QCD sum rules from viewpoint of inverse problem

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QCD sum rules

Shifman, Vainshtein and Zakharov, Nucl. Phys. B 147, 385 (1979); B 147, 448 (1979).

○ QCD sum rules are useful in extracting hadronic parameters, *e.g.*, mass and decay constant.

○ It combines the operator product expansion (OPE) and the dispersion relation.

✔ Quark-hadron duality is assumed.

-duality violation is hard to quantify

-sometimes the numerical stability is not guaranteed

Contents

○ Conventional QCD sum rules

- operator product expansion (OPE)
- quark-hadron duality

○ Dispersion relation as an inverse problem

- mass, decay constant and width of ground stale
- excited states
- comparison with conventional QCD sum rules

ρ^0 mesons

- ○ Spin-1 bound state that consists of $(\bar{u}u - \bar{d}d)/\sqrt{2}$

(PDG, MeV unit)	mass	width
$\rho(770)$	775.26 ± 0.25	149.1 ± 0.8
$\rho(1450)$	1465 ± 25	400 ± 60
$\rho(1700)$	1720 ± 20	250 ± 100

spectral function

$$\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\boxed{\Pi(q^2)}$$

calculablable in OPE

$$J_\mu = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/\sqrt{2}.$$

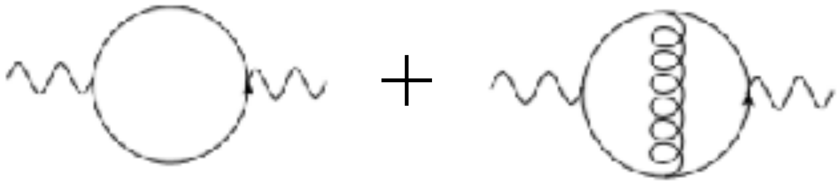
for $q^2 \ll 0$

optical theorem: $\sigma(e^+e^- \to \underset{(I=1)}{\text{hadrons}})(s) = \frac{16\pi^2\alpha^2}{s}\mathrm{Im}\Pi(s)$

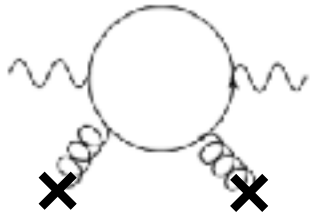
OPE for spectral function

$$\Pi(q^2) = \sum_i \underset{\text{short-distance}}{\overset{\text{Wilson coefficient}}{C_i(q^2)}} \; \underset{\text{long-distance}}{\overset{\text{matrix element}}{\langle 0|\mathcal{O}_i|0\rangle}}$$

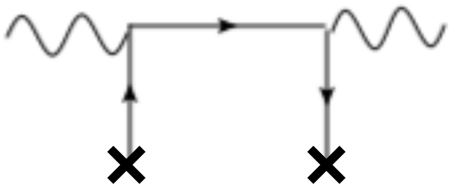
$$\Pi(q^2) = \frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right)\ln\left(\frac{\mu^2}{-q^2}\right)\langle 0|\mathbf{1}|0\rangle \Big\} \text{ pQCD:}$$



$$+\frac{1}{12\pi\boxed{(q^2)^2}}\langle 0|\alpha_s G^2|0\rangle \Big\} \text{ gluon condensate:}$$



$$+\frac{2}{\boxed{(q^2)^2}}\langle 0|m_q\bar{q}q|0\rangle \Big\} \; q\bar{q} \text{ condensate:}$$



$$+\frac{224\pi}{81}\frac{\alpha_s}{\boxed{(q^2)^3}}\kappa\langle 0|\bar{q}q|0\rangle^2 \Big\} \text{ four-quark condensate}$$

$$\langle 0|\left(\bar{q}q\right)^2|0\rangle \to \kappa\langle 0|\left(\bar{q}q\right)|0\rangle^2$$

(κ is factorization violation)

- ○ Power series of $1/q^2$
- ○ Large $|q^2|$ ➡ Higher power corrections are suppressed.

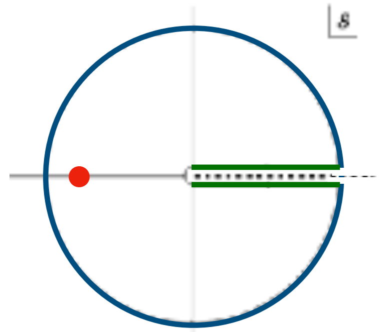
input parameters at $\mu = 1$ GeV (condensate fixed by data) S. Narison, *Nucl.Part.Phys.Proc.* 258-259 (2015) 189-194.

$\alpha_s = 0.5$ $\Lambda_{\mathrm{QCD}} = 0.353$ GeV $\langle m_q\bar{q}q\rangle = 0.007 \times (-0.246)^3$ GeV^4 $\langle\alpha_s GG\rangle = 0.07$ GeV^4 $\alpha_s\langle\bar{q}q\rangle^2 = 1.49\times10^{-4}$ GeV^6

conventional QCD sum rules

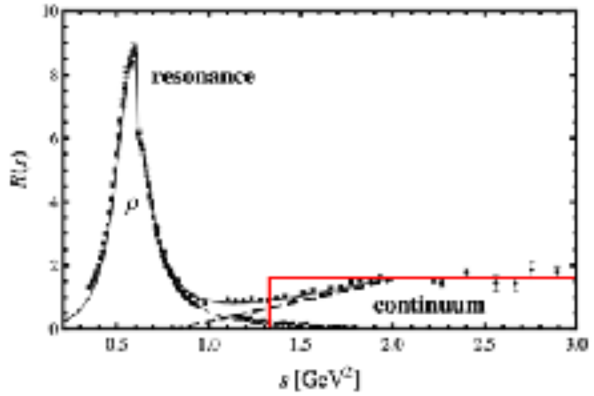
(1) OPE: $\Pi(q^2) = \frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right)\ln\left(\frac{\mu^2}{-q^2}\right)\langle 0|\mathbf{1}|0\rangle + \frac{1}{12\pi(q^2)^2}\langle 0|\alpha_s G^2|0\rangle + \frac{2}{(q^2)^2}\langle 0|m_q\bar{q}q|0\rangle + \frac{224\pi}{81}\frac{\alpha_s}{(q^2)^3}\kappa\langle 0|\bar{q}q|0\rangle^2$ ···(★) $q^2 < 0$ (spacelike)

