TensoriaCalc

A User-Oriented Mathematica Package for Semi-Riemannian Tensor Calculus

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Introduction

We describe TensoriaCalc, a tensor calculus package written to be smoothly consistent with the Wolfram Language, so as to ensure ease of usage. It allows multiple metrics to be defined in a given session; and, once a metric is computed, associated standard differential geometry operations to be carried out – covariant derivatives, Hodge duals, index raising and lowering, derivation of geodesic equations, etc. Other non-metric operations, such as the Lie and exterior derivatives, coordinate transformation on tensors, etc. are also part of its built-in functionality.

TensoriaCalc has been published

Website:

http://www.stargazing.net/yizen/Tensoria.html

GitHub:

https://github.com/tensoria/TensoriaCalc

arXiv:2512.18796 **[gr-qc]**

Declare a Metric, 2-Sphere for example

2-Sphere: $(dl)^2 = (d\theta)^2 + \mathrm{Sin}[\theta]\,(d\phi)^2$

```
g2Sphere = Metric[a_, b_, (dθ)^2 + Sin[θ]^2 (dϕ)^2, Coordinates → {θ, ϕ}, StartIndex → 1]
% // Head
```

g_{ab}

```
Tensor
```

2-Sphere metric: $\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Sin}[\theta]^2 \end{pmatrix}$

```
Metric[a_, b_, {{1, 0}, {0, Sin[θ]^2}}, Coordinates → {θ, ϕ}, StartIndex → 1]
```

g_{ab}

Declare a Metric, 2-Sphere for example

```
Metric[a⁻, b⁻, (∇θ)^2 + Sin[θ]^-2 (∇ϕ)^2, Coordinates → {θ, ϕ}, StartIndex → 1]
```

g^{ab}

Declare a Metric, 2-Sphere for example

Geometrical objects:

```
Christoffel[a⁻, m₋, n₋, g2Sphere]
Riemann[a⁻, b₋, m₋, n₋, g2Sphere, TooltipDisplay → TensorComponents]
Ricci[a⁻, b₋, g2Sphere]
```

$\Gamma^{a}{}_{mn}$

$R^{a}{}_{bmn}$

$R^{a}{}_{b}$

Declare a Metric, 2-Sphere for example

TensorComponents can extract tensor components from a Tensor object.

```
g2Sphere
% // TensorComponents // MatrixForm
```

g_{ab}

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Declare a Metric, 2-Sphere for example

$$\Gamma^{\alpha}{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\lambda}(\partial_{\mu} g_{\nu\lambda} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu})$$

```
Christoffel[a⁻, m₋, n₋, g2Sphere]
% // TensorComponents // MatrixForm
```

$$\Gamma^{a}{}_{mn}$$

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix} \\ \begin{pmatrix} 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} \mathrm{Cot}[\theta] \\ 0 \end{pmatrix} \end{pmatrix}$$

Declare a Metric, 2-Sphere for example

$\Gamma^{\alpha}{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$

NonZeroTensorComponents extracts non-zero tensor components from a Tensor object.

```
Christoffel[a⁻, m₋, n₋, g2Sphere]
% // NonZeroTensorComponents
```

$\Gamma^{a}{}_{mn}$

$\{\Gamma^{\phi}{}_{\phi\theta} \to \mathrm{Cot}[\theta],\ \Gamma^{\phi}{}_{\theta\phi} \to \mathrm{Cot}[\theta],\ \Gamma^{\theta}{}_{\phi\phi} \to -\mathrm{Cos}[\theta]\ \mathrm{Sin}[\theta]\}$

Declare a Metric, 2-Sphere for example

$$R^{\alpha}{}_{\beta\mu\nu} \equiv \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\beta}$$

```
Riemann[a⁻, b₋, m₋, n₋, g2Sphere, TooltipDisplay → TensorComponents]
% // NonZeroTensorComponents
```

$R^{a}{}_{bmn}$

$\left\{R^{\phi}{}_{\theta\phi\theta} \to 1,\ R^{\theta}{}_{\phi\phi\theta} \to -\mathrm{Sin}[\theta]^2,\ R^{\phi}{}_{\theta\theta\phi} \to -1,\ R^{\theta}{}_{\phi\theta\phi} \to \mathrm{Sin}[\theta]^2\right\}$

