Hadronic vacuum polarisation from lattice QCD

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Anomalous magnetic moment

- Fermions have magnetic moments

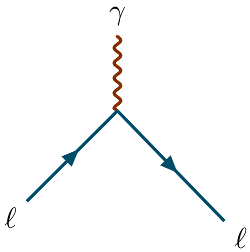
$$\vec{\mu} = g\frac{Q}{2m}\vec{S}$$

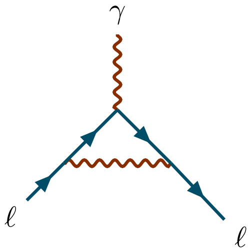
- Free, spin-½ particles have $g = 2$
- Quantum corrections

Electron	2.002 319 304 361 8(5)
Muon	2.002 331 841 8(12)

- Anomalous magnetic moment

$$a = (g - 2)/2$$





Muon anomalous magnetic moment

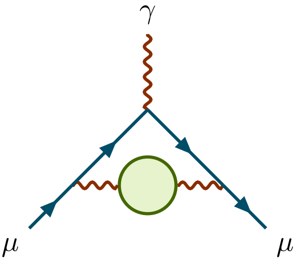
- Experimentally measured at Brookhaven National Laboratory
- Theoretical prediction from QED and phenomenology

Experiment	11 659 208.9(63) $\times 10^{-10}$
Theory	11 659 181.0(43) $\times 10^{-10}$

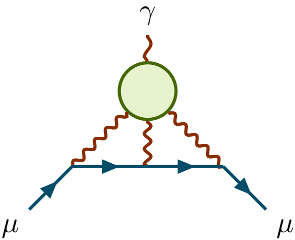
- 3.7σ discrepancy between experiment and theory
- Could this discrepancy be a signal of new physics?
- Upgraded experiment underway at Fermilab
  - Expect 4$\times$ reduction in errors
- New experiment using different methodology at J-PARC

Muon anomalous magnetic moment

$a_\mu^{QED} \times 10^{10}$	11 658 471.8931	$\pm$	0.0104
$a_\mu^{EW} \times 10^{10}$	15.36	$\pm$	0.10
$a_\mu^{HVP} \times 10^{10}$	684.5	$\pm$	4.0
$a_\mu^{HLbL} \times 10^{10}$	9.2	$\pm$	1.8
$a_\mu \times 10^{10}$	11 659 208.9	$\pm$	6.3



- Errors dominated by QCD corrections
  - Hadronic vacuum polarisation
  - Hadronic light-by-light scattering



Hadronic Vacuum Polarisation

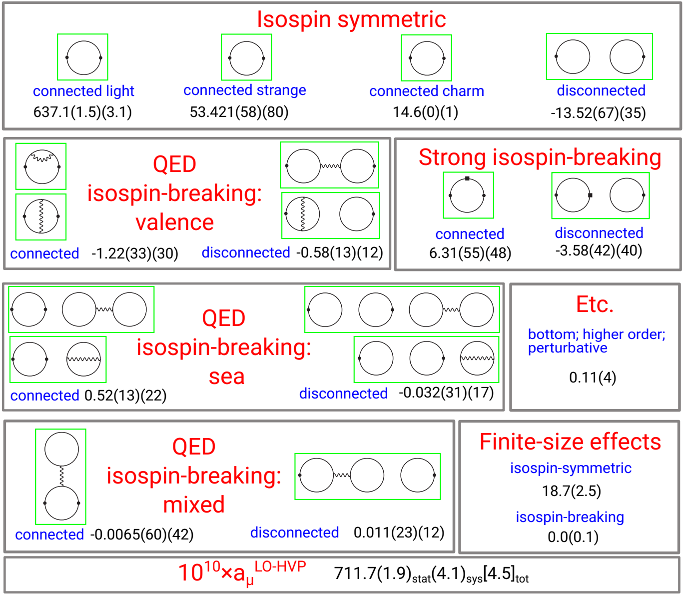
Lattice calculation of HVP contribution

$$G(t) = \frac{1}{3} \sum_{\mu=1,2,3} \int \langle J_\mu(\vec{x}, t) J_\mu(0) \rangle \, d^3x$$

$$a_\mu^{HVP,LO} = \alpha^2 \int_0^\infty K(t) G(t) dt$$

- Aim to reach sub-percent precision
- Unprecedented level of precision for a lattice study
- Need to address a lot of important contributions
  - QED corrections and strong isospin breaking
  - Very precise determination of lattice spacing
  - Effects of finite volume and finite time extent
  - Discretisation errors

Anatomy of precision lattice QCD



QED and strong isospin breaking

- Perform computations in isospin-symmetric limit
- Compute first-order derivative with respect to up and down quark mass
- Compute derivative with respect to sea and valence quark charge to second order
- Final results are Taylor expansion around isospin-symmetric, zero-charge point

Scale setting

- $K(t)$ depends on the square of the muon mass m_μ
- To compute a_μ on the lattice, require m_μ in lattice units
- Relative error in lattice spacing doubled in a_μ
- Use $M_\Omega = 1672.45(29)\,\mathrm{MeV}$ to set lattice spacing
- Include QED and strong isospin breaking