(2) Dispersion relation:



$$\Pi(q^2) = \frac{1}{\pi}\int_{s_i}^{\infty} ds \frac{\mathrm{Im}\Pi(s)}{s-q^2} + \text{(external circular contour)} \quad \cdots(\blacklozenge)$$

Shape of $\mathrm{Im}\Pi(s)$:



Y. Kwon, M. Procura and W. Weise, Phys. Rev. C**78**, 055203 (2008).

resonance continuum

Parametrization: $\mathrm{Im}\Pi(s) = \pi f_V^2 \delta(s-m_V^2) + \pi\rho^h(s)\theta(s-s_0)$

quark-hadron duality for continuum

$\pi\rho^h(s) = \mathrm{Im}[\Pi(-s)]|_{-s\to s}$ ⬅ analytic continuation (spacelike →timelike)

Comparison between (★) and (◆) leads to,

$$\frac{f_V^2}{m_V^2-q^2} = \frac{1}{4\pi^2}(1+\alpha_s)\ln\left(\frac{q^2-s_0}{q^2}\right) + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

Alternatively, the Borel transformed version is,

$$f_V^2 e^{-m_V^2/M^2} = \frac{1}{\pi}\int_{s_i}^{s_0} ds \mathrm{Im}\Pi^{\mathrm{pert}}(s) e^{-s/M^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{M^2} + 2\frac{\langle m_q\bar{q}q\rangle}{M^2} - \frac{112\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{M^4}.$$

- ○ The relations are regarded as constraints among f_V, m_V, etc.
- ○ It is hard to quantify the uncertainty in quark-hadron duality.

$$\hat{B}_M \equiv \lim_{\substack{Q^2, n\to\infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!}(Q^2)^n\left(-\frac{d}{dQ^2}\right)^n$$

$$Q^2 \equiv -q^2$$

QCD sum rules treated as an inverse problem

$D^0 - \bar{D}^0$ mixing [2001.04079]

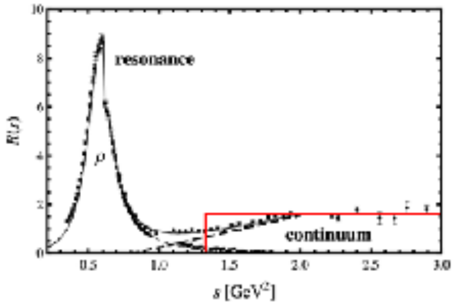
muon g-2 [2004.06451]

Inverse problem: $$\frac{1}{\pi}\int_0^{\Lambda}\frac{\overset{\text{unknown}}{\boxed{\mathrm{Im}\Pi(s)}}}{s-q^2}ds = \omega(q^2)$$ (Λ : cut-off above which the duality is valid)

Calculable input (r.h.s.): $$\omega(q^2) \;=\; \frac{1}{4\pi^2}(1+\alpha_s/\pi)\ln\frac{q^2-\Lambda}{q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

integral equation: often has multiple solutions.

Parametrization of solution: $$\mathrm{Im}\Pi(q^2) = \pi f_V^2\delta(q^2 - m_V^2) + \underbrace{\pi\rho^h(q^2)}_{\text{no step function}}$$



○ The continuum part is expanded by shifted Legendre polynomials:

$$\rho^h(q^2) = b_0\tilde{P}_0(q^2/\Lambda) + b_1\tilde{P}_1(q^2/\Lambda) + b_2\tilde{P}_2(q^2/\Lambda) + b_3\tilde{P}_3(q^2/\Lambda)$$

Strategy

① $$\frac{1}{\pi}\int_0^{\Lambda}\frac{\overset{\text{substitute}}{\boxed{\mathrm{Im}\Pi(s)}}}{s-q^2}ds = \overset{\text{theoretical input}}{\boxed{\omega(q^2)}}$$

② Fit unknown constants so as to reproduce the theoretical input.

unknowns: $f_V, m_V, \Lambda, b_0, b_1, b_2, b_3$

Ground state

How to handle multiple solutions:

(1) Fix (Λ, m_V). (2) Fit the other parameters, f_V, b_0, b_1 via the least square method.

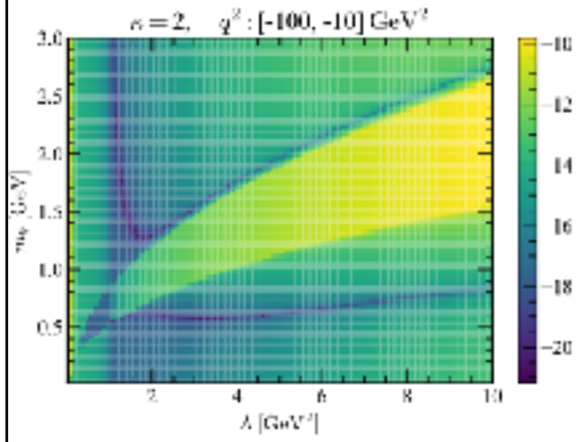
Residual sum of square (RSS): $\sum_{i=1}^{20}\left(\frac{1}{\pi}\int_0^{\Lambda}\frac{\mathrm{Im}\Pi(s)}{s-q_i^2}ds - \omega(q_i^2)\right)^2$

$q^2 : [-100, -10]\ \mathrm{GeV}^2$

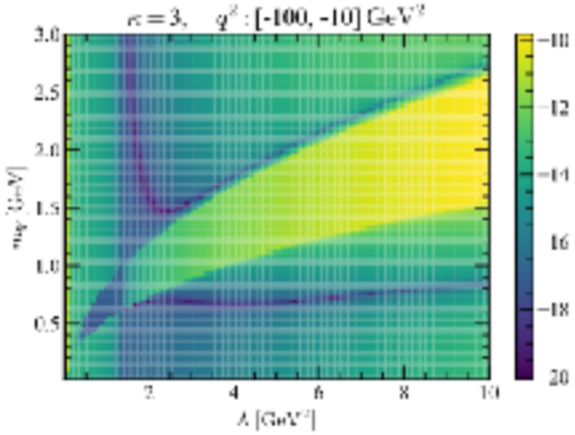
$\Lambda - m_V$ planes

The color represents $\log_{10}(\mathrm{RSS})$.

