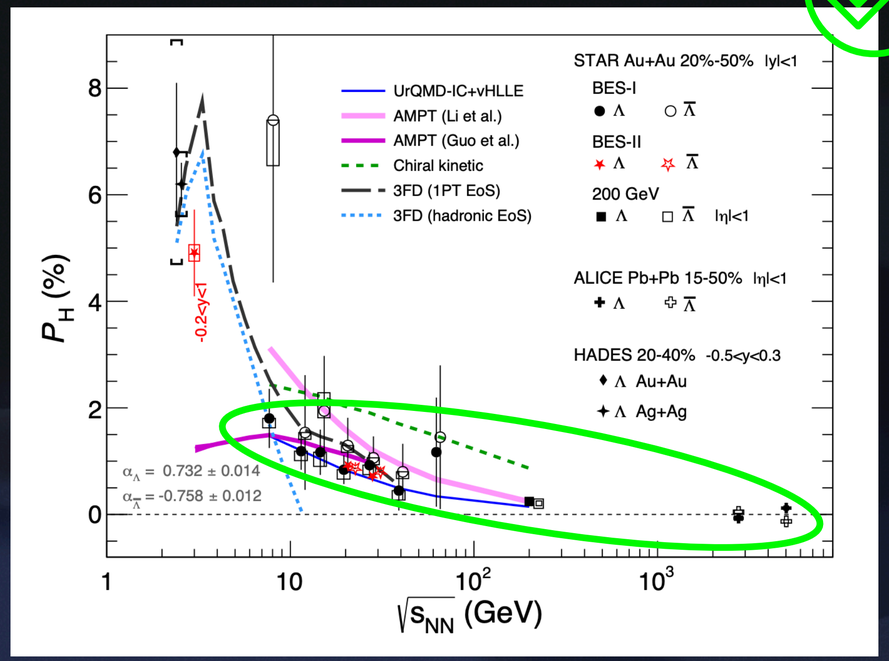
At global equilibrium polarization is only driven by thermal vorticity ϖ

$$S^{\mu}_{\varpi}(k) = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}\frac{\int \mathrm{d}\Sigma \cdot k\, n_{\mathrm{F}}(1 - n_{\mathrm{F}}\varpi_{\nu\rho})}{\int \mathrm{d}\Sigma \cdot k\, n_{\mathrm{F}}}$$

$$\varpi_{\mu\nu} = \frac{1}{2}\left(\partial_{\nu}\beta_{\mu} - \partial_{\nu}\beta_{\mu}\right)$$

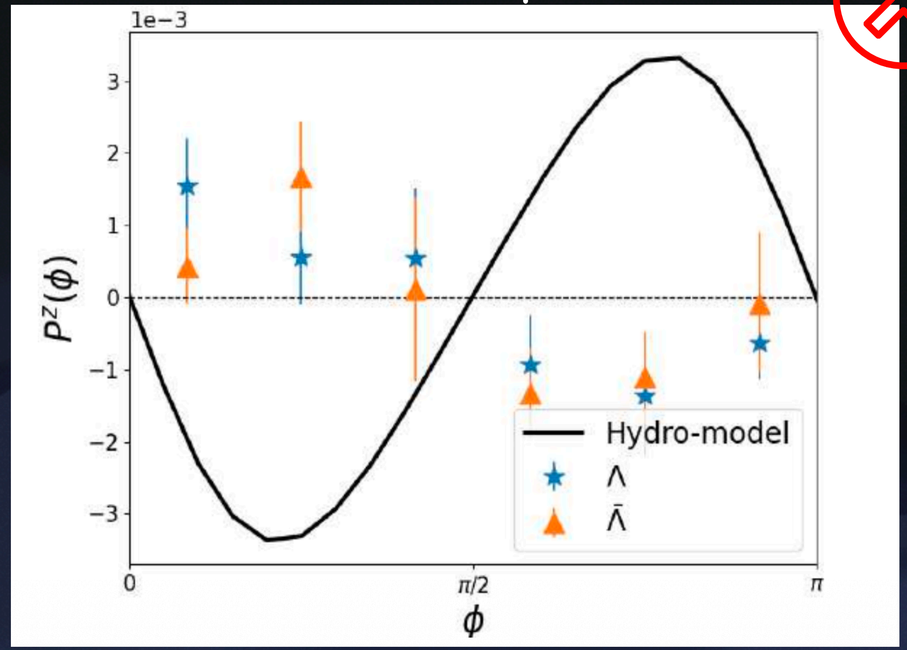
$$\beta_{\mu} = \beta u_{\mu}$$

Dependence on energy



F.Becattini, M.Buzzegoli, T.Niida, S.Pu, Tang, Wang, *Int.J.Mod.Phys.E* 33 (2024) 06, 2430006

