Moments of pion distribution amplitude using OPE on lattice

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Outline

- Pion distribution amplitude and its moments
- Moments using lattice OPE with a valance heavy quark [1]
- Lattice correlators
- Exploratory numerical results
- Summary

[1] proposed by Detmold and Lin (Phys.Rev. D73 (2006)).

Pion LC wave function/distribution amplitude:

$$\langle \pi^+(p)|\overline{d}(\frac{z}{2})\ \gamma_5\gamma_\mu\ u(-\frac{z}{2})|0\rangle = -ip_\mu f_\pi \int_0^1 d\xi\ e^{i(\overline{\xi}p\frac{z}{2}-\xi p\frac{z}{2})}\phi_\pi(\xi)$$
$$\overline{\xi} = 1-\xi$$

fraction ξ of pion momentum is carried by u quark.

(Mellin) Moments:

$$a_n = \int_0^1 d\xi\ \xi^n\ \phi_\pi(\xi).$$

OPE:

$$\langle \pi^+(p)|O^{\mu_1..\mu_n}|0\rangle = f_\pi\ a_{n-1}\ [p^{\mu_1}\dots p^{\mu_n} - \text{Traces}]$$
$$O^{\mu_1..\mu_n} = \overline{\psi}\gamma^{\{\mu_1}\gamma^5(iD^{\mu_2})\dots(iD^{\mu_n\}})\psi - \text{Traces}$$

In the isospin limit $m_u = m_d$:

$$\phi_\pi(\xi) = \phi_\pi(\overline{\xi})$$

$\Longrightarrow$ Odd moments vanish $\rightarrow$ lowest non-trivial moment is a_2.

Lattice calculation:

- second moment is calculated quite precisely (Braun et. al., 2015).
- going beyond is challenging because:
  - operator mixing (reduced symmetry: O(4) $\rightarrow$ H(4)).

A different approach is needed.

Euclidean OPE with a valance heavy quark

We need to calculate the matrix elements:

$$\int d^4x \; e^{iqx} \; \langle \pi^+(p) | \, T[A^{\mu}_{\mathbf{\Psi},\psi}(x) A^{\nu}_{\mathbf{\Psi},\psi}(0)] | 0 \rangle$$

Heavy-light currents:

$$A^{\mu}_{\mathbf{\Psi},\psi} = \overline{\mathbf{\Psi}} \gamma^{\mu} \gamma^5 \psi + \overline{\psi} \gamma^{\mu} \gamma^5 \mathbf{\Psi}$$

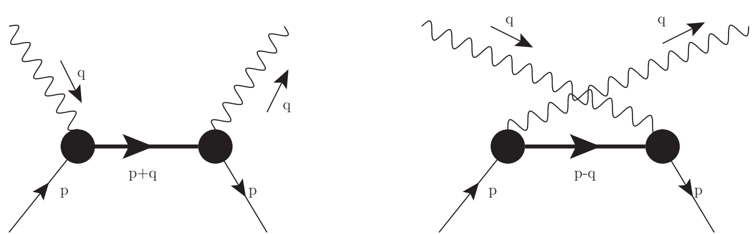
ψ: light quarks, $\mathbf{\Psi}$: fictitious, relativistic, valance quark which is heavy.

- Simplify the lattice calculation:
  - no disconnected contribution.
  - easy to invert the quark matrix.
- No effect of the heavy quark on DA.

Scale hierarchy required:

$$\Lambda_{QCD} << m_{\mathbf{\Psi}} << \frac{1}{a}$$

$\Longrightarrow$ Fine lattices are required.
Also, need to take the continuum limit.



VV type operator:

$$\int d^4x\ e^{iqx}\ T[A^{\mu}_{\mathbf{\Psi},\psi}(x)A^{\nu}_{\mathbf{\Psi},\psi}(0)] = \sum_k C_k(q)\mathcal{O}_k(0) \ \text{ for large q}$$

Euclidean OPE:

$$\int d^4x\ e^{iqx}\ T[A^{\mu}_{\mathbf{\Psi},\psi}(x)A^{\nu}_{\mathbf{\Psi},\psi}(0)] = \overline{\psi}\gamma^{\mu}\frac{-i(i\not{D}+\not{q})+m_{\Psi}}{(iD+q)^2+m_{\Psi}^2}\gamma^{\nu}\psi + \overline{\psi}\gamma^{\nu}\frac{-i(i\not{D}-\not{q})+m_{\Psi}}{(iD-q)^2+m_{\Psi}^2}\gamma^{\mu}\psi$$