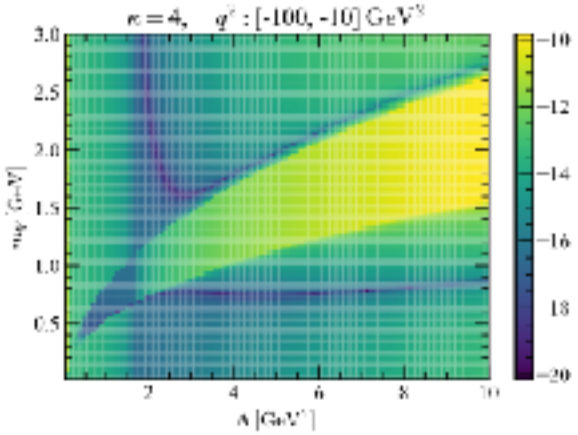
Blue → better fitting



$\kappa = 2$



$\kappa = 3$

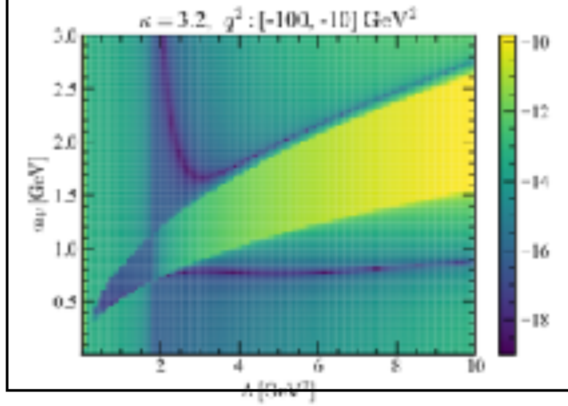


$\kappa = 4$

Two minima structures

lower one: $m_V \sim 0.8$ GeV

upper one: excited state



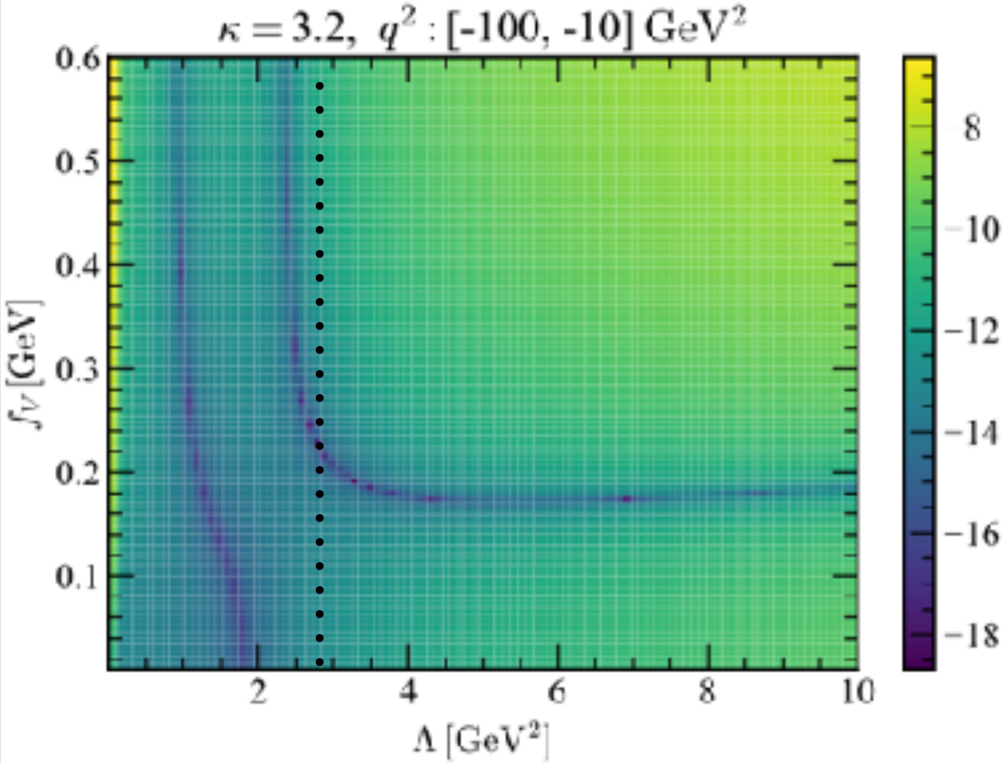
$\kappa = 3.2$

- ○ $m_\rho = 0.78$ GeV is reproduced for $2\ \mathrm{GeV}^2 < \Lambda < 6\ \mathrm{GeV}^2$.
- ○ $\kappa = 3.2$ is consistent with the phenomenological result, 3.0 ± 0.2.

Narison Phys.Lett.B 673 (2009) 30-36.

Ground state (cont’d)

$\Lambda - f_V$ plane $\kappa = 3.2,\ m_V = 0.78$ GeV fixed to reduce the excited state’s influence



○ A global minimum of RSS at $\Lambda = 2.8$ GeV^2 leads to $f_V = 0.22$ GeV.

Decay constant in previous works

$f_{\rho^\pm} = 208.5 \pm 5.5 \pm 0.9$ MeV Lattice QCD W. Sun, Chin.Phys.*C* 42 (2018) 6, 063102.

$f_\rho = 223$ MeV Bethe-Salpeter Equation, Z. G. Wang, S. L Wan, Phys.Rev.C 76 (2007) 025207.

$f_\rho = 246,\ 215$ MeV Light-front quark model H. M. Choi, C. R. Ji, Phys.Rev.D 75 (2007) 034019.

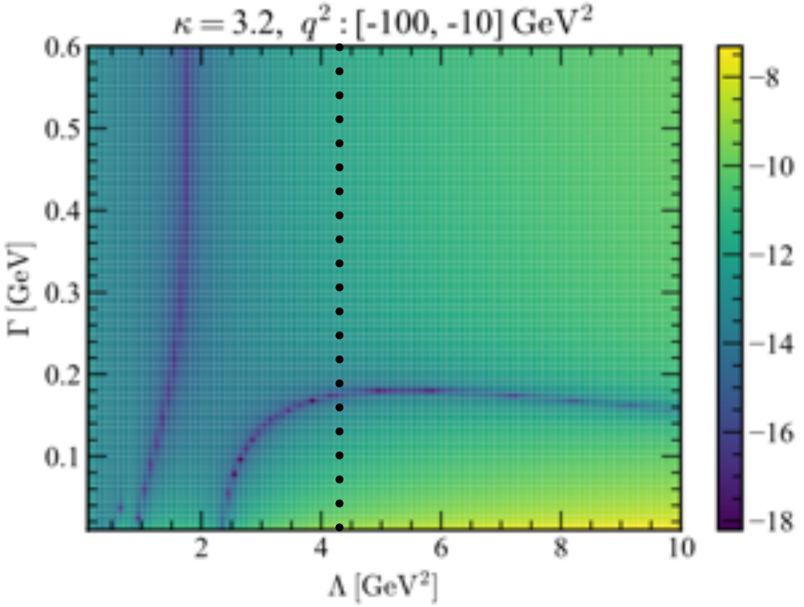
linear, harmonic osc.