Azimuthal dependence



Shear induced polarization

At local equilibrium also contribution from thermal shear $\xi_{\mu\nu} = (1/2)(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu)$

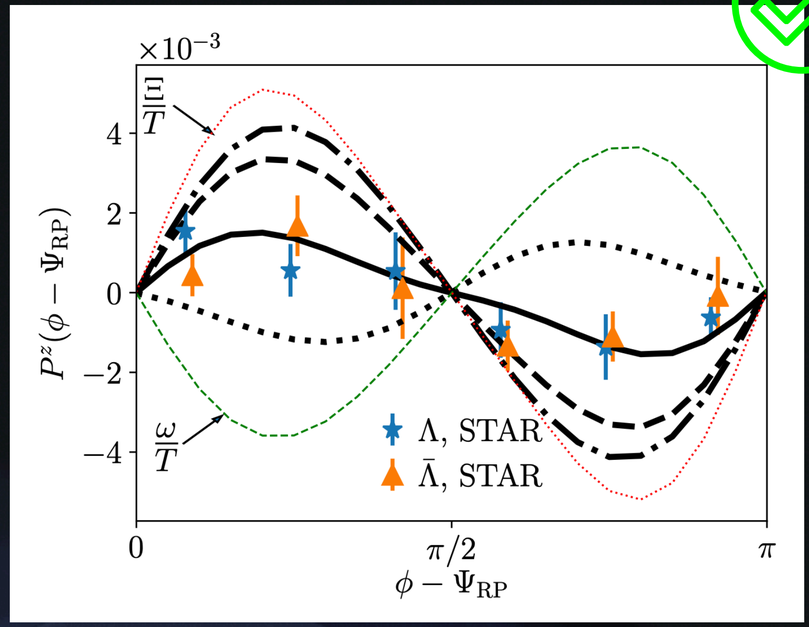
Under different geometric approximations:

$$(1)\quad S^\mu_\xi(k) \simeq -\frac{1}{6m}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}\Sigma\cdot k\, n_{\mathrm{F}}(k)\frac{k_\nu \hat{t}_\rho}{k^0}k^\lambda \xi_{\lambda\rho}$$

F.Becattini, M.Buzzegoli, A.Palermo, *Phys.Lett.B* 820 (2021) 136519

$$(2)\quad S^\mu_\xi(k) \simeq -\frac{1}{6m}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}\Sigma\cdot k\, n_{\mathrm{F}}(k)\frac{k_\nu u_\rho}{\varepsilon}k^\lambda \xi_{\lambda\rho}$$

B.Fu, S.Y.F.Liu, L.Pang, H.Song, Y.Yin, *Phys.Rel.Lett.* 127 (2021) 14, 142301



F.Becattini, M.Buzzegoli, G.Inghirami, I.Karpenko, A.Palermo, *Phys.Rev.Lett.* 127 (2021) 27, 272302

New formula valid for **any** geometry

$$(3)\quad S^\mu_\xi(k) \simeq -\frac{1}{6m}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}\Sigma\cdot k\, n_{\mathrm{F}}(k)\frac{k_\nu \hat{n}_\rho}{|k\cdot\hat{n}|}k^\lambda \xi_{\lambda\rho}$$

Vanishes at global equilibrium

X.L. Sheng, F.Becattini, D.Roselli, *Phys.Rev.C* 114 (2026)3,3

Expectation values in Milne coordinates

Symmetries of the metric fix the expectation values form

$$\langle \hat{T}^{\mu\nu} \rangle = \mathscr{E}(\tau) u^\mu u^\nu + \mathscr{P}_{\mathrm{T}}(\tau)\left(\hat{x}^\mu \hat{x}^\nu + \hat{y}^\mu \hat{y}^\nu\right) + \mathscr{P}_{\mathrm{L}}(\tau)\hat{\eta}^\mu \hat{\eta}^\nu + \frac{\mathscr{T}(\tau)}{2}\left(\hat{x}^\mu \hat{y}^\nu - \hat{x}^\nu \hat{x}^\mu\right)$$

$$\langle \widehat{\mathcal{S}}^{\lambda\mu\nu} \rangle = \varepsilon^{\lambda\mu\nu\sigma} \mathcal{S}(\tau) \hat{\eta}_\sigma$$

Symmetries allows for a non-vanishing spin-torque

$$\partial_\tau(\tau \mathcal{S}(\tau)) = \tau \mathscr{T}(\tau)$$

Numerically we found

$$\mathscr{T}(\tau) \sim 10^{-15}\,\mathrm{GeV} \ll \tau \mathcal{S}$$

Renormalization

Normal ordering define **vacuum subtracted** expectation values which are finite

$$\langle : \widehat{O}(x) : \rangle \equiv \mathrm{Tr}(\hat{\rho}\,\widehat{O}(x)) - \langle 0|\,\widehat{O}(x)\,|0\rangle$$