Taylor expansion:

$$\frac{-i(i\not{D}+\not{q})+m_{\Psi}}{(iD-q)^2+m_{\Psi}^2} = -\frac{-i(i\not{D}+\not{q})+m_{\Psi}}{Q^2+D^2-m_{\Psi}^2}\sum_{n=0}^{\infty}\left(\frac{-2iq.D}{Q^2+D^2-m_{\Psi}^2}\right)^n,\ Q^2=-q^2$$

Higher twist terms $\rightarrow$ expansion parameter: $\left(\frac{-2iq.D+D^2}{Q^2-m_\Psi^2}\right)$ $\rightarrow$ Extra powers of p^2 $\rightarrow$ small.

Antisymmetric in μ and ν:

$$\int d^4x\ e^{iqx}\ T[A^\mu_{\Psi,\psi}(x)A^\nu_{\Psi,\psi}(0)] = \frac{i}{2}\sum_{n=0,even}^{\infty}\frac{1}{(\tilde{Q}^2)^{n+1}}\overline{\psi}\gamma_\lambda\gamma_5(i\not{D}_{\mu_1})\dots(i\not{D}_{\mu_n})\psi\varepsilon_{\mu\nu\rho\lambda}$$

$$\tilde{Q}^2 = Q^2 - m_\psi^2$$

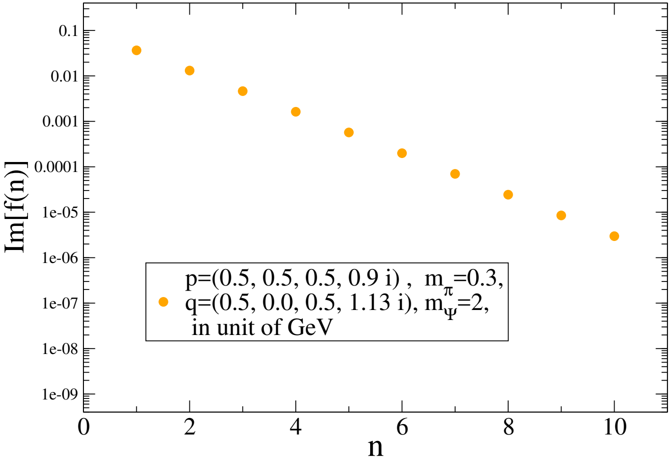
$$U_A^{[\mu\nu]}(p,q) = \int d^4x\ e^{iqx}\ \langle\pi^+(p)|T[A^\mu_{\Psi,\psi}(x)A^\nu_{\Psi,\psi}(0)]|0\rangle = \sum_{n=0,\ even}^{\infty} a_n\ f(n)\ f_\pi$$

$$f(n) = \frac{i}{2}\frac{\xi^{n+1}}{n+1}\left[\frac{2\eta C_n^2(\eta)(q^\rho p^\lambda)}{p.q}\right]\varepsilon_{\mu\nu\rho\lambda}, \xi = \frac{\sqrt{p^2q^2}}{\tilde{Q}^2},\ \eta = \frac{p.q}{\sqrt{p^2q^2}}$$

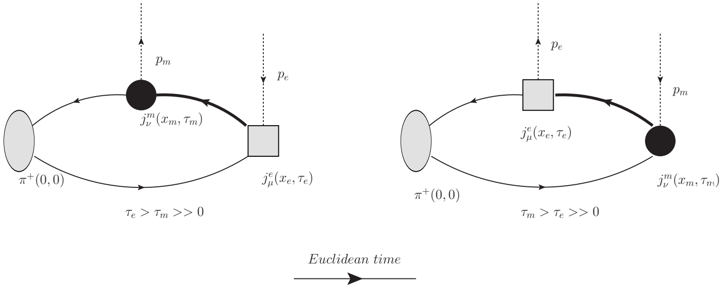
$$\uparrow$$

$$f(n) \sim \left(\frac{1}{\tilde{Q}^2}\right)^{(n+1)}$$

- Analytic continuation is in the Wilson coefficients.
- For simplicity, Wilson coefficients are set to one.
- Identical result for the VV type correlator.