Width of $\rho(770)$

Breit-Wigner-like term + polynomials

$$\mathrm{Im}\Pi(q^2) = \frac{1}{24\pi}\frac{m_V^4 + m_V^2\Gamma^2}{(q^2 - m_V^2)^2 + m_V^2\Gamma^2} + \pi\rho^h(q^2),$$

$\Lambda - \Gamma$ plane :



$m_V = 0.78\ \mathrm{GeV}$ (fixed)

Global minimum at $(\Lambda, \Gamma) = (4.3\ \mathrm{GeV}^2, \boxed{0.17\ \mathrm{GeV}})$

PDG: $\Gamma = (0.1491 \pm 0.0008)\ \mathrm{GeV}$

○ Difficulty in reproducing width is also reported in the Baysian-based analysis.

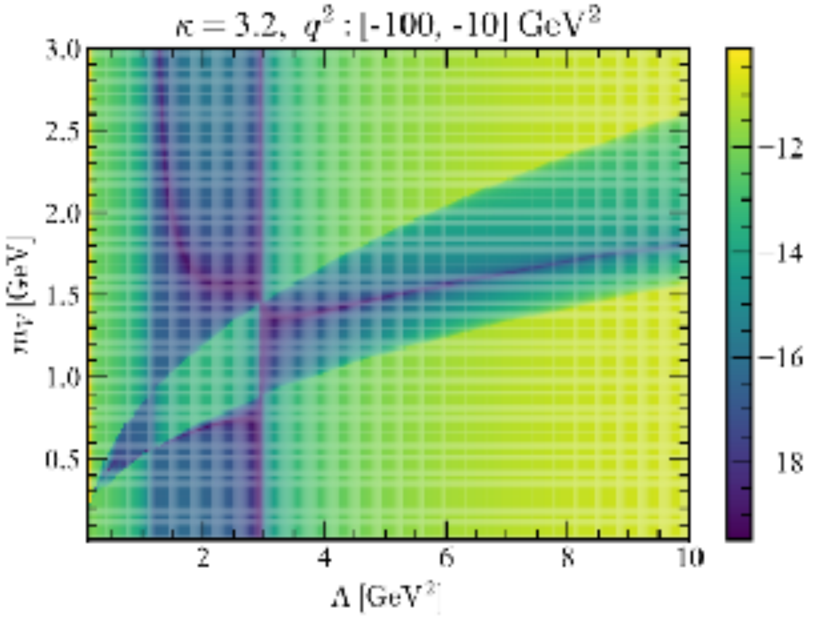
P. Gubler and M. Oka, Prog.Theor.Phys. 124 (2010) 995-1018.

Radially excited state

$$\boxed{\mathrm{Im}\Pi(q^2) = \underbrace{\pi f^2_{\rho(770)}\delta(q^2 - m^2_{\rho(770)})}_{\text{ground state}} + \underbrace{\pi f^2_V\delta(q^2 - m^2_V)}_{\text{1st excited state}} + \pi\rho^h(q^2)}$$

$\Lambda - m_V$ plane

ground state is set to $m_{\rho(770)} = 0.78$ GeV and $f_{\rho(770)} = 0.22$ GeV



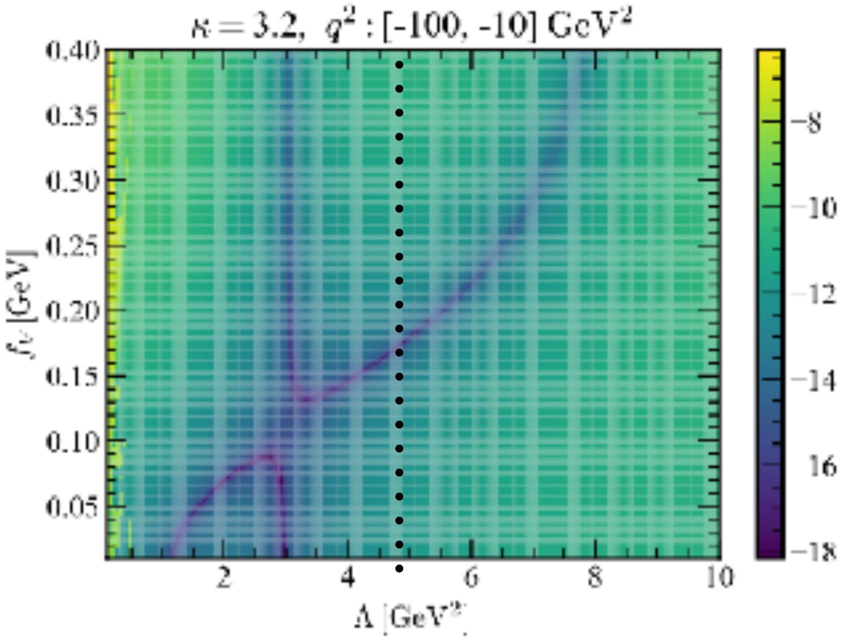
○ A global minimum exists for $\Lambda = 3 - 5$ GeV^2 , indicating an excited state near $m_V = 1.46$ GeV.

○ There may exist $\rho(1570)$ with a similar mass, which has been speculated as OZI suppresed decay of $\rho(1700)$.

→ A more precise OPE input may help clarify this issue.

$\Lambda - f_V$ plane

$m_V = 1.46$GeV.



global minimum: $(\Lambda, f_V) = (4.8\ \text{GeV}^2, 0.19\ \text{GeV})$

Decay constants in the literature

$f_{\rho(2S)} = (184 \pm 14)$MeV Double pole sum rule M.S. Maior de Sousa, R.Rodrigues da Silva Braz.J.Phys. 46 (2016) 6, 730-739.

$f_{\rho(2S)} = (185 \pm 78)$MeV Lattice QCD T. Yamazaki, et al, Phys. Rev. D**65** 014501 (2002).

$f_{\rho(2S)} = 155$MeV Rainbow-ladder truncation in Dyson-Scwinger Eq., S.x Qin, Phys. Rev. C 85, 035202 (2012).

