Mapping the transverse spin of nucleon in position space

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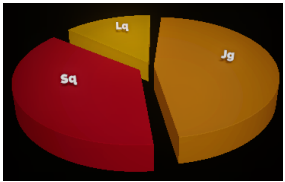
Based on: Lorcé, Mukherjee, Singh, Won [Phys. Lett. B 868 (2025)]
+Lorcé, Mukherjee, Singh, Won [arXiv: 2609.08902]



Nucleon spin: what is still missing?

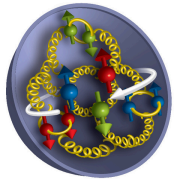
Origin of nucleon spin

Individual partonic contributions (Spin sum rule)



=1/2

Distribution of spin in position space



Credit: bnl.gov

Nucleon spin: what is still missing?

Spin sum rule

Spatial distribution

Nucleon spin: what is still missing?

Longitudinal

Spin sum rule — well understood

Spatial distribution — well understood

R. L. Jaffe, A. Manohar, Nucl. Phys. B 337 (1990)
X.-D. Ji, Phys. Rev. Lett. 78 (1997)
Chen *et al.*, Phys. Rev. Lett. 100 (2008)
M. Wakamatsu, Phys. Rev. D 83 (2011)
Y. Hatta, Phys. Rev. D 84 (2011)
E. Leader, C. Lorcé, Phys. Rept. 541 (3) (2014)
C. Lorce, L. Mantovani, B. Pasquini, Phys. Lett. B 776(2018)

Nucleon spin: what is still missing?

	Longitudinal	Transverse
Spin sum rule	✅ well understood	✅ recently understood Lorcé EPJC 81 (5) (2021) 413
Spatial distribution	✅ well understood	

Nucleon spin: what is still missing?

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Spin sum rule	✓ well understood	✓ recently understood Lorcé EPJC 81 (5) (2021) 413
Spatial distribution	✓ well understood	**This work** missing piece

Our goal: map the transverse spin in position space
aligns with one of the goals of the Electron-Ion Collider

Challenge with the transverse angular momentum (AM)

$$\vec{J} = \vec{L} + \vec{S} = \vec{R} \times \vec{P} + \vec{S}$$

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Longitudinal spin sum rule

$$\langle J_a^L \rangle = s_L/2 \qquad a = q, g$$

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Transverse spin sum rule

$\langle J_a^T \rangle = s_T/2$	*Bakker*, *Leader*, *Trueman*, *PRD* 70, 114001 (2004) *Harindranath*, *Mukherjee*, *Ratabole*, *PRD* 63, 045006 (2001)
$\langle J_a^T \rangle = \gamma \times s_T/2$	*X.Ji*, *Yuan*, *PLB* 810, 135786 (2020)
$\langle J_a^T \rangle = \gamma^{-1} \times s_T/2$	*C.Lorce*, *EPJC* 81 (5) (2021) 413.

where $\gamma = p^0/M$

Why this apparent contradiction - Answer lies in the Pivot

Center of

- $+$ energy
- $\times$ spin
- $\bullet$ mass

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$$R_E^\mu = \frac{1}{P^0}\int d^3r\; r^\mu T^{00}(r)$$

$R_M^\mu =$ Center of energy in the rest frame of system

$$R_c^\mu = \frac{P^0 R_E^\mu + M R_M^\mu}{P^0 + M}$$

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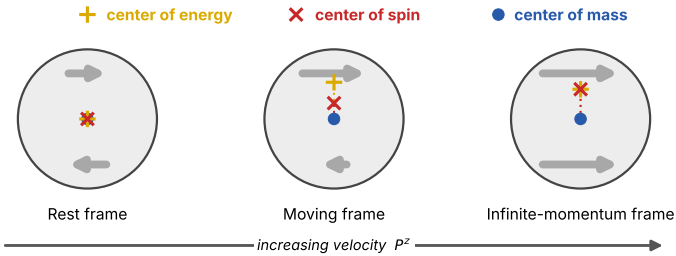
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C. Lorce, EPJC 81 (5) (2021) 413.

Transverse spin sum rule

$\langle J_a^T \rangle = \frac{1}{2} s_T$	Center of spin
$\langle J_a^T \rangle = \gamma \frac{1}{2} s_T$	Center of Mass
$\langle J_a^T \rangle = \gamma^{-1} \frac{1}{2} s_T$	Center of energy

Position-space OAM density: an extra term

$$\langle L^i \rangle(x) = \epsilon^{ijk} x^j \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3} e^{i\Delta\cdot x} \langle T^{0k} \rangle \longrightarrow \textbf{Energy-Momentum tensor}$$

where $\Delta = p' - p$ and $P = \dfrac{p' + p}{2}$.

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$$\langle L^i\rangle(\boldsymbol{x}) = \epsilon^{ijk} \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3}\, e^{i\Delta\cdot x} \left[-i\frac{\partial\langle T^{0k}\rangle}{\partial\Delta^j} + \underbrace{\frac{x^0}{2}\frac{P^j}{P^0}\langle T^{0k}\rangle}_{\text{time dependent}}\right]$$

no probabilistic interpretation

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no probabilistic interpretation

Breit frame ($\boldsymbol{P} = 0$): $\langle L^i\rangle(\boldsymbol{x}) = -i\epsilon^{ijk}\int \dfrac{d^3\mathbf{\Delta}}{(2\pi)^3}\, e^{-i\mathbf{\Delta}\cdot\boldsymbol{x}}\, \dfrac{\partial\langle T^{0k}\rangle}{\partial\Delta^j}$

The time-dependent term is proportional to $\boldsymbol{P}$, so it vanishes at $\boldsymbol{P} = 0$.