The Milne Hamiltonian is time-dependent so it is its l.l.e

$$\widehat{H}(\tau_1)\,|0_{\tau_1}\rangle = 0 \implies \widehat{H}(\tau_2)\,|0_{\tau_1}\rangle \neq 0$$

Normal ordering respect to time-dependent vacuum spoil continuity equations

$$\nabla_\mu \Big(\mathrm{Tr}(\hat{\rho}\,\hat{T}^{\mu\nu}(x)) - \langle 0_\tau|\,\hat{T}^{\mu\nu}(x)\,|0_\tau\rangle \Big) \neq 0$$

Normal ordering defined with respect to the fixed initial Milne time

$$: T^{\mu\nu} := \mathrm{Tr}(\hat{\rho}\,\hat{T}^{\mu\nu}(x)) - \mathrm{Tr}(\hat{\rho}\,\hat{T}^{\mu\nu}(x))\Big|_{\zeta=\mathfrak{S}=0} \longrightarrow \nabla_\mu : T^{\mu\nu}(x) := 0$$

$$\lim_{T\to 0}\,(\lim_{\zeta\to 0}\lim_{\mathfrak{S}\to 0})\hat{\rho} = |0_{\tau_0}\rangle\langle 0_{\tau_0}|$$

Dirac equation in Milne coordinates: solution

Dirac equation in Milne coordinates:

$$\left[\mathrm{i}\gamma^0\partial_\tau + \mathrm{i}\boldsymbol{\gamma}_\mathrm{T}\cdot\nabla_\mathrm{T} + \frac{\mathrm{i}\gamma^3}{\tau}(\partial_\eta - \frac{1}{2}\gamma^3\gamma^0) - m\right]\psi(\tau, \mathbf{x}_\mathrm{T}, \eta) = 0$$

General solution

$$\psi(\tau, x, y, \eta) = (2\pi)^{-3/2}\int \mathrm{d}^2p_\mathrm{T}\mathrm{d}\mu\, \mathrm{e}^{\mathrm{i}(\mathbf{x}_\mathrm{T}\cdot\mathbf{p}_\mathrm{T}+\mu\eta)}\,\widehat{A}_r(\mathbf{p}_\mathrm{T}, \mu)U_r(\tau, \mathbf{p}_\mathrm{T}, \mu) + \mathrm{h.c}$$

Milne spinor

$$U_r(\tau, \mathbf{p}_\mathrm{T}, \mu) = -\frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}\pi\nu/2}\mathrm{H}^{(2)}_\nu(m_\mathrm{T}\tau)u_r(\mathbf{p}_\mathrm{T})$$

Transverse Minkowski spinor

$$u_r(\mathbf{p}_\mathrm{T}) = \frac{m + m_\mathrm{T}\gamma^0 + \mathbf{p}_\mathrm{T}\cdot\boldsymbol{\gamma}_\mathrm{T}}{\sqrt{2(m+m_\mathrm{T})}}\xi_r$$

matrix valued Hankel functions $\nu = \frac{1}{2}\gamma^3\gamma^0 - \mathrm{i}\mu\mathrm{I}$:

$$\mathrm{H}^{(2)}_\nu(z) = \begin{pmatrix} \mathrm{H}^{(2)}_{\frac{1}{2}-\mathrm{i}\mu}(z) & 0 & 0 & 0 \\ 0 & \mathrm{H}^{(2)}_{-\frac{1}{2}-\mathrm{i}\mu}(z) & 0 & 0 \\ 0 & 0 & \mathrm{H}^{(2)}_{-\frac{1}{2}-\mathrm{i}\mu}(z) & 0 \\ 0 & 0 & 0 & \mathrm{H}^{(2)}_{\frac{1}{2}-\mathrm{i}\mu}(z) \end{pmatrix}$$

Milne creation/annihilation operator

$$\widehat{A}^\dagger_r(\mathbf{p}_\mathrm{T}, \mu) \quad \widehat{A}_r(\mathbf{p}_\mathrm{T}, \mu)$$

Milne and Minkowski operators are related

$$\hat{a}_r(p) = \frac{1}{\sqrt{2\pi m_\mathrm{T}\cosh\vartheta}}\int_{-\infty}^{+\infty}\mathrm{d}\mu\, \mathrm{e}^{\mathrm{i}\mu\vartheta}\,\widehat{A}_r(\mathbf{p}_\mathrm{T}, \mu)$$