Further excited states

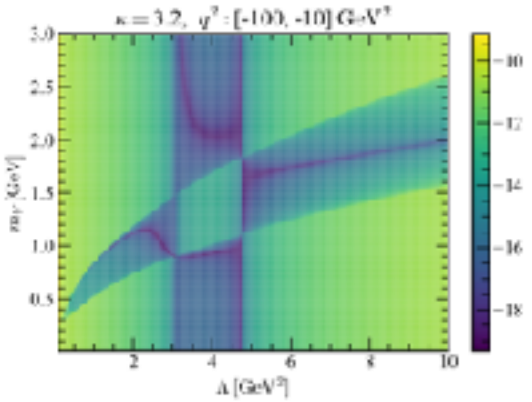
○ Iteratively the excited state's mass and decay constant are investigated.

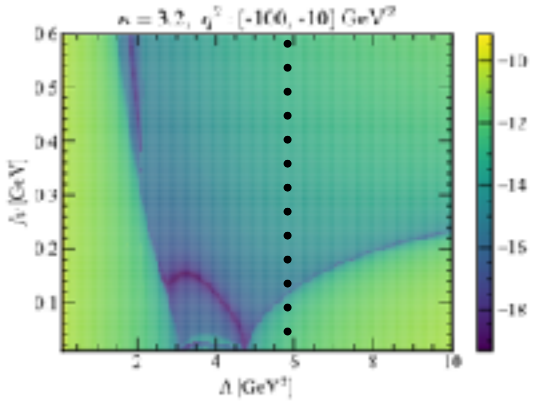
$$\mathrm{Im}\Pi(s) = \pi f^2_{\rho(770)}\delta(s - m^2_{\rho(770)}) + \pi f^2_{\rho(1450)}\delta(s - m^2_{\rho(1450)}) + \pi f_V^2\delta(s - m_V^2) + \pi\rho(s)$$

additional pole

$\Lambda - m_V$ plane **$\Lambda - f_V$ plane**

Triple pole

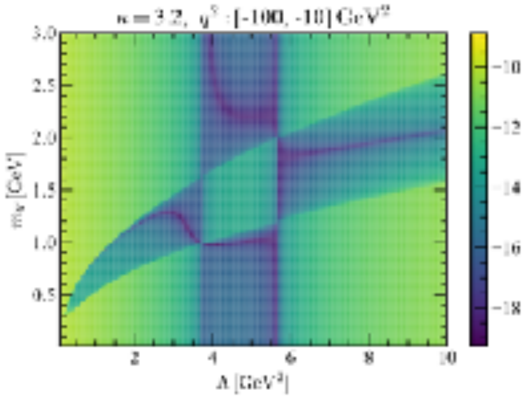


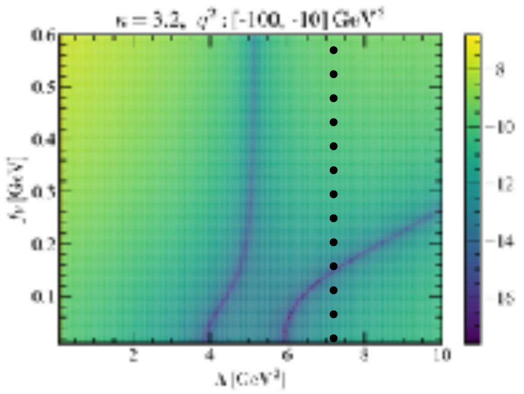


○ A global minimum on $\Lambda - m_V$ plane exists at $\Lambda \sim 5\mathrm{GeV}^2$.
gives $m_V = 1.7\mathrm{GeV}$.

○ A global minimum on $\Lambda - f_V$ plane at $\Lambda = 5.8\mathrm{GeV}^2$
gives $f_V = 0.14\mathrm{GeV}$.

Quadrapole

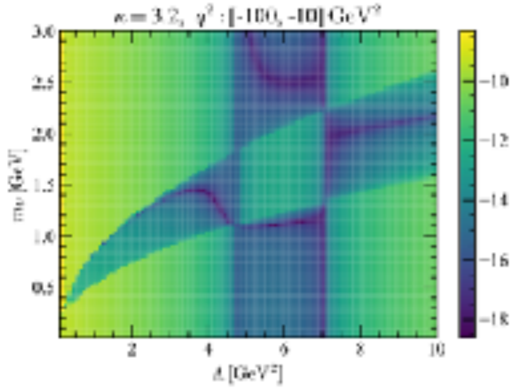




○ A global minimum on $\Lambda - m_V$ plane exists at $\Lambda \sim 6 - 7\mathrm{GeV}^2$.
gives $m_V = 1.9\mathrm{GeV}$.

○ A global minimum on $\Lambda - f_V$ plane at $\Lambda = 7.1\mathrm{GeV}^2$
gives $f_V = 0.14\mathrm{GeV}$.

Quintuple



○ A global minimum on $\Lambda - m_V$ plane exists at $\Lambda \sim 6 - 7\mathrm{GeV}^2$,
supporting $m_V = 2\mathrm{GeV}$.

($\rho(2150)$ is well-established while $\rho(2000)$ needs confimation experimentally.)

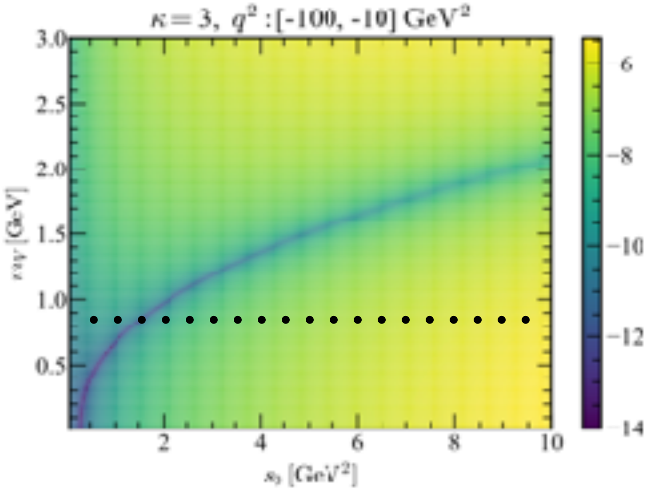
Conventional QCD sum rules

Parametrization

$$\mathrm{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \pi\rho^h(q^2)\theta(q^2 - s_0)$$