Elastic frame

$$P.\Delta = 0 \implies \Delta^0 = \frac{\boldsymbol{P}.\boldsymbol{\Delta}}{P^0}$$

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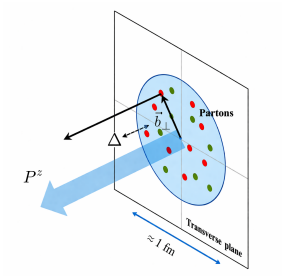
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Spatial distribution of OAM in elastic frame

$$\langle L^a \rangle(\boldsymbol{b}_\perp, P^z) = \epsilon^{ajk} \int \frac{d^2\boldsymbol{\Delta}}{(2\pi)^2} e^{-i\boldsymbol{\Delta}_\perp \cdot \boldsymbol{b}_\perp} \left[-i \frac{\partial \langle T^{0k} \rangle}{\partial \Delta^j} \right]_{\Delta^z = 0}$$

where $a = 1, 2$.

Lorcé, Mukherjee, **Singh**, Won, PLB 868 (2025)

Spatial distribution of boost in elastic frame

$$\langle K^a \rangle(\boldsymbol{b}_\perp) = i \int \frac{d^2\boldsymbol{\Delta}^\perp}{(2\pi)^2} e^{-i\boldsymbol{\Delta}_\perp \cdot \boldsymbol{b}_\perp} \left. \frac{\partial \langle T^{00} \rangle}{\partial \Delta^a} \right|_{\Delta^z = 0} .$$

Lorcé, Mukherjee, **Singh**, Won, arXiv: 2609.08902

Spin-1/2 energy-momentum tensor

$$\mathcal{M}_{s's}[\hat{T}_a^{\mu\nu}] = \bar{u}'\Big[\frac{P^\mu P^\nu}{M}A_a + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M}D_a + Mg^{\mu\nu}\tilde{C}_a + \frac{iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M}J_a - \frac{iP^{[\mu}\sigma^{\nu]\rho}\Delta_\rho}{2M}S_a\Big]u$$

Generalized intrinsic spin tensor

$$\mathcal{M}_{s's}[\hat{S}_q^{\mu\alpha\beta}] = \frac{1}{2}\epsilon^{\mu\alpha\beta\lambda}\bar{u}'\Big[\gamma_\lambda\gamma_5 G_A^q + \frac{\Delta_\lambda\gamma_5}{2M}G_P^q\Big]u$$

Multipole ansatz

$$F_a(t) = \frac{F_a(0)}{\left(1 - t/\Lambda_{F_a}^2\right)^{n_F}}$$

Hackett, Pefkou, and Shanahan, PRL 132, 251904 (2024)
Lorcé, Moutarde, and Trawinski, EPJC 79, 89 (2019)

Angular momentum multipole decomposition

$$J^i_{\perp,a}\left(b_\perp, P^z; s', s\right) = \int dr^z\, J^i_{\perp,a}\left(r, P^z; s', s\right)$$

$$= \int \frac{d^2\Delta_\perp}{(2\pi)^2}\, e^{-i\Delta_\perp\cdot b_\perp} \Big[\delta_{s's}\, i\epsilon^{ij}_\perp \sqrt{\tau}\, \underline{X^j_1}\, \tilde{J}^U_{1,a}(t)$$

$$+ (\sigma_\perp)^i_{s's}\, \underline{X_0}\, \tilde{J}^T_{0,a}(t) + (\sigma_\perp)^j_{s's}\, \tau\, \underline{X^{ij}_2}\, \tilde{J}^T_{2,a}(t)\Big]_{t=-\Delta^2_\perp}$$

where $X_0 = 1$ denotes **monopole**,

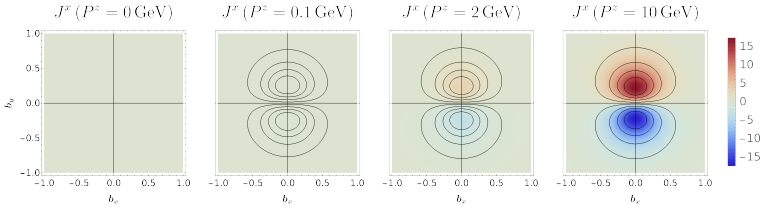
$X^i_1 = \Delta^i_\perp / |\Delta_\perp|$ denote **dipole**,

$X^{ij}_2 = \Delta^i_\perp \Delta^j_\perp / |\Delta_\perp|^2 - \delta^{ij}_\perp/2$ denotes **quadrupole**

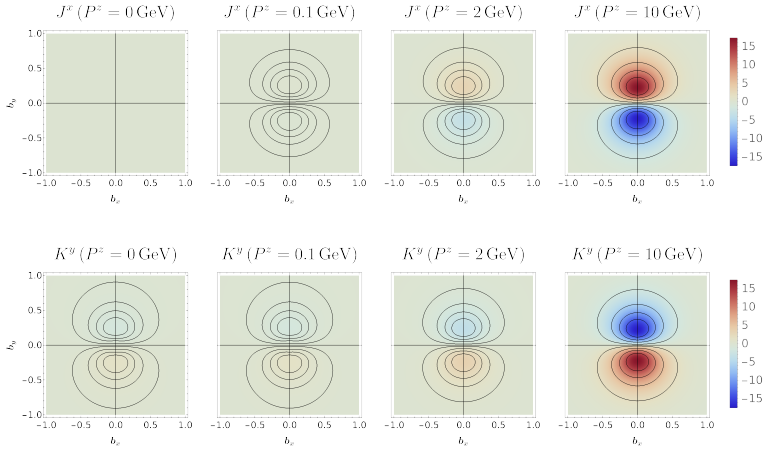
Boost multipole decomposition

$$
\begin{aligned}
&K^i\left(b_\perp, P^z; s', s\right) \\
&= \int \frac{d^2\Delta_\perp}{(2\pi)^2}\, e^{-i\Delta_\perp\cdot b_\perp}\Big[\delta_{s's} i\sqrt{\tau} X_1^i \tilde{K}^U_{1,a}(t) \\
&+ \epsilon_\perp^{ij}\left(\sigma_\perp\right)^j_{s's} X_0 \tilde{K}^T_{0,a}(t) + \epsilon_\perp^{jk}\left(\sigma_\perp\right)^j_{s's}\tau X_2^{ki}\tilde{K}^T_{2,a}(t)\Big]_{t=-\Delta_\perp^2}.
\end{aligned}
$$