Declare a Metric, 2-Sphere for example

Specified indices with coordinates

```
Riemann[θ⁻, ϕ₋, ϕ₋, θ₋, g2Sphere, TooltipDisplay → TensorComponents]
Riemann[1⁻, 2₋, 2₋, 1₋, g2Sphere, TooltipDisplay → TensorComponents]
Riemann[a⁻, b₋, m₋, n₋, g2Sphere, TooltipDisplay → TensorComponents] /.
 {a → θ, b → ϕ, m → ϕ, n → θ}
Riemann[a⁻, b₋, m₋, n₋, g2Sphere, TooltipDisplay → TensorComponents] /.
 {a → 1, b → 2, m → 2, n → 1}
```

$-\mathrm{Sin}[\theta]^2$

$-\mathrm{Sin}[\theta]^2$

$-\mathrm{Sin}[\theta]^2$

$-\mathrm{Sin}[\theta]^2$

Declare a Metric, 2-Sphere for example

$R_{\beta\nu} \equiv R^{\alpha}{}_{\beta\alpha\nu}$

```
Ricci[a⁻, b₋, g2Sphere]
% // TensorComponents // MatrixForm
```

$R^{a}{}_{b}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Schwarzschild Black Hole

General Spherical Symmetric Metric

```
dlsq = Exp[2 ψ[t, r]] (1 - (2 GN M[t, r]) / r) (dt)^2 - (dr)^2/ (1 - (2 GN M[t, r]) / r) -
   r^2 ((dθ)^2 + Sin[θ]^2 (dϕ)^2)
g = Metric[α_, β_, dlsq, Coordinates → {t, r, θ, ϕ}]
% // TensorComponents // MatrixForm
```

$$-\frac{(\mathrm{d}r)^2}{1-\frac{2\,GN\,M[t,r]}{r}}+e^{2\psi[t,r]}(\mathrm{d}t)^2\left(1-\frac{2\,GN\,M[t,r]}{r}\right)-r^2\left((\mathrm{d}\theta)^2+(\mathrm{d}\phi)^2\,\mathrm{Sin}[\theta]^2\right)$$

$$g_{\alpha\beta}$$

$$\begin{pmatrix} \frac{e^{2\psi[t,r]}(r-2\,GN\,M[t,r])}{r} & 0 & 0 & 0 \\ 0 & -\frac{r}{r-2\,GN\,M[t,r]} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Schwarzschild Black Hole

Vacuum solution

$G^{\mu}{}_{\nu}[g_{\alpha\beta}] = 8\,\pi G_N\, T^{\mu}{}_{\nu} = 0$

```
Einstein[μ⁻, ν₋, g]
% == ZeroTensor[{μ⁻, ν₋}, Coordinates → {t, r, θ, ϕ}]
EinsteinFieldEqns = TensorEquations[%, OutputForm → Tensor]
EinsteinFieldEqns // TensorComponents // MatrixForm
```

$$G^{\mu}{}_{\nu}$$

$$G^{\mu}{}_{\nu} == 0^{\mu}{}_{\nu}$$

$$(G==0)^{\mu}{}_{\nu}$$

$$\begin{pmatrix} \frac{2\,GN\,M^{(0,1)}[t,r]}{r^2} == 0 & \frac{2\,e^{-2\psi[t,r]}\,GN\,M^{(1,0)}[t,r]}{(r-2\,GN\,M[t,r])^2} == 0 & \\ -\frac{2\,GN\,M^{(1,0)}[t,r]}{r^2} == 0 & \frac{2\,GN\,M^{(0,1)}[t,r]-2\,(r-2\,GN\,M[t,r])\,\psi^{(0,1)}[t,r]}{r^2} == 0 & \\ \text{True} & \text{True} & \frac{-2\,GN\,M^{(0,1)}[t,r]+(r-2\,GN\,M[t,r])\,\psi^{(0,1)}[t,r]-(2\,r-GN\,M[t,}{} \\ \text{True} & \text{True} & \end{pmatrix}$$

Schwarzschild Black Hole

$\{G^t{}_t = 0,\ G^t{}_r = 0\}$

```
{EinsteinFieldEqns /. {μ → t, ν → t}, EinsteinFieldEqns /. {μ → t, ν → r}}
MSol = DSolve[%, M, {t, r}][[1]]
```

$$\left\{\frac{2\,\mathrm{GN}\,M^{(0,1)}[t,r]}{r^2} == 0,\ \frac{2\,e^{-2\psi[t,r]}\,\mathrm{GN}\,M^{(1,0)}[t,r]}{(r-2\,\mathrm{GN}\,M[t,r])^2} == 0\right\}$$

$\{M \to \mathrm{Function}[\{t, r\}, c_1]\}$

M is constant

Similarly we can show ψ only depends on t not on r.