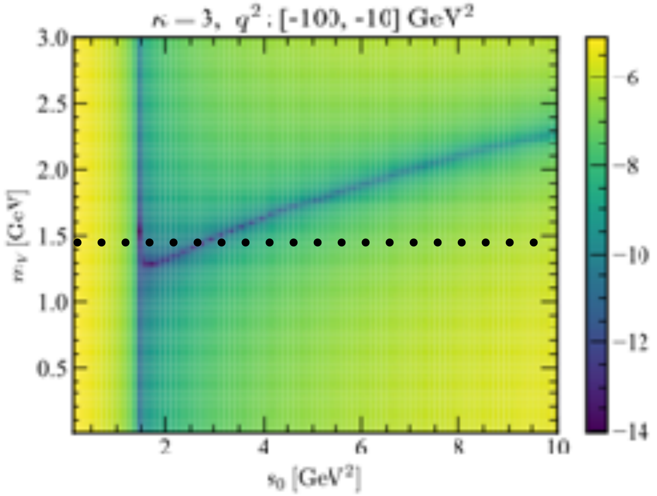
duality: $\pi\rho^h(s) = \mathrm{Im}[\Pi(-s)]|_{-s\to s}$ s_0 : free parameter.

$s_0 - m_V$ planes



single pole

- ○ Around $m_V \sim 0.78$GeV, there are no global minimum or plateau.
- ○ The above mass leads to $f_V = 0.20$GeV.

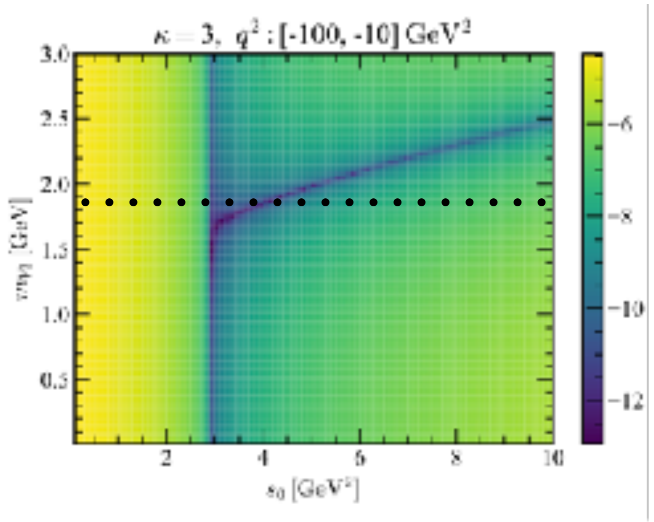


double pole

with $m_{\rho(770)} = 0.78$GeV, $f_{\rho(770)} = 0.20$GeV

- ○ Around $m_V \sim 1.46$ GeV, again there are no global minimum or plateau.
- ○ By fixing $m_V = 1.46$ GeV, one obtains $f_V = 0.21$GeV.

larger than that of ground state



triple pole

with $m_{\rho(1450)} = 1.46$GeV, $f_{\rho(1450)} = 0.21$GeV

- ○ Around $m_V \sim 1.72$ GeV, as similar to above, neither global minimum nor plateau exists.

➡ Sensible solutions do not exist.
The continuum part in the sum rules could be oversimplified.

Summary

- Regarding the QCD sum rule as an inverse problem, we extracted rho meson's decay constants, masses and width.

$$f_{\rho(770)}(f_{\rho(1450)}, f_{\rho(1700)}, f_{\rho(1900)}) \approx 0.22\ \ (0.19,\ 0.14,\ 0.14)\ \text{GeV}$$

$$m_{\rho(770)}(m_{\rho(1450)}, m_{\rho(1700)}, m_{\rho(1900)}) \approx 0.78\ (1.46, 1.70, 1.90)\ \text{GeV}$$

$$\Gamma_{\rho(770)} \approx 0.17\ \text{GeV}$$

- It is indicated that conventional duality assumption could be oversimplified since the excited state search does not lead to the global minimum or plateau.

Backup

Correlator

Definition of correlator:

$$\Pi_{\mu\nu}(q\ ) = i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu\underbrace{(x)J_\nu}_{\text{insertion of complete system}}(0)]|0\rangle = \boxed{(q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)}$$

ground state: $\langle 0|J_\mu|V^\lambda\rangle = f_V m_V \epsilon^\lambda_\mu.$

$$2\mathrm{Im}\Pi_{\mu\nu}(q^2) = \sum_n \overbrace{\langle 0|J_\mu|n\rangle}\langle n|J_\nu|0\rangle d\Phi_n(2\pi)^4\delta(q-p_n)$$

$$= \boxed{(q_\mu q_\nu - g_{\mu\nu}m_V^2)2\pi f_V^2\delta(q^2-m_V^2)} + \cdots \qquad \text{(narrow width approximation)}$$

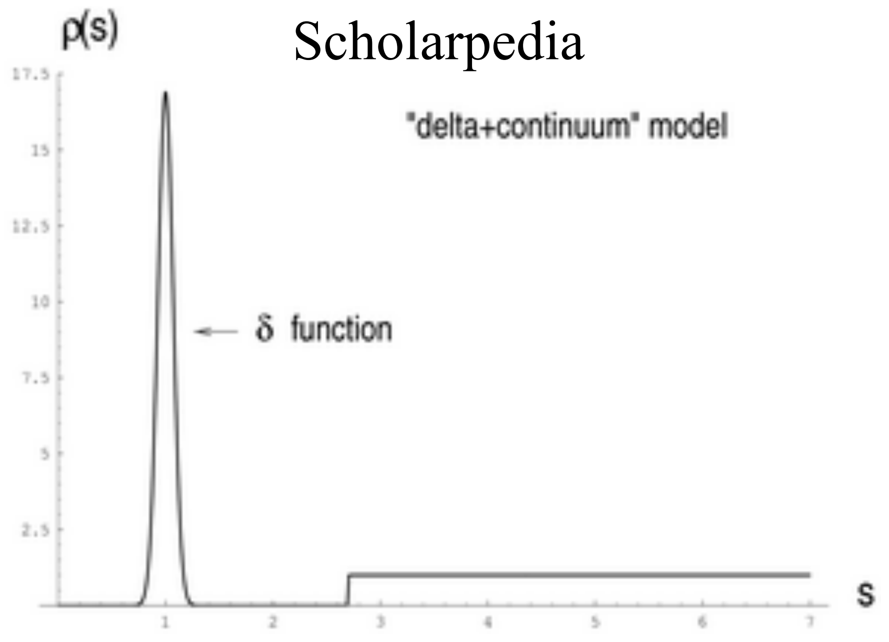
Thus, $\mathrm{Im}\Pi(q^2) = \pi f_V^2\delta(q^2-m_V^2) + \ \cdots$

spin sum: $\sum_\lambda \epsilon^\lambda_\mu \epsilon^{\lambda *}_\nu = \frac{1}{m_V^2}(q_\mu q_\nu - m_V^2 g_{\mu\nu})$