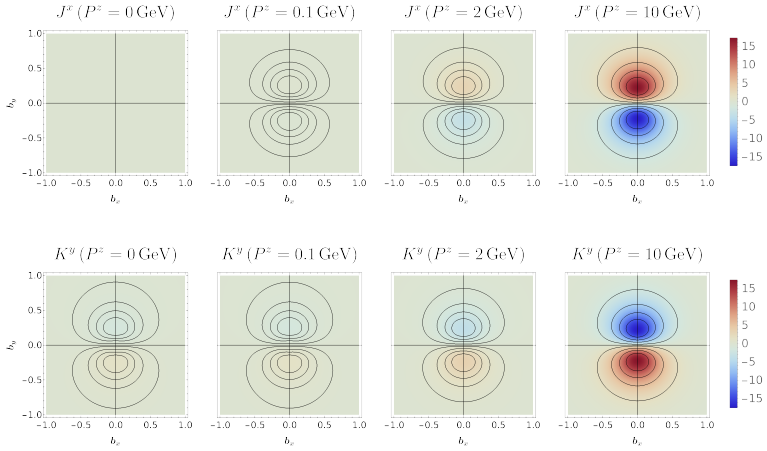
Unpolarized target: transverse angular momentum and boost



Unpolarized target: transverse angular momentum and boost

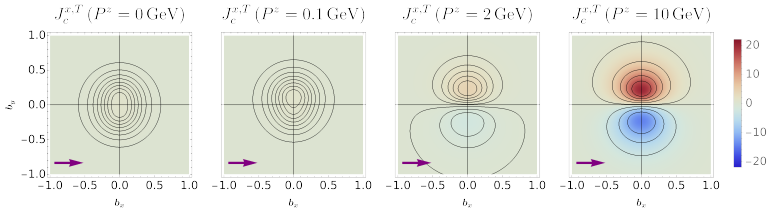


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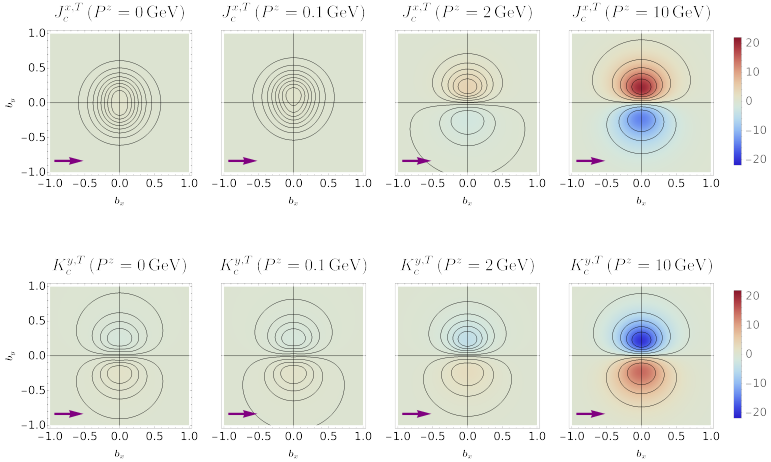


$$\int d^2b_\perp\, J^x(P^z, b_\perp) = 0 \qquad \int d^2b_\perp\, K^y(P^z, b_\perp) = 0$$

Transversely polarized target: Total transverse AM and boost



Transversely polarized target: Total transverse AM and boost



$$\int d^2b_\perp J_c^{x,T}(P^z, b_\perp, s', s) = \frac{\sigma_\perp^x}{2} \qquad \int d^2b_\perp K_c^{y,T}(P^z, b_\perp, s', s) = -\frac{(P \times \sigma_\perp)^y}{2(E_P + M)}$$

Transverse AM and boosts at different pivots

For spin-1/2 targets, the transverse AM and boost distributions depend on the pivots as follows:

$$J^i_X(\boldsymbol{b}_\perp) = J^i(\boldsymbol{b}_\perp) - \epsilon^{ijk} R^j_X P^k(\boldsymbol{b}_\perp),$$
$$K^i_X(\boldsymbol{b}_\perp) = K^i(\boldsymbol{b}_\perp) + R^i_X E(\boldsymbol{b}_\perp),$$

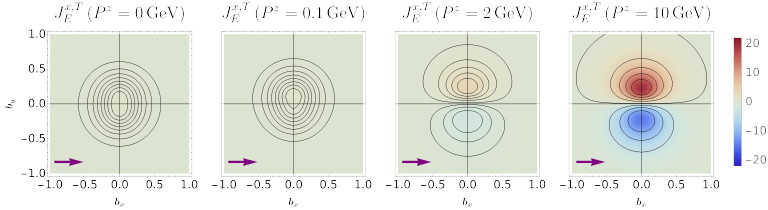
with

$$\boldsymbol{R}_E = \mathcal{R} + \frac{\boldsymbol{P}\times\boldsymbol{s}}{2E_P(E_P+M)},$$
$$\boldsymbol{R}_M = \mathcal{R} - \frac{\boldsymbol{P}\times\boldsymbol{s}}{2M(E_P+M)},$$
$$\boldsymbol{R}_c = \mathcal{R} := \boldsymbol{0},$$