Therefore, we have proven the Birkhoff's theorem.

```
SubtractSides[EinsteinFieldEqns /. {μ → t, ν → t}, EinsteinFieldEqns /. {μ → r, ν → r}]
ψSol = DSolve[%, ψ, {t, r}][[1]]
```

$$\frac{2\,\mathrm{GN}\,M^{(0,1)}[t,r]}{r^2} - \frac{2\,\mathrm{GN}\,M^{(0,1)}[t,r] - 2\,(r - 2\,\mathrm{GN}\,M[t,r])\,\psi^{(0,1)}[t,r]}{r^2} == 0$$

$\{\psi \to \mathrm{Function}[\{t, r\}, c_1[t]]\}$

Schwarzschild Black Hole

Re-define time to let $e^{2\,\psi[t]}\,\mathbb{d}t \to \mathbb{d}t$

```
dlsq = (((dlsq /. ψSol) /. ⅇ^(2 ℂ₁[t]) → 1) /. (MSol /. {ℂ₁ → M})) /. {M → rs/(2 GN)}
g = Metric[α_, β_, dlsq, Coordinates → {t, r, θ, ϕ}, FlatMetric → {1, -1, -1, -1},
  OrthonormalFrameFieldIndices → {α̂, β}]
% // TensorComponents // MatrixForm
```

$$-\frac{(\mathbb{d}r)^2}{1-\frac{rs}{r}} + \left(1-\frac{rs}{r}\right)(\mathbb{d}t)^2 - r^2\left((\mathbb{d}\theta)^2 + (\mathbb{d}\phi)^2\,\mathrm{Sin}[\theta]^2\right)$$

$$g_{\alpha\beta}$$

$$\begin{pmatrix} 1-\frac{rs}{r} & 0 & 0 & 0 \\ 0 & -\frac{r}{r-rs} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Schwarzschild Black Hole

Schwarzschild Metric is time independent.

```
PartialD[g, t]
% // TensorComponents // MatrixForm
%% // ZeroTensorQ
```

$\partial_t[g]_{\alpha\beta}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
True
```

Schwarzschild Black Hole

Time translation U={1,0,0,0}
Killing equation $\pounds_V\, g_{\alpha\beta} = \nabla_{\{\alpha} g_{\beta\}} = 0$

```
NonMetricTensor[{μ⁻}, {1, 0, 0, 0}, "U", Coordinates → {t, r, θ, ϕ}]
% // TensorComponents // MatrixForm
ZeroTensorQ[LieD[%%, g]]
```

U^{μ}

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

True

Schwarzschild Black Hole

3 rotational Killing vectors

$(J^{\beta\alpha})^{\mu} = -x^{[\beta} p^{\alpha]\,\mu}$

$(J^{a})^{\mu} \equiv (1/2)\,\epsilon^{abc}(J^{bc})^{\mu}$, a=1, 2, 3

```
J1 = NonMetricTensor[{μ⁻}, -I (y∇z - z∇y), "J¹", Coordinates → {t, x, y, z}]
J2 = NonMetricTensor[{μ⁻}, -I (z∇x - x∇z), "J²", Coordinates → {t, x, y, z}]
J3 = NonMetricTensor[{μ⁻}, -I (x∇y - y∇x), "J³", Coordinates → {t, x, y, z}]
```

$J^{1\mu}$

$J^{2\mu}$

$J^{3\mu}$

Schwarzschild Black Hole

$t = t$

$x = r\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi]$

$y = r\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]$

$z = r\,\mathrm{Cos}[\theta]$