Conventional local duality

the case with duality:

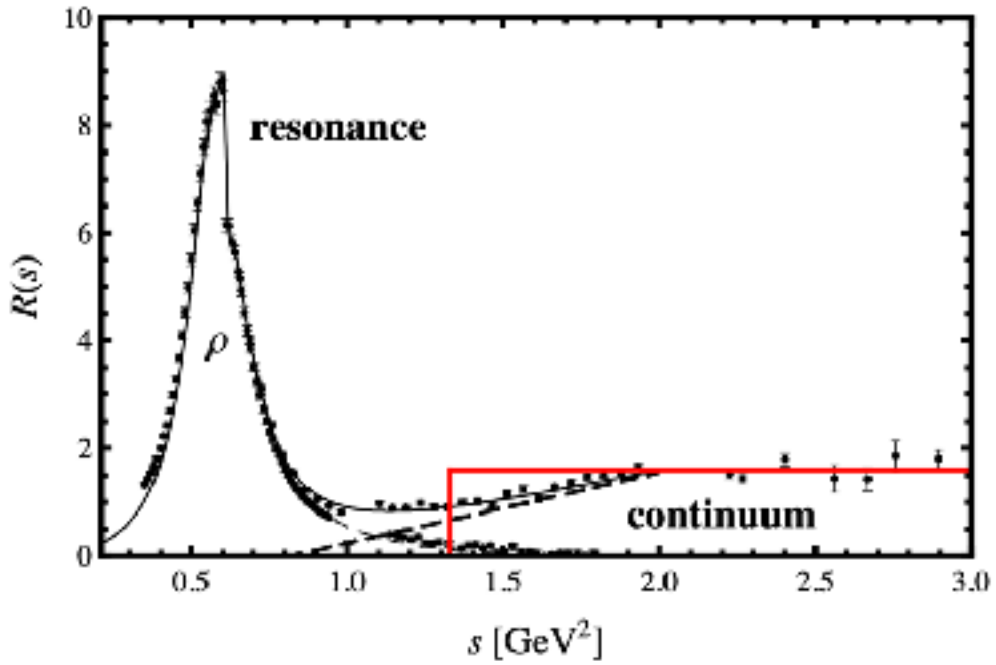
$$\mathrm{Im}\Pi(s) = \pi f_V^2 \delta(s - m_V^2) + \text{(const)}\times \theta(s - s_0)$$



Scholarpedia

(broadened for visualization)

more realistic case:



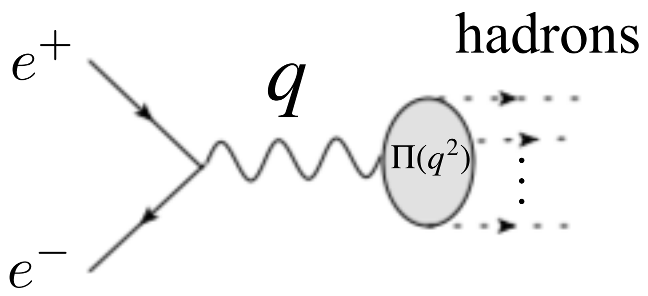
Example: Y. Kwon, M. Procura and W. Weise, Phys. Rev. C**78**, 055203 (2008).

(data are smooth, at least not constant)

OPE and duality assumption

$$\begin{cases} \text{theory (OPE):} & \Pi(q^2) = \sum_i C_i(q^2)\,\langle 0|\mathcal{O}_i|0\rangle \quad \text{valid for} \quad \overset{\text{negative}}{q^2 < 0} \\ \text{experiment:} & \sigma(e^+e^- \to \text{hadrons})(q^2) \propto \text{Im}\Pi(q^2) \quad \overset{\text{positive}}{q^2 > 0} \end{cases}$$

Center of mass frame $\begin{cases} p_+^\mu = (E_{\text{CM}}, \mathbf{p}) \\ p_-^\mu = (E_{\text{CM}}, -\mathbf{p}) \end{cases}$



$$q = p_+ + p_-$$

$$q^2 = (p_+ + p_-)^2 = 4E_{\text{CM}}^2 > 0$$

Trouble: we cannot use the OPE for $e^+e^- \to$ hadrons.

A trick for obtaining imaginary part

(1) Making a formula based on the OPE for $q^2 < 0$:

negative

$$\Pi(\underset{\text{negative}}{q^2}) = \frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right)\ln\left(\frac{\mu^2}{-q^2}\right)\langle 0|\mathbf{1}|0\rangle + \frac{1}{12\pi(q^2)^2}\langle 0|\alpha_s G^2|0\rangle + \frac{2}{(q^2)^2}\langle 0|m_q\bar{q}q|0\rangle + \frac{224\pi}{81}\frac{\alpha_s}{(q^2)^3}\kappa\langle 0|\bar{q}q|0\rangle^2$$

(2) Flipping the sign, $q^2 \to -q^2$ (assumption of local quark-hadron duality):

$$\Pi(\underset{\text{positive}}{-q^2}) = \frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right)\ln\left(\frac{\mu^2}{+q^2}\right)\langle 0|\mathbf{1}|0\rangle + \frac{1}{12\pi(q^2)^2}\langle 0|\alpha_s G^2|0\rangle + \frac{2}{(q^2)^2}\langle 0|m_q\bar{q}q|0\rangle - \frac{224\pi}{81}\frac{\alpha_s}{(q^2)^3}\kappa\langle 0|\bar{q}q|0\rangle^2$$