Im[f(n)]
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
0
2
4
6
8
10
n
p=(0.5, 0.5, 0.5, 0.9 i) , m_π=0.3,
q=(0.5, 0.0, 0.5, 1.13 i), m_Ψ=2,
in unit of GeV

Lattice correlators: the three point-function



$$C_3^{\mu\nu}(\tau_m,\tau_e;\vec{p}_m,\vec{p}_e) = \int d^3x_m \int d^3x_e$$
$$e^{i\vec{p}_e\cdot\vec{x}_e} \mathrm{e}^{-i\vec{p}_m\cdot\vec{x}_m} \langle 0|\mathrm{T}[A_e^\mu(\vec{x}_e,\tau_e)A_m^\nu(\vec{x}_m,\tau_m)\mathscr{O}_\pi^\dagger(\vec{0},0)]|0\rangle$$

Two different choices:

- shift the operator $\mathrm{T}[A_e^\mu(\vec{x}_e,\tau_e)A_m^\nu(\vec{x}_m,\tau_m)]$ by $(\vec{x}_m,\tau_m)$.
- shift the operator $\mathrm{T}[A_e^\mu(\vec{x}_e,\tau_e)A_m^\nu(\vec{x}_m,\tau_m)]$ by $(\vec{x}_e,\tau_e)$.

$$
\begin{aligned}
C_3^{\mu\nu}(\tau_m,\tau_e;\vec{p}_m,\vec{p}_e) \quad = \quad & \frac{1}{2V_3E_\pi}\delta_{\vec{n},\vec{n}_e-\vec{n}_m} \times e^{-E_\pi\tau_m}\langle\pi(\vec{p})|\mathcal{O}_\pi^\dagger(\vec{0},0)|0\rangle \\
& \times \int d^3x\ e^{i\vec{p}_e\cdot\vec{x}}\ \langle 0|\mathrm{T}[A_e^\mu(\vec{x},\tau)A_m^\nu(\vec{0},0)]|\pi(\vec{p})\rangle \\
& \vec{x}=\vec{x}_e-\vec{x}_m, \tau=\tau_e-\tau_m:
\end{aligned}
$$

The two point function:

$$
\begin{aligned}
C_\pi(\tau_\pi;\vec{p}_\pi) \quad = \quad & \int d^3x\ e^{i\vec{p}_\pi\cdot\vec{x}}\left\langle 0\middle|\mathcal{O}_\pi(\vec{x},\tau)\mathcal{O}_\pi^\dagger(\vec{0},0)\middle|0\right\rangle \\
& \xrightarrow{\tau_\pi\to\infty} \frac{\left|\left\langle\pi(\vec{p}_\pi)\middle|\mathcal{O}_\pi^\dagger(\vec{0},0)\middle|0\right\rangle\right|^2}{2V_3E_\pi}\times \mathrm{e}^{-E_\pi\tau_\pi}.
\end{aligned}
$$

We can form the ratio

$$
\begin{aligned}
R_{3,m}^{\mu\nu}(\tau;\vec{q},\vec{p}) &= \frac{C_3^{\mu\nu}(\tau_m,\tau_e;\vec{p}_e,\vec{p}_m)}{C_\pi(\tau_m;\vec{p})} \times \left\langle \pi(\vec{p}_\pi) \middle| \mathscr{O}_\pi^\dagger(\vec{0},0) \middle| 0 \right\rangle \\
&= \int \mathrm{d}^3x\ \mathrm{e}^{i\vec{q}\cdot\vec{x}} \left\langle 0 \middle| T[A_e^\mu(\vec{x},\tau)\,A_m^\nu(\vec{0},0)] \middle| \pi(\vec{p}_\pi) \right\rangle_{\tau=\tau_e-\tau_m;\vec{q}=\vec{p}_e;\vec{p}=\vec{p}_e-\vec{p}_m}
\end{aligned}
$$