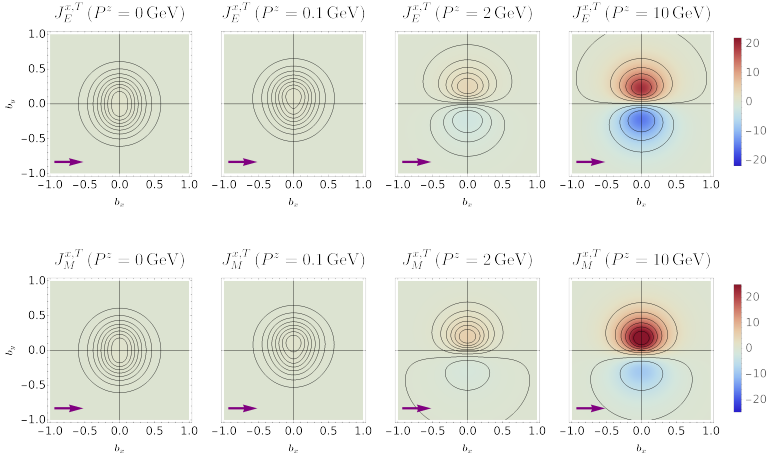
with $E_P = \sqrt{(P^z)^2 + M^2}$.

For spin-0 targets, $\boldsymbol{R}_E = \boldsymbol{R}_M = \boldsymbol{R}_c = \mathcal{R}$.

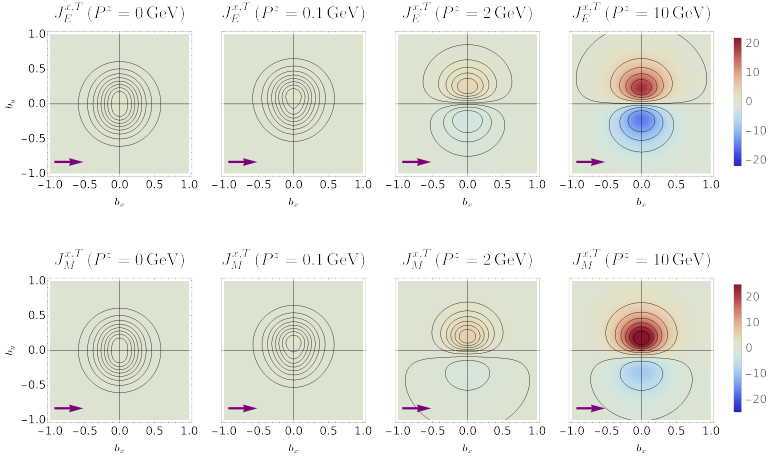
Transverse AM distributions about the center of energy and mass



Transverse AM distributions about the center of energy and mass

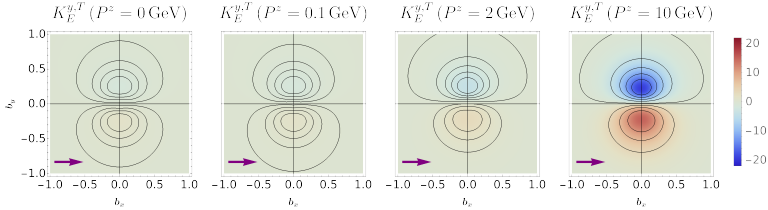


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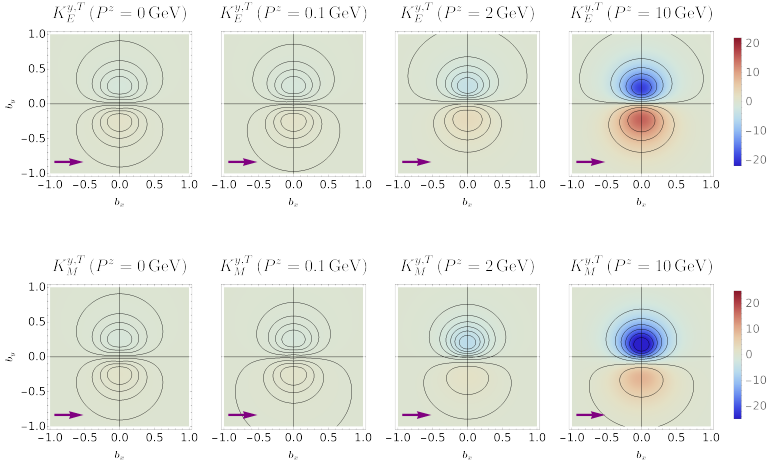


$$\int d^2b_\perp\, J_E^{x,T}(P^z, b_\perp, s', s) = \frac{M}{E_P}\frac{\sigma_\perp^x}{2} \qquad \int d^2b_\perp\, J_M^{x,T}(P^z, b_\perp, s', s) = \frac{E_P}{M}\frac{\sigma_\perp^x}{2}$$

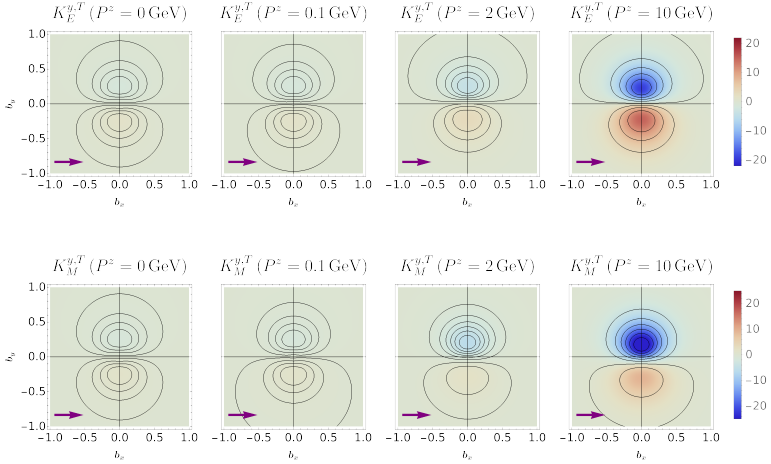
Transverse boost distributions at different pivots