```
Mink4dTotrθϕ = Sequence[{t → t, x → r Sin[θ] Cos[ϕ], y → r Sin[θ] Sin[ϕ], z → r Cos[θ]},
   {t, r, θ, ϕ}];
CoordinateTransformation[Mink4dTotrθϕ]
```

$$\Big\{ dt \to dt,\ dx \to r\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,d\theta + \mathrm{Cos}[\phi]\,dr\,\mathrm{Sin}[\theta] - r\,d\phi\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi],$$

$$dy \to r\,\mathrm{Cos}[\phi]\,d\phi\,\mathrm{Sin}[\theta] + r\,\mathrm{Cos}[\theta]\,d\theta\,\mathrm{Sin}[\phi] + dr\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi],$$

$$dz \to \mathrm{Cos}[\theta]\,dr - r\,d\theta\,\mathrm{Sin}[\theta],\ t \to t,\ x \to r\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta],\ y \to r\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi],$$

$$z \to r\,\mathrm{Cos}[\theta],\ \nabla t \to \nabla t,\ \nabla x \to \frac{\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\nabla\theta}{r} + \mathrm{Cos}[\phi]\,\nabla r\,\mathrm{Sin}[\theta] - \frac{\mathrm{Csc}[\theta]\,\nabla\phi\,\mathrm{Sin}[\phi]}{r},$$

$$\nabla y \to \frac{\mathrm{Cos}[\phi]\,\mathrm{Csc}[\theta]\,\nabla\phi}{r} + \frac{\mathrm{Cos}[\theta]\,\nabla\theta\,\mathrm{Sin}[\phi]}{r} + \nabla r\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi],\ \nabla z \to \mathrm{Cos}[\theta]\,\nabla r - \frac{\nabla\theta\,\mathrm{Sin}[\theta]}{r} \Big\}$$

Schwarzschild Black Hole

```
J1
% // TensorComponents // MatrixForm
J1 = CoordinateTransformation[%%, Mink4dTotrθϕ]
% // TensorComponents // MatrixForm
```

$J1^{\mu}$

$$\begin{pmatrix} 0 \\ 0 \\ i\, z \\ -i\, y \end{pmatrix}$$

$J1^{\mu}$

$$\begin{pmatrix} 0 \\ 0 \\ i\, \mathrm{Sin}[\phi] \\ i\, \mathrm{Cos}[\phi]\, \mathrm{Cot}[\theta] \end{pmatrix}$$

Schwarzschild Black Hole

```
J2
% // TensorComponents // MatrixForm
J2 = CoordinateTransformation[%%, Mink4dTotrθϕ]
% // TensorComponents // MatrixForm
```

$J^{2\mu}$

$$\begin{pmatrix} 0 \\ -i\, z \\ 0 \\ i\, x \end{pmatrix}$$

$J^{2\mu}$

$$\begin{pmatrix} 0 \\ 0 \\ -i\, \mathrm{Cos}[\phi] \\ i\, \mathrm{Cot}[\theta]\, \mathrm{Sin}[\phi] \end{pmatrix}$$

Schwarzschild Black Hole

```
J3
% // TensorComponents // MatrixForm
J3 = CoordinateTransformation[%%, Mink4dTotrθϕ]
% // TensorComponents // MatrixForm
```

$J^{3\mu}$

$$\begin{pmatrix} 0 \\ i\,y \\ -i\,x \\ 0 \end{pmatrix}$$

$J^{3\mu}$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -i \end{pmatrix}$$

Schwarzschild Black Hole

Killing equation $\pounds_V\, g_{\alpha\beta} = \nabla_{\{\alpha} g_{\beta\}} = 0$

```
ZeroTensorQ[LieD[J1, g]]
ZeroTensorQ[LieD[J2, g]]
ZeroTensorQ[LieD[J3, g]]
```

```
True
```

```
True
```

```
True
```

Schwarzschild Black Hole

Curvature Singularity
ContractTensors contracts tensors with repeated indices.