The second choice:

$$
\begin{aligned}
R_{3,e}^{\mu\nu}(\tau;\vec{q},\vec{p}) &= \frac{C_3^{\mu\nu}(\tau_m,\tau_e;\vec{p}_e,\vec{p}_m)}{C_\pi(\tau_e;\vec{p})} \times \left\langle \pi(\vec{p}_\pi) \middle| \mathscr{O}_\pi^\dagger(\vec{0},0) \middle| 0 \right\rangle \\
&= \int \mathrm{d}^3x\ \mathrm{e}^{i\vec{q}\cdot\vec{x}} \left\langle 0 \middle| T[A_e^\mu(\vec{x},\tau)\,A_m^\nu(\vec{0},0)] \middle| \pi(\vec{p}_\pi) \right\rangle_{\tau=\tau_m-\tau_e;\vec{q}=-\vec{p}_m;\vec{p}=\vec{p}_e-\vec{p}_m}
\end{aligned}
$$

Perform the temporal Fourier transform:

$$
\int \mathrm{d}\tau\ \mathrm{e}^{iq_4\tau}\ R_{3,m/e}^{\mu\nu}(\tau;\vec{q},\vec{p}) = U_A^{\mu\nu}(q,p)
$$

exploratory numerical results

Quenched calculation:

- Hugely reduces the computational cost.
- Neglecting the effect of the fermion loops.

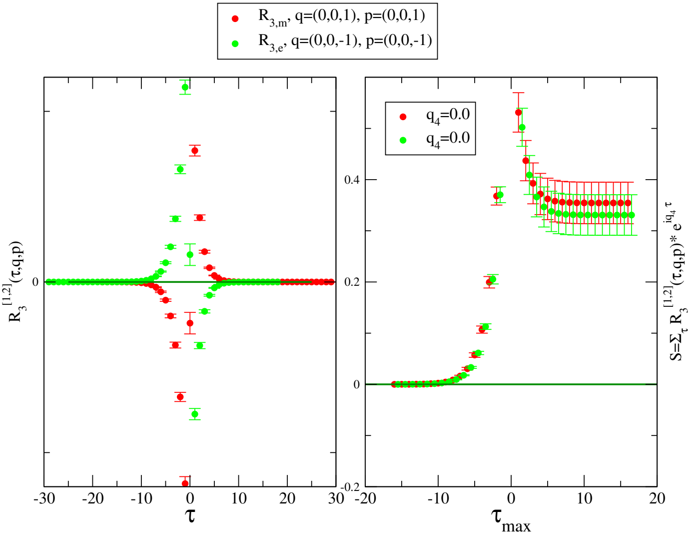
Nonperturbative Clover action for valance quarks. $\rightarrow$ Wilson quarks with less cut-off artifacts ($\mathcal{O}(a^2)$).

Modarate light quark mass with $m_\pi = 450$ MeV.

fine lattice: $a^{-1} = 4$ GeV (lattices upto $a^{-1} = 8$ GeV are available).

Moderate physical box size: $L = 2.4$ fm, $T = 4.8$ fm.

$m_\Psi \sim 2$ GeV, sample size$= 332 \times 4$, $U_A^{\mu\nu}(q,p) \sim \varepsilon_{\mu\nu\rho\sigma} q_\rho p_\sigma$, $\mu = 1, \nu = 2$



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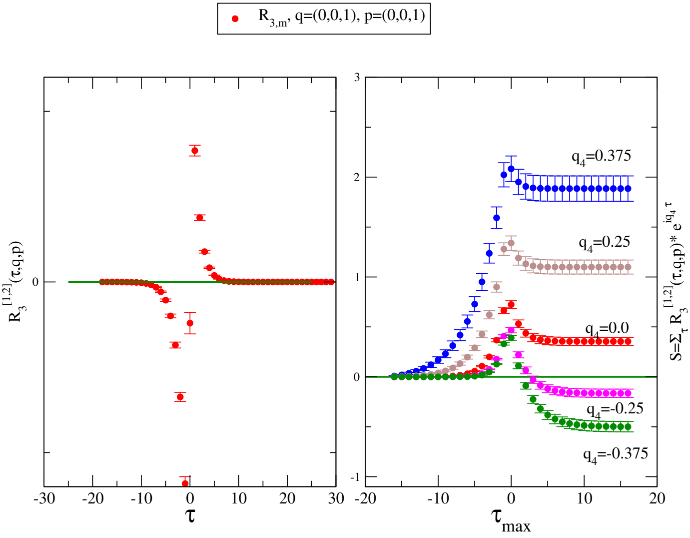