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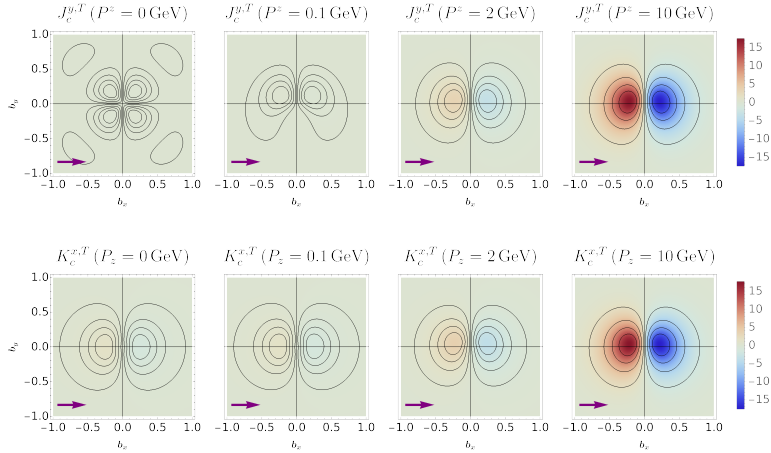


Transverse boost distributions at different pivots



$$\int d^2b_\perp\, K_E^{y,T}(P^z, b_\perp, s', s) = 0 \qquad \int d^2b_\perp\, K_M^{y,T}(P^z, b_\perp, s', s) = -\frac{(P \times \sigma_\perp)^y_{s's}}{2M}$$

Pivot-independent components for nucleon polarized in x-direction



Transverse LF AM multipole decomposition

$$\mathcal{J}^a(b_\perp, P^+; \lambda', \lambda) = \frac{M}{P^+} \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\Delta_\perp \cdot b_\perp}$$
$$\times \left[i\delta_{\lambda'\lambda} \epsilon_\perp^{ab} \sqrt{\tau} X_1^b \tilde{\mathcal{J}}_1 + \sigma^a_{\lambda'\lambda} \tilde{\mathcal{J}}_0 + \sigma^b_{\lambda'\lambda} \tau X_2^{ba} \tilde{\mathcal{J}}_2 \right]$$

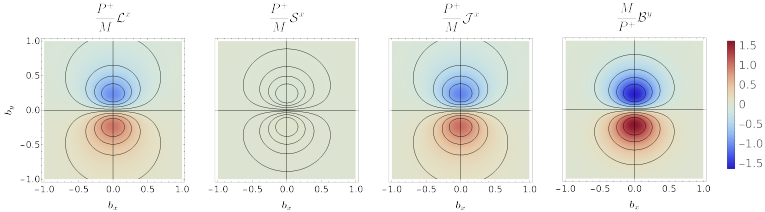
Transverse LF boost multipole decomposition

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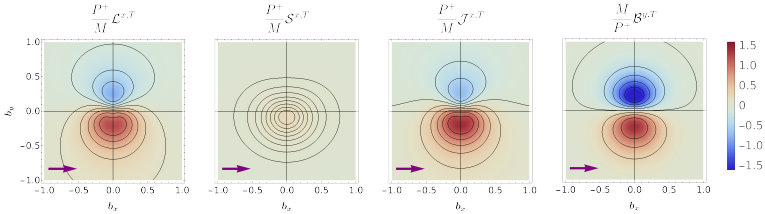
Lorcé, Mukherjee, **Singh**, Won, arXiv:2609.08902

Transverse LF angular momentum and boost distribution

Unpolarized nucleon



Transversely polarized nucleon



LF boost is directly obtained from EF boost

$$\frac{M}{P^+}\mathcal{B}^a(b_\perp, P^+; \lambda', \lambda) = \lim_{P^z \to \infty} \frac{M}{P^z} K^a_{E/c}(b_\perp, P^z; s', s)$$

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LF TAM is a combination of EF TAM and boost

$$\frac{P^+}{M}\mathcal{J}^a(b_\perp, P^+; \lambda', \lambda) = \lim_{P^z\to\infty} \frac{P^z}{M}\Big[J^a_{E/c}(b_\perp, P^z; s', s) + \epsilon^{ab}_\perp K^b_{E/c}(b_\perp, P^z; s', s)\Big]$$

Conclusion

- Spatial distributions of angular momentum provide richer picture of hadron spin.
- Studying them in **elastic frame** helps interpolates between the **Breit frame** and the **infinite momentum frame** pictures.
- **Spin-1/2** targets have complicated spatial structure of transverse angular momentum and boost consisting of monopole, dipole and quadrupole contributions.
- Different **transverse spin sum rules** arise from the choice of pivots .
- **Light-front** picture of transverse angular momentum and boost distribution can be obtained from the corresponding quantities in the instant form.

Conclusion

- Spatial distributions of angular momentum provide richer picture of hadron spin.
- Studying them in **elastic frame** helps interpolates between the **Breit frame** and the **infinite momentum frame** pictures.
- **Spin-1/2** targets have complicated spatial structure of transverse angular momentum and boost consisting of monopole, dipole and quadrupole contributions.
- Different **transverse spin sum rules** arise from the choice of pivots .
- **Light-front** picture of transverse angular momentum and boost distribution can be obtained from the corresponding quantities in the instant form.