```
Riemann[α₋, β₋, μ₋, ν₋, g] × Riemann[α⁻, β⁻, μ⁻, ν⁻, g]
(ContractTensors[%]) // FullSimplify
```

$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$

$\frac{12\,\mathrm{rs}^2}{r^6}$

```
Weyl[α₋, β₋, μ₋, ν₋, g] × Weyl[α⁻, β⁻, μ⁻, ν⁻, g]
(ContractTensors[%]) // FullSimplify
```

$C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu}$

$\frac{12\,\mathrm{rs}^2}{r^6}$

Schwarzschild Black Hole

MoveIndices move indices to the target position (upstairs or downstairs)

```
Riemann[α₋, β₋, μ₋, ν₋, g]
MoveIndices[%, {α⁻, β₋, μ⁻, ν₋}, g]
```

$R_{\alpha\beta\mu\nu}$

$R^{\alpha\ \mu}{}_{\ \beta\ \nu}$

Schwarzschild Black Hole

Tidal force $\sim \frac{1}{r^3}$

```
MoveIndices[Riemann[α_, β_, μ_, ν_, g], OrthonormalBasis, g] // FullSimplify
NonZeroTensorComponents[%]
```

$R_{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}}$

$$\Big\{R_{\hat{1}\hat{0}\hat{1}\hat{0}} \to \frac{rs}{r^3},\ R_{\hat{0}\hat{1}\hat{1}\hat{0}} \to -\frac{rs}{r^3},\ R_{\hat{2}\hat{0}\hat{2}\hat{0}} \to -\frac{rs}{2\,r^3},\ R_{\hat{0}\hat{2}\hat{2}\hat{0}} \to \frac{rs}{2\,r^3},\ R_{\hat{3}\hat{0}\hat{3}\hat{0}} \to -\frac{rs}{2\,r^3},\ R_{\hat{0}\hat{3}\hat{3}\hat{0}} \to \frac{rs}{2\,r^3},$$

$$R_{\hat{1}\hat{0}\hat{0}\hat{1}} \to -\frac{rs}{r^3},\ R_{\hat{0}\hat{1}\hat{0}\hat{1}} \to \frac{rs}{r^3},\ R_{\hat{2}\hat{1}\hat{2}\hat{1}} \to \frac{rs}{2\,r^3},\ R_{\hat{1}\hat{2}\hat{2}\hat{1}} \to -\frac{rs}{2\,r^3},\ R_{\hat{3}\hat{1}\hat{3}\hat{1}} \to \frac{rs}{2\,r^3},\ R_{\hat{1}\hat{3}\hat{3}\hat{1}} \to -\frac{rs}{2\,r^3},$$

$$R_{\hat{2}\hat{0}\hat{0}\hat{2}} \to \frac{rs}{2\,r^3},\ R_{\hat{0}\hat{2}\hat{0}\hat{2}} \to -\frac{rs}{2\,r^3},\ R_{\hat{2}\hat{1}\hat{1}\hat{2}} \to -\frac{rs}{2\,r^3},\ R_{\hat{1}\hat{2}\hat{1}\hat{2}} \to \frac{rs}{2\,r^3},\ R_{\hat{3}\hat{2}\hat{3}\hat{2}} \to -\frac{rs}{r^3},\ R_{\hat{2}\hat{3}\hat{3}\hat{2}} \to \frac{rs}{r^3},$$

$$R_{\hat{3}\hat{0}\hat{0}\hat{3}} \to \frac{rs}{2\,r^3},\ R_{\hat{0}\hat{3}\hat{0}\hat{3}} \to -\frac{rs}{2\,r^3},\ R_{\hat{3}\hat{1}\hat{1}\hat{3}} \to -\frac{rs}{2\,r^3},\ R_{\hat{1}\hat{3}\hat{1}\hat{3}} \to \frac{rs}{2\,r^3},\ R_{\hat{3}\hat{2}\hat{2}\hat{3}} \to \frac{rs}{r^3},\ R_{\hat{2}\hat{3}\hat{2}\hat{3}} \to -\frac{rs}{r^3}\Big\}$$

Schwarzschild Black Hole

GeodesicSystem returns a List consisting of:
The affinely parameterized geodesic Lagrangian

$$\frac{1}{2} g_{\alpha\beta}\, u^{\alpha}[\lambda]\, u^{\beta}[\lambda]$$

The non-affinely parametrized geodesic Lagrangian

$$\left| g_{\alpha\beta}\, u^{\alpha}[s]\, u^{\beta}[s] \right|^{\frac{1}{2}}$$

List of affinely parametrized geodesic equations

$$\frac{d}{d\lambda} u^{\alpha}[\lambda] + \Gamma^{\alpha}{}_{\mu\nu}\, u^{\mu}[\lambda]\, u^{\nu}[\lambda] = 0$$

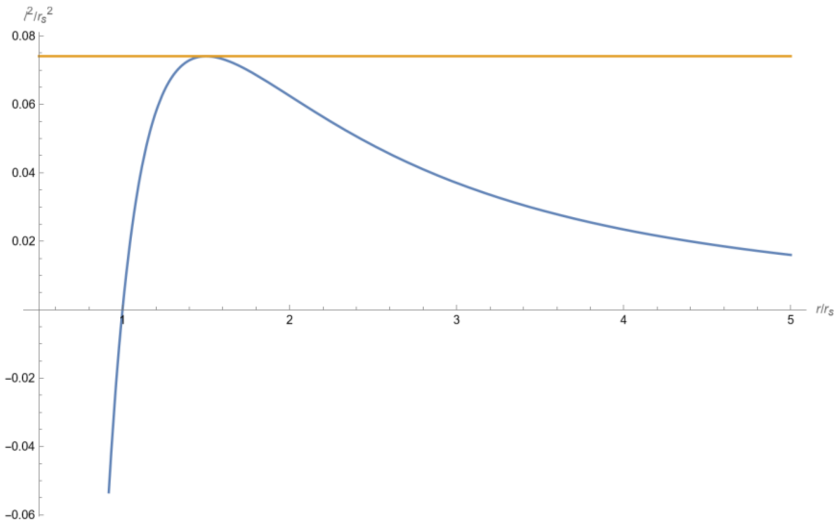
List of non-affinely parametrized geodesic equations

$$\frac{d}{ds} u^{\alpha}[s] + \Gamma^{\alpha}{}_{\mu\nu}\, u^{\mu}[s]\, u^{\nu}[s] = u^{\alpha}[s] \frac{d}{ds} \ln \left| g_{\alpha\beta}\, u^{\alpha}[s]\, u^{\beta}[s] \right|^{\frac{1}{2}}$$