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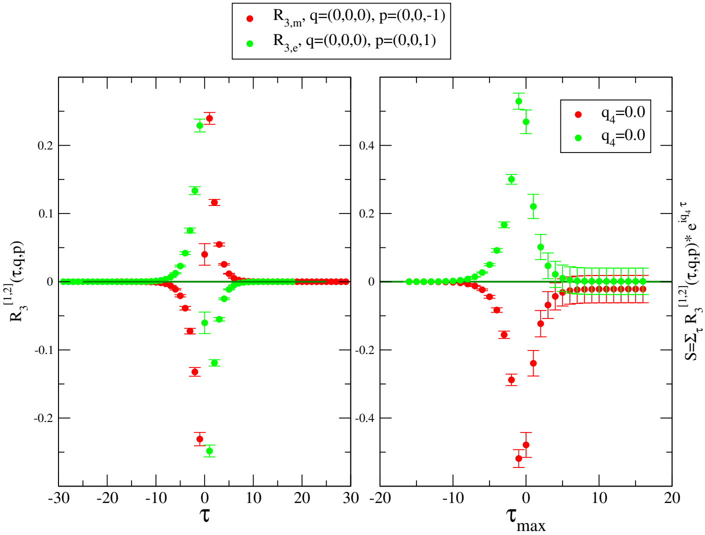

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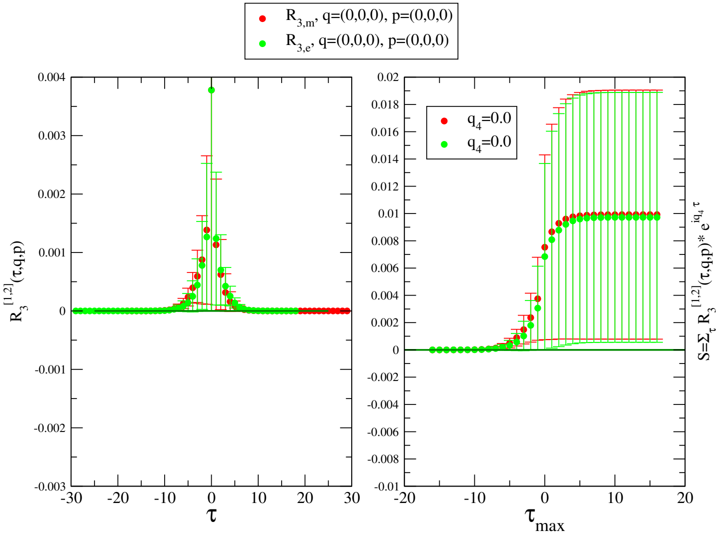


parton distribution function

OPE of the forward Compton scattering tensor in Euclidean space:

$$T^{\mu\nu}_{\Psi,\psi}(p,q) = \sum_S \int d^4x \; e^{iq.x} \; \langle p,S | \, T[V^{\mu}_{\Psi,\psi}(x) V^{\nu}_{\Psi,\psi}(0)] | p,S \rangle$$

In target rest frame:

$$T^{\{\mu\nu\}}_{\Psi,\psi}(p,q) = \sum_{n=2,4,\ldots} A^n \; f(n)$$

A^n s are moments.

f(n) is completely known:

$$f(n) \;\; = \;\; -\sqrt{q_0^2 - Q^2}\,\xi^n \{ \frac{2}{q_0} [C_n \frac{\tilde{Q}^2}{Q^2} \frac{(n-1)\eta\, C^{(2)}_{n-1}(\eta) - 4\eta^2 C^{(3)}_{n-2}(\eta)}{n(n-1)} + \ldots$$

$\uparrow$

kinamatic variables and Wilson coefficients (known perturbatively)

Summary

- I have described a new method to calculate the moments of pion distribution amplitude by studying the Euclidean OPE on lattice. The method is free from operator mixing problem of the traditional approach.
- A valance heavy quark is used to make lattice calculation simpler and to give more flexibility.
- I have shown our preliminary numerical data and demonstrate our strategy to analyze those. The data is qualitatively consistent with OPE.
- The method described here can also be used to study parton distribution function of the pion and nucleon.

Thanks for your attention!