Mapping the transverse spin of hadrons in position space

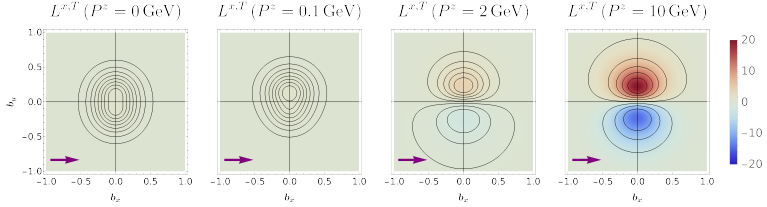
Thank you



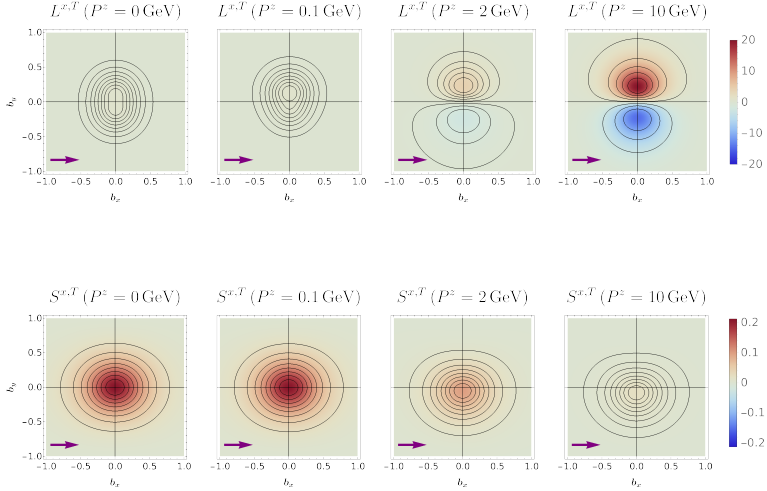
Backup Slides



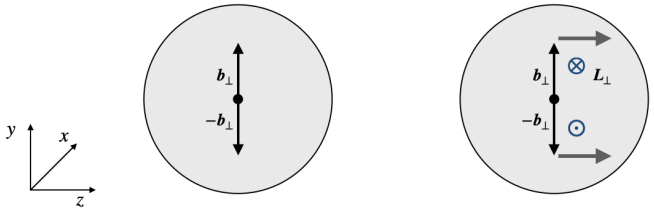
Transversely polarized target: Transverse OAM and intrinsic spin



Transversely polarized target: Transverse OAM and intrinsic spin



Interpretation of spin-0 AM distribution



Rest frame Moving frame

$$d\boldsymbol{L} = \boldsymbol{r} \times dm\, \boldsymbol{v} \quad \text{where } dm = \rho(\boldsymbol{r}) d^3\boldsymbol{r}$$

$$\downarrow$$

Vanishes by the definition of COM $\longrightarrow \int d^3\boldsymbol{r}\, \boldsymbol{r}\rho(\boldsymbol{r}) = 0$

Form Factor Constraints

- As a result of the Poincaré symmetry, the total EMT form factors are constrained as
$$A(0) = 1, \quad J(0) = \frac{1}{2}, \quad \bar{C}(t) = 0.$$
Ji:1996ek,Leader:2013jra,Teryaev:1999su
- In contrast, $D(0)$ is not fixed by any symmetry, but is conjectured to be negative in QCD motivated by stability arguments.
Burkert:2023wzr,Perevalova:2016dln,Polyakov:2018zvc,Lorce:2025oot
- $$S_q = \frac{1}{2} G_A^q, \qquad S_G = 0.$$
- In the present work, we utilize the simple multipole Ansatz for the form factors A, D, J, S fitted to Lattice QCD calculations for the pion and the nucleon, to conduct a numerical analysis and illustrate our results.
Hackett:2023nkr, Hackett:2023rif

$$g\left(b_\perp, P^z; s', s\right) = \int \frac{d^2\Delta_\perp}{(2\pi)^2}\, e^{-i\Delta_\perp \cdot b_\perp}$$

$$\left[\delta_{s's}\tilde{g}_0^U(t) + i\epsilon_\perp^{ij}\sigma_{s's}^i\sqrt{\tau}X_1^j\tilde{g}_1^T(t)\right]_{t=-\Delta_\perp^2},$$

with $g = E, P^z$

$$\tilde{E}^U = \frac{P^0\left[P^0 + M(1+\tau)\right]}{\left(P^0+M\right)(1+\tau)}$$

$$\times\left[E + \frac{2\tau\left(P^z\right)^2}{P^0\left[P^0+M(1+\tau)\right]}J + \left(\frac{P^z}{P^0}\right)^2 F\right],$$

$$\tilde{E}^T = -\frac{P^0P^z}{\left(P^0+M\right)(1+\tau)}$$

$$\times\left[E - \frac{2\left[P^0+M(1+\tau)\right]}{P^0}J + \left(\frac{P^z}{P^0}\right)^2 F\right],$$

$$\tilde{P}^{z,U} = \frac{P^z\left[P^0 + M(1+\tau)\right]}{\left(P^0 + M\right)(1+\tau)}\left[E + F + \frac{\tau P^2}{P^0\left[P^0 + M(1+\tau)\right]}\left(\frac{2\left(P^z\right)^2}{P^2}J - L\right)\right],$$

$$\tilde{P}^{z,T} = \frac{\left(P^z\right)^2}{\left(P^0 + M\right)(1+\tau)}\left[-E - F + \frac{P^2\left[P^0 + M(1+\tau)\right]}{P^0\left(P^z\right)^2}\left(\frac{2\left(P^z\right)^2}{P^2}J - L\right)\right],$$

with

$$E_a = A_a - \tau B_a + \bar{C}_a + \tau D_a,$$

$$J_a = \frac{A_a + B_a}{2},$$

$$F_a = -\tau D_a - \bar{C}_a.$$

TAM multipole amoplitudes

$$\tilde{J}_0 = -2MP^z \frac{d}{dt}\left[\frac{P^0}{M}\frac{1}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}\tau(A-2J)\right] - \frac{\beta}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}\frac{\tau}{4}D + \frac{\cos\theta}{\gamma}\frac{1}{2}\left(J - S + \frac{1}{2}G_A\right)$$
$$+ \frac{M}{4P^0}(G_A - \tau G_P) - 2M^2\frac{d}{dt}\left[2\tau J - \frac{\cos\theta}{\gamma}\tau(J+S)\right],$$