```
geodesicsystem = GeodesicSystem[g, AffineParameter → λ, NonAffineParameter → t] /.
   θ → Function[{λ}, Pi / 2];
```

Schwarzschild Black Hole

```
LNullPEρ[ρ_] = (LNullKEPE⟦1⟧ - 1/2 r'[λ]^2) /. r[λ] → rs ρ;
Plot[{FullSimplify[LNullPEρ[ρ] (rs^2/ℓ^2)], 2/27}, {ρ, 0.5, 5},
 AxesLabel → {"r/rs", "ℓ^2/rs^2"}]
```



Kerr-Newman Black Hole

```
id = # &;
ρ = (r^2 + a^2 Cos[θ]^2)^(1/2);
Δ = r^2 - rs r + a^2 + e^2;
FS = FullSimplify[#, {0 ≤ ϕ ≤ 2 Pi, 0 ≤ θ ≤ Pi, r ≥ a > 0, G > 0, M > 0, rs > 0, e ∈ Reals}] &;
dτsq = -(((dr)^2/Δ + (dθ)^2) ρ^2 - (dt - a Sin[θ]^2 (dϕ))^2 Δ/ρ^2 + ((r^2 + a^2) (dϕ) - a (dt))^2 Sin[θ]^2/ρ^2);
g = Metric[μ_, ν_, dτsq, TensorName → "gKN", Coordinates → {t, r, θ, ϕ},
  ChristoffelOperator → FS, RiemannOperator → id, RicciOperator → FS,
  RicciScalarOperator → FS, MetricOperator → FS, TooltipStyle → {"Large"}]
% // TensorComponents // MatrixForm
```

$g_{KN}{}_{\mu\nu}$

$$\begin{pmatrix} 1+\frac{2\left(e^2-r\,rs\right)}{a^2+2r^2+a^2\,\mathrm{Cos}[2\theta]} & 0 & 0 & \frac{a\left(-e^2+r\,rs\right)\mathrm{Sin}[\theta]^2}{r^2+a^2\,\mathrm{Cos}[\theta]^2} \\ 0 & -\frac{r^2+a^2\,\mathrm{Cos}[\theta]^2}{a^2+e^2+r^2-r\,rs} & 0 & 0 \\ 0 & 0 & -r^2-a^2\,\mathrm{Cos}[\theta]^2 & 0 \\ \frac{a\left(-e^2+r\,rs\right)\mathrm{Sin}[\theta]^2}{r^2+a^2\,\mathrm{Cos}[\theta]^2} & 0 & 0 & \frac{-\left(a^2+r^2\right)^2\mathrm{Sin}[\theta]^2+a^2\left(a^2+e^2+r^2-r\,rs\right)\mathrm{Sin}[\theta]^4}{r^2+a^2\,\mathrm{Cos}[\theta]^2} \end{pmatrix}$$

Kerr-Newman Black Hole