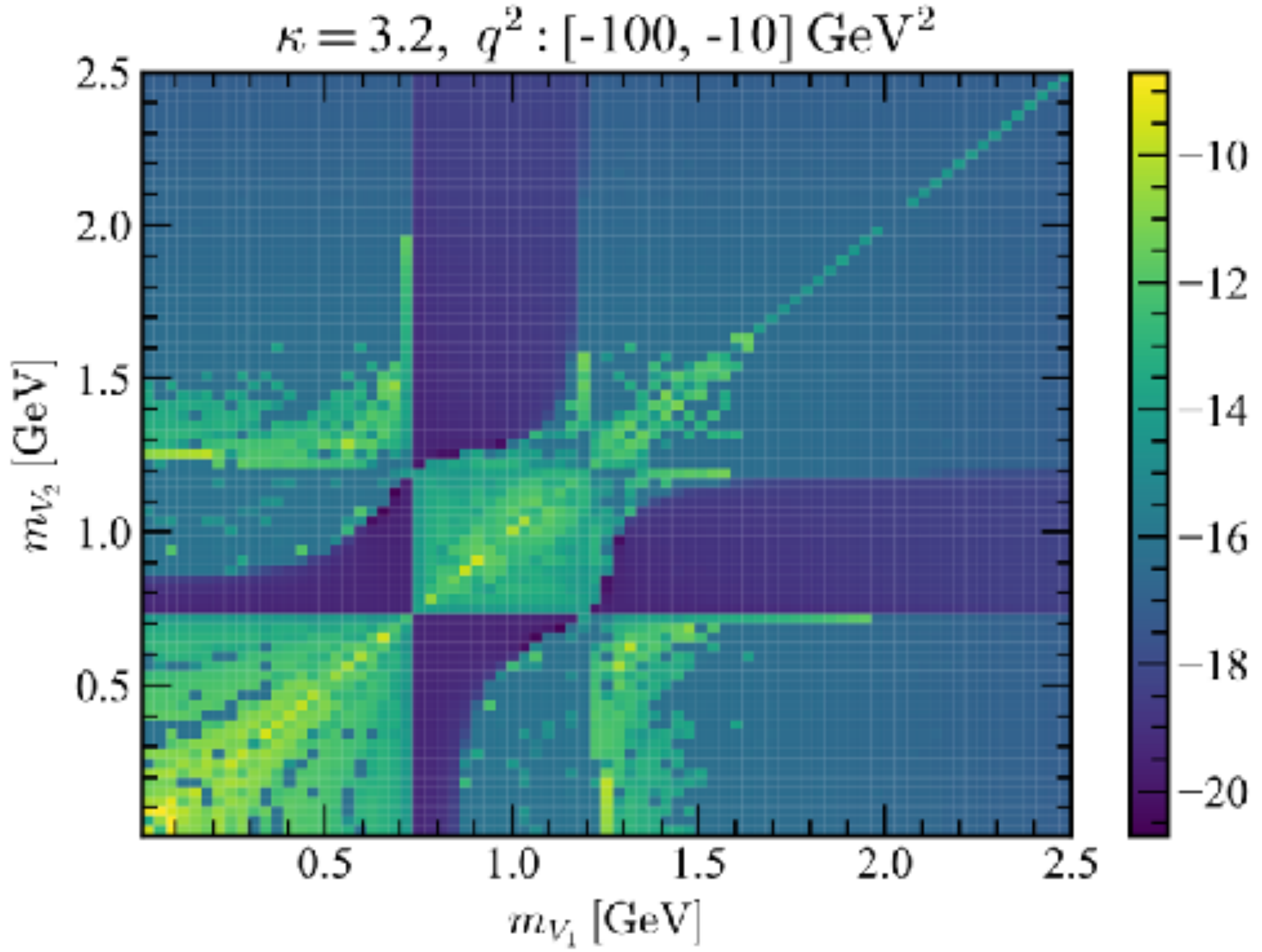
for large $|q^2|$

(3) Taking the imaginary part of ImΠ($-q^2$).

Double-pole parametrization with free m_{V_1} and m_{V_2}

$$\mathrm{Im}\Pi(q^2) = \pi f_{V_1}\delta(q^2 - m_{V_1}^2) + \pi f_{V_2}\delta(q^2 - m_{V_2}^2) + \pi\rho(q^2)$$

m_{V_1}-m_{V_2} plane



Coefficients in polynomial expansion

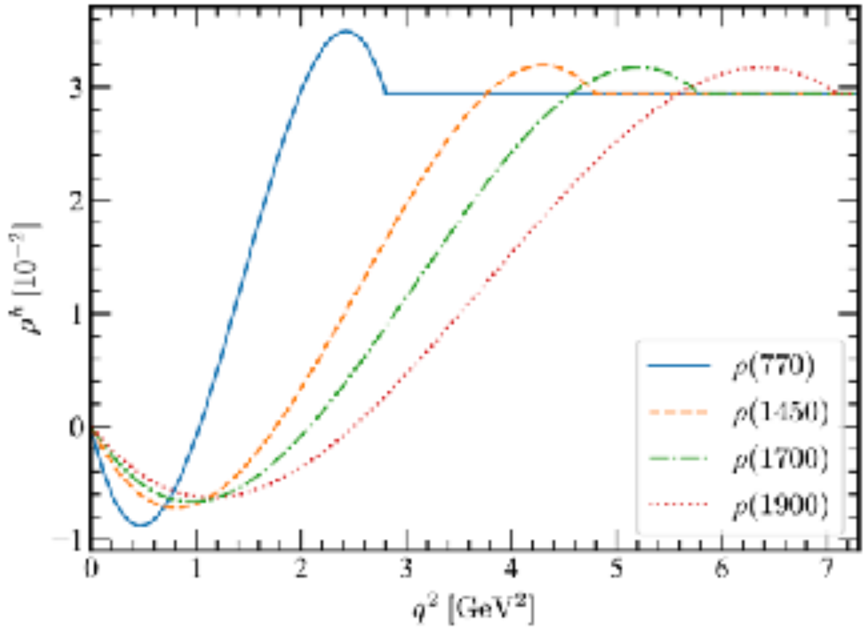
$$\rho^h(q^2) = b_0\tilde{P}_0(q^2/\Lambda) + b_1\tilde{P}_1(q^2/\Lambda) + b_2\tilde{P}_2(q^2/\Lambda) + b_3\tilde{P}_3(q^2/\Lambda)$$

up to $\rho(1S)$: $b_0 = 0.0126, \quad b_1 = 0.0276, \quad b_2 = 0.0022, \quad b_3 = -0.0128.$

up to $\rho(2S)$: $b_0 = 0.0104, \quad b_1 = 0.0248, \quad b_2 = 0.0033, \quad b_3 = -0.0101,$

up to $\rho(3S)$: $b_0 = 0.0106, \quad b_1 = 0.0244, \quad b_2 = 0.0031, \quad b_3 = -0.0099,$

up to $\rho(4S)$: $b_0 = 0.0118, \quad b_1 = 0.0242, \quad b_2 = 0.0028, \quad b_3 = -0.0095,$



For further improvement, the boundary conditions for slopes need to be considered.