$$\tilde{J}_1 = 4MP^z \frac{d}{dt}\left(\frac{P^0}{M}\frac{\cos\theta}{\gamma}A + \frac{1}{\beta\gamma}\frac{\sin\theta}{\sqrt{\tau}}2\tau J\right) + \beta\frac{\cos\theta}{\gamma}\frac{1}{2}D + \frac{1}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}\frac{1}{2}(J - S + G_A)$$
$$+ 4M^2\frac{d}{dt}\left[\frac{1}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}\tau(J+S)\right],$$

$$\tilde{J}_2 = 4MP^z \frac{d}{dt}\left[\frac{P^0}{M}\frac{1}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}(A-2J)\right] + \frac{\beta}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}\frac{1}{2}D + \frac{M}{2P^0}\left(\frac{1}{\beta\gamma}\frac{\sin\theta}{\sqrt{\tau}}G_A - G_P\right)$$
$$+ 4M^2\frac{d}{dt}\left[2J - \frac{\cos\theta}{\gamma}(J+S)\right].$$

where

$$\cos\theta = \frac{P^0 + M(1+\tau)}{(P^0+M)\sqrt{1+\tau}}, \qquad \sin\theta = -\frac{\sqrt{\tau}P^z}{(P^0+M)\sqrt{1+\tau}},$$

and

$$\gamma = \frac{P^0}{\sqrt{P^2}} = \frac{P^0}{P^0_{\rm BF}} = \frac{P^0}{M\sqrt{1+\tau}}, \qquad \beta = \frac{P^z}{P^0}.$$

Boost multipole amplitudes

$$\tilde{K}_0 = -2MP^z \frac{d}{dt}\left[\frac{P^0}{M}\frac{1}{\beta\gamma}\frac{\sin\theta}{\sqrt{\tau}}\tau A + \tau J - \frac{M}{2P^z}\frac{1}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}\tau F\right],$$

$$\tilde{K}_1 = 4M^2 \frac{d}{dt}\left[-\frac{(P^0)^2}{M^2}\frac{\cos\theta}{\gamma}A + 2\tau J + \frac{\cos\theta}{\gamma}F\right],$$

$$\tilde{K}_2 = 4MP^z \frac{d}{dt}\left[\frac{P^0}{M}\frac{1}{\beta\gamma}\frac{\sin\theta}{\sqrt{\tau}}A + 2J - \frac{M}{P^z}\frac{1}{\gamma}\frac{\sin\theta}{\sqrt{\tau}}F\right],$$

with $F = -\bar{C} - \tau D$.

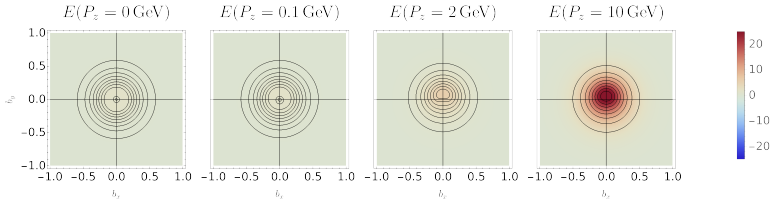
Spin sum rule at center of energy pivot

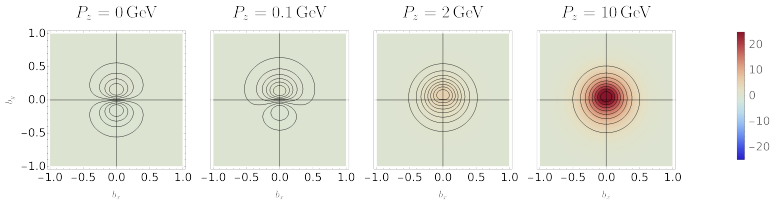
$$
\begin{aligned}
J^i_{\perp,E}(P^z;s',s) &= \int d^2b_\perp \left[J^i_\perp\left(b_\perp,P^z;s',s\right) - \epsilon^{ij}_\perp R^j_E P^z\left(b_\perp,P^z;s',s\right)\right] \\
&= \frac{(\sigma_\perp)^i_{s's}}{2} - \frac{\epsilon^{ij}_\perp \epsilon^{jkl} P^k (\sigma_\perp)^l_{s's}}{2E_P(E_P+M)}\left[P^z(E(0)+F(0))\right] \\
&= \frac{M}{E_P}\frac{(\sigma_\perp)^i_{s's}}{2}
\end{aligned}
$$

Spin sum rule at center of mass pivot

$$
\begin{aligned}
J^i_{\perp,M}(P^z;s',s) &= \int d^2b_\perp \left[J^i_\perp\left(b_\perp,P^z;s',s\right) - \epsilon^{ij}_\perp R^j_M P^z\left(b_\perp,P^z;s',s\right)\right] \\
&= \frac{(\sigma_\perp)^i_{s's}}{2} + \frac{\epsilon^{ij}_\perp \epsilon^{jkl} P^k (\sigma_\perp)^l_{s's}}{2M(E_P+M)}\left[P^z(E(0)+F(0))\right] \\
&= \frac{E_P}{M}\frac{(\sigma_\perp)^i_{s's}}{2}
\end{aligned}
$$