```
A = NonMetricTensor[{ν_}, {r e/ρ^2, 0, 0, -(a r e Sin[θ]^2)/ρ^2}, "A", Coordinates → {t, r, θ, ϕ}]
% // TensorComponents // MatrixForm
```

A_{ν}

$$\begin{pmatrix} \frac{e\, r}{r^2+a^2 \operatorname{Cos}[\theta]^2} \\ 0 \\ 0 \\ -\frac{a\, e\, r \operatorname{Sin}[\theta]^2}{r^2+a^2 \operatorname{Cos}[\theta]^2} \end{pmatrix}$$

Kerr-Newman Black Hole

```
F = ⅆμ_ A
% // TensorComponents // MatrixForm
```

$$(\mathbb{d}A)_{\mu\nu}$$

$$\begin{pmatrix}
0 & \frac{2 e r^2}{\left(r^2+a^2 \cos[\theta]^2\right)^2}-\frac{e}{r^2+a^2 \cos[\theta]^2} & -\frac{2 a^2 e r \cos[\theta] \sin[\theta]}{\left(r^2+a^2 \cos[\theta]^2\right)^2} & 0 \\
-\frac{2 e r^2}{\left(r^2+a^2 \cos[\theta]^2\right)^2}+\frac{e}{r^2+a^2 \cos[\theta]^2} & 0 & 0 & \frac{2 a e r^2 \sin[\theta]^2}{\left(r^2+a^2 \cos[\theta]^2\right)^2}-\frac{a e \sin[\theta]^2}{r^2+a^2 \cos[\theta]^2} \\
\frac{2 a^2 e r \cos[\theta] \sin[\theta]}{\left(r^2+a^2 \cos[\theta]^2\right)^2} & 0 & 0 & -\frac{2 a e r \cos[\theta] \sin[\theta]}{r^2+a^2 \cos[\theta]^2}-\frac{2 a^3 e r \cos[\theta] \sin[\theta]^3}{\left(r^2+a^2 \cos[\theta]^2\right)^2} \\
0 & -\frac{2 a e r^2 \sin[\theta]^2}{\left(r^2+a^2 \cos[\theta]^2\right)^2}+\frac{a e \sin[\theta]^2}{r^2+a^2 \cos[\theta]^2} & \frac{2 a e r \cos[\theta] \sin[\theta]}{r^2+a^2 \cos[\theta]^2}+\frac{2 a^3 e r \cos[\theta] \sin[\theta]^3}{\left(r^2+a^2 \cos[\theta]^2\right)^2} & 0
\end{pmatrix}$$

Kerr-Newman Black Hole

Maxwell's eqns in vacuum

$\nabla^{\mu} F_{\mu\nu} = J_{\nu} = 0$

```
CovariantD[μ⁻, F, g] // FS
% // ZeroTensorQ
```

$(\nabla \mathbb{d}\mathbf{A})^{\underline{\mu}}{}_{\underline{\mu}\nu}$

```
True
```

Kerr-Newman Black Hole

G=2T

```
G = Einstein[α⁻, β⁻, g, EinsteinOperator → FS];
T = ElectromagneticStressEnergyTensor[α⁻, β⁻, F, g] // FS;
TensorEquations[G, 2 T, OutputForm → BooleanQ]
```

```
True
```

Kerr-Newman Black Hole

Let's calculate Carter constant

$l^\mu = \left\{\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right\}$

$n^\nu = \left\{\frac{r^2+a^2}{2\Sigma}, \frac{-\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right\}$