Lattice Gauge Theory

In QCD at low energy coupling is too strong to investigate with perturbative techniques.

First principle calculation based on path integral formalism.

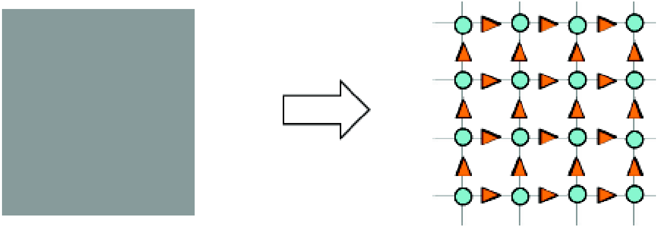
Path integral in Euclidean space:

$$\langle \mathscr{O}[\bar{\psi}, \psi, A_\mu] \rangle = \frac{1}{Z} \int \mathscr{D}A_\mu \mathscr{D}\bar{\psi}\mathscr{D}\psi \;\; \mathscr{O}[\bar{\psi}, \psi, A_\mu] \;\; e^{-S[\psi, \bar{\psi}, A_\mu]}$$

Functional integral over contineous space-time of infinite extent $\Longrightarrow$ infinite number of dof's.

Discretize 4-dim. space-time for finite degree of freedom $\Rightarrow$ 4-dim. lattice:

Ultraviolet regulator with cutoff scale $\sim \frac{1}{a}$, a is the lattice spacing.



$$\psi(x) \longrightarrow \psi(n)$$
$$A_\mu(x) \longrightarrow U_\mu(n), \text{ SU(3) group valued link variable}$$

Path integral on 4-dim. lattice:

$$\langle \mathcal{O}[\bar{\psi},\psi,U_\mu]\rangle = \frac{1}{Z}\int \prod_{n,\mu} dU_\mu(n)\bar{\psi}(n)\psi(n) \;\; \mathcal{O}[\bar{\psi},\psi,U_\mu] \;\; e^{-\int d^4x \; \mathcal{L}_{lattice}[\psi,\bar{\psi},U]}$$

Quark fields are Grassmann (anticommuting) numbers $\Rightarrow$ Integrated analytically

$$\langle \bar{\mathcal{O}}[U_\mu]\rangle = \frac{1}{Z}\int \prod_{n,\mu} dU_\mu(n) \;\; \bar{\mathcal{O}}[U_\mu] \;\; e^{-S^{eff}_{lattice}[U]}$$

Average over the configurations after importance sampling gives expectation value:

$$\langle \bar{\mathcal{O}}[U_\mu]\rangle \quad = \quad \frac{1}{N}\sum_{i=1}^{N} \bar{\mathcal{O}}[U^i_\mu] + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

$\Downarrow$

statistical error

Hadron mass using lattice gauge therory

In continuum notation:

$$\langle \mathcal{O}(x,t)\mathcal{O}^\dagger(0)\rangle = \frac{1}{Z}\int [d\psi][d\bar{\psi}][dU]\mathcal{O}(x,t)\mathcal{O}^\dagger(0)e^{-S[\psi,\bar{\psi},A_\mu]}$$

Euclidean correlation function:

$$\begin{aligned}
C(t) &= \int d^3x\ \langle \mathcal{O}(x,t)\mathcal{O}^\dagger(0)\rangle \\
&= \int d^3x\ \langle e^{Ht}e^{i\hat{P}x}\mathcal{O}(0)e^{-Ht}e^{-i\hat{P}x}\mathcal{O}^\dagger(0)\rangle \\
&= \int d^3x \int d^3p \sum_n \langle e^{Ht}e^{i\hat{P}x}\mathcal{O}(0)e^{-Ht}e^{-i\hat{P}x}|E_n,p\rangle\langle E_n,p|\mathcal{O}^\dagger(0)\rangle \\
&= \int d^3x \int d^3p \sum_n \langle \mathcal{O}(0)|E_n,p\rangle\langle E_n,p|\mathcal{O}^\dagger(0)\rangle e^{-E_n t}e^{-ip.x} \\
&= \sum_n |\langle 0|\mathcal{O}^\dagger(0)|E_n,0\rangle|^2 e^{-M_n t}
\end{aligned}$$

M_n can be extracted by fitting lattice data for $C(t)$.