Longitudinal momentum and energy distributions





Genuine LF spin operator

Genuine LF spin operator is defined as

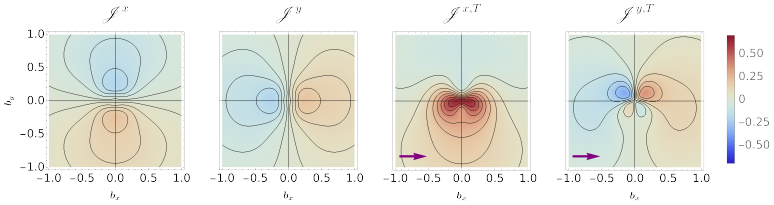
$$\hat{J}^a = \frac{1}{M}(\hat{W}^a - \hat{P}^a \hat{J}^3),$$

where

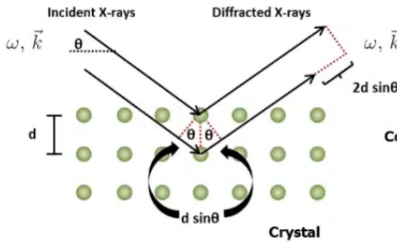
$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{J}_{\nu\alpha}\hat{P}_\beta \qquad \text{and} \qquad \hat{J}^3 = \frac{\hat{W}^+}{\hat{P}^+}$$

and it's spatial distribution is

$$J^a(b_\perp; \lambda', \lambda) = \frac{P^+}{M}\mathcal{L}^a(b_\perp, P^+; \lambda', \lambda) - \epsilon^{ab}\frac{P^-}{M}\mathcal{B}^b(b_\perp, P^+; \lambda', \lambda)$$



Example: X-ray diffraction

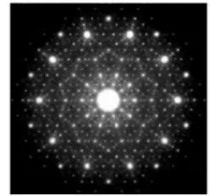


Constructive interference (Bragg's law)

$$2d \sin\theta = n\lambda$$

$$\lambda \sim d$$

Diffraction pattern



$$\propto |A_{\text{scatt}}|^2$$

Scattered amplitude

$$A_{\text{scatt}} \propto F(\vec{q}) = \int \mathrm{d}^3 r \, e^{i\vec{q}\cdot\vec{r}} \, \rho(\vec{r}) \qquad \vec{q} = \vec{k} - \vec{k}'$$

Form factor — Scatterer distribution

Credit: Prof. Cedric Lorce

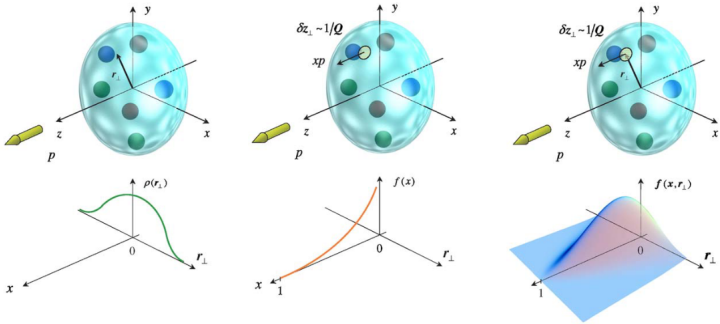
$$\langle O \rangle = \int \frac{d^3P}{(2\pi)^3} d^3\mathcal{R}\, \rho_\psi(\vec{\mathcal{R}}, \vec{P}) \langle O \rangle_{\vec{\mathcal{R}},\vec{P}}$$

$$\rho_\psi(\vec{\mathcal{R}}, \vec{P}) = \int d^3z\, e^{-i\vec{P}.\vec{z}} \psi^* \left(\vec{R} - \frac{\vec{z}}{2}, s'\right) \psi \left(\vec{R} + \frac{\vec{z}}{2}, s\right) \qquad \textbf{Phase space density}$$

$$= \int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}.\vec{R}} \psi^* \left(\vec{P} + \frac{\vec{q}}{2}, s'\right) \psi \left(\vec{P} - \frac{\vec{q}}{2}, s\right)$$

$$\langle O \rangle_{\vec{\mathcal{R}},\vec{P}} = \int \frac{d^3\Delta}{(2\pi)^3} e^{i\vec{\Delta}.\vec{\mathcal{R}}} \frac{\langle P + \frac{\Delta}{2} | O | P - \frac{\Delta}{2} \rangle}{\sqrt{4(P^0)^2 - (\Delta^0)^2}} \qquad \textbf{System localized in phase space}$$

$$\langle \int d^3r\, r^j\, O(r) \rangle_{\vec{\mathcal{R}},\vec{p}} = \mathcal{R}^j \frac{\langle p|O(0)|p\rangle}{2p^0} + \frac{1}{2p^0} \left[-i \frac{\partial}{\partial \Delta^j} \left\langle p + \frac{\Delta}{2} | O | p - \frac{\Delta}{2} \right\rangle \right]_{\vec{\Delta}=\vec{0}}$$



A.V. Belitsky, A.V. Radyushkin Physics Reports 418 (2005) 1 - 387

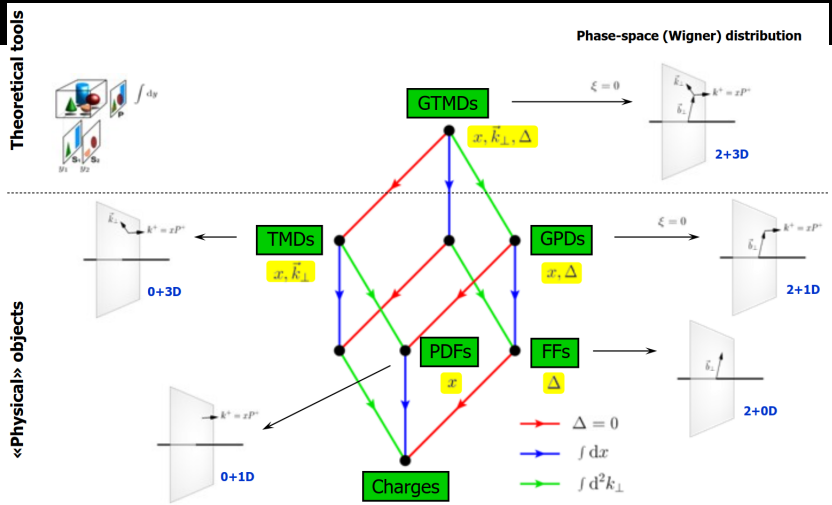


Figure: Map of all the parton distributions

Lorcé, Pasquini, Vanderhaeghen, JHEP 05 (2011) 041