```
lμU = NonMetricTensor[{μ⁻}, {(r^2 + a^2)/Δ, 1, 0, a/Δ}, "l", Coordinates → {t, r, θ, ϕ}]

nνU = NonMetricTensor[{ν⁻}, {(r^2 + a^2)/(2 ρ^2), -Δ/(2 ρ^2), 0, a/(2 ρ^2)}, "n", Coordinates → {t, r, θ, ϕ}]
```

l^μ

n^ν

Kerr-Newman Black Hole

$K^{\mu\nu} = \rho^2\, l^{\{\mu}\, n^{\nu\}} + r^2\, g^{\mu\nu}$

```
Kμν =
 TensorName[
  (TensorsProduct[ρ² (TensorsProduct[{lμU, nνU}] + TensorsProduct[{nνU, lμU}]) -
      r² Metric[μ⁻, ν⁻, g, MetricOperator → FS] , Tensor] // FS), "K"]
% // TensorComponents // MatrixForm
```

$K^{\mu\nu}$

$$\begin{pmatrix} \frac{a^2\left(a^4+3a^2r^2+r^2\left(e^2+2r^2-r\,rs\right)+\left(a^4+a^2r^2+r^2\left(-e^2+r\,rs\right)\right)\mathrm{Cos}[2\theta]\right)}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)} & 0 & 0 & \frac{a^5+3a^3r^2+2a\,r^2\left(e^2+r^2-r\,rs\right)+a^3\left(a^2+r^2\right)\mathrm{Cos}[2\theta]}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)} \\ 0 & -\frac{a^2\left(a^2+e^2+r\left(r-rs\right)\right)\mathrm{Cos}[\theta]^2}{r^2+a^2\mathrm{Cos}[\theta]^2} & 0 & 0 \\ 0 & 0 & \frac{1}{1+\frac{a^2\mathrm{Cos}[\theta]^2}{r^2}} & 0 \\ \frac{a^5+3a^3r^2+2a\,r^2\left(e^2+r^2-r\,rs\right)+a^3\left(a^2+r^2\right)\mathrm{Cos}[2\theta]}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)} & 0 & 0 & \frac{2a^4\mathrm{Cos}[\theta]^2+2r^2\left(a^2+e^2+r^2-r\,rs\right)\mathrm{Csc}[\theta]^2}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)} \end{pmatrix}$$

Kerr-Newman Black Hole

Killing equation $\nabla^{\{\alpha}K^{\mu\nu\}} = 0$

```
SymmetrizeIndices[CovariantD[α⁻, Kμν, g]]
ZeroTensorQ[%]
```

$(\nabla K)^{\{\alpha\mu\nu\}}$

```
True
```

Kerr-Newman Black Hole

$K = K^{\mu\nu} p_\mu p_\nu$

```
pμ = NonMetricTensor[{μ_}, {-En, pr[λ], pθ[λ], l}, "p", Coordinates → {t, r, θ, ϕ}];
Kμν pμ (pμ /. μ → ν)
ContractTensors[%]
```

$$p_\mu p_\nu K^{\mu\nu}$$

$$-\frac{2\,\mathrm{En}\,l\left(a^5+3a^3r^2+2ar^2\left(e^2+r^2-r\,rs\right)+a^3\left(a^2+r^2\right)\mathrm{Cos}[2\theta]\right)}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)}+$$
$$\frac{a^2\,\mathrm{En}^2\left(a^4+3a^2r^2+r^2\left(e^2+2r^2-r\,rs\right)+\left(a^4+a^2r^2+r^2\left(-e^2+r\,rs\right)\right)\mathrm{Cos}[2\theta]\right)}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)}+$$
$$\frac{l^2\left(2a^4\mathrm{Cos}[\theta]^2+2r^2\left(a^2+e^2+r^2-r\,rs\right)\mathrm{Csc}[\theta]^2\right)}{\left(a^2+e^2+r\left(r-rs\right)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)}-$$
$$\frac{a^2\left(a^2+e^2+r\left(r-rs\right)\right)\mathrm{Cos}[\theta]^2\,\mathrm{pr}[\lambda]^2}{r^2+a^2\mathrm{Cos}[\theta]^2}+\frac{\mathrm{p\theta}[\lambda]^2}{1+\frac{a^2\mathrm{Cos}[\theta]^2}{r^2}}$$

Thank you