Chiral perturbation theory

- Systematic expansion based on the symmetries of QCD
- Formulated in terms of hadronic degrees of freedom
- Expansion in powers of m_q and p^2

LO Lagrangian

$$\frac{F^2}{4}\,\mathrm{Tr}\Big(D_\mu U D_\mu U^\dagger\Big) - 2B\frac{F^2}{4}\,\mathrm{Tr}\Big(U\mathcal{M} + \mathcal{M}U^\dagger\Big) + \frac{m_0^2}{12}(U+D)^2$$

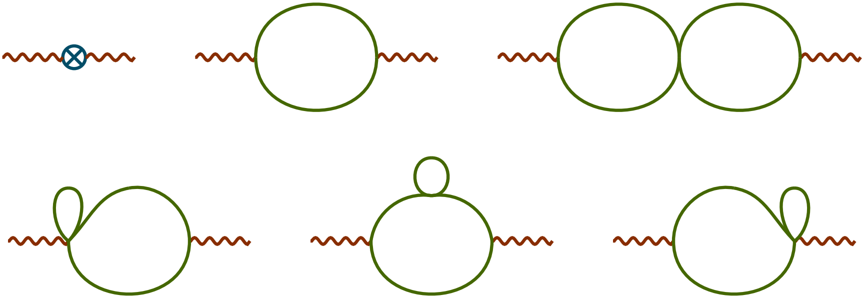
- In $SU(2)$: four mesons in a 2×2 matrix

$$U = \exp\left(\frac{i}{F}\begin{bmatrix} U & \pi^+ \\ \pi^- & D \end{bmatrix}\right)$$

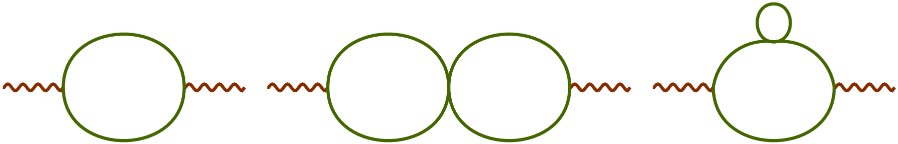
- The $U_A(1)$ anomaly contributes to the singlet mass

Chiral perturbation theory

χPT	LO	NLO	NNLO
Continuum	B	$L_1, \ldots, L_{10}, H_1$	$C_1, \ldots, C_{90}$



Finite size effects



- Integrate over loop momenta

$$\frac{1}{3}\int\frac{d^3p}{(2\pi)^3}\frac{e^{-2E_pt}}{E_p^2}p^2\left[1+\frac{16}{F^2}L_9E_p^2-\frac{2}{F^2}\int\frac{d^3r}{(2\pi)^3}\frac{1}{2E_r}\right.$$
$$\left.+\frac{1}{3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{E_r}\frac{r^2}{E_r^2-E_p^2}\right]$$

- Finite volume/time effects:

$$\int\frac{d^3p}{(2\pi)^3}\to\frac{1}{L^3}\sum_p,\qquad\int\frac{dp_4}{2\pi}\to\frac{1}{T}\sum_{p_4}$$

Finite size effects

- Compare to alternative model: Meyer-Lellouch-Lüscher-Gounaris-Sakurai model
- At reference volume $L_{ref} = 6.272\,\text{fm}$, $T_{ref} = 9.408\,\text{fm}$:
- Dedicated lattice study with $L_{big} = 10.752\,\text{fm}$, $T_{big} = 10.752\,\text{fm}$

	NLO	NNLO	LLGS	Lattice
$a_\mu(L_{big}, T_{big}) - a_\mu(L_{ref}, T_{ref})$	11.6	15.7	17.8	18.1(24)
$a_\mu(\infty, \infty) - a_\mu(L_{big}, T_{big})$	0.3	0.6	—	—

- Take lattice value plus small correction from NNLO χPT

$$a_\mu(\infty, \infty) - a_\mu(L_{ref}, T_{ref}) = 18.1(24) + 0.6(3) = 18.7(24)$$

Discretisation errors

- Short-distance discretisation errors dominated by $O(a^2)$ effects
- Long-distance effects arise due to "taste breaking"
- Staggered action gives rise to 4 "fermion doublers"
- Mesons consequently have 16 tastes
- Each element of meson matrix $\to$ 4×4 taste matrix

Taste basis

$$\gamma_\alpha \in \{ \quad \gamma_5, \quad \gamma_{\mu 5} = i\gamma_\mu\gamma_5, \quad \gamma_{\mu\nu} = i\gamma_\mu\gamma_\nu, \quad \gamma_\mu, \quad \gamma_I = 1 \quad \}$$

Staggered chiral perturbation theory

χPT	LO	NLO
Continuum	B	$L_1, \ldots, L_{10}, H_1$
Staggered	$\mathcal{C}_1, \mathcal{C}_{2V}, \mathcal{C}_{2A}, \ldots, \mathcal{C}_6$	*(222 LECs)*

Staggered chiral perturbation theory

TABLE XVI. The eight a^2 source operators with one $D_\mu \tilde{F}$ and one Lie derivative. Notation as in Table XV.

Operator	Keep?
119. $\mathrm{Str}(D_\mu \tilde{F}_L \Sigma^\dagger \tilde{F}_L D_\mu \Sigma^\dagger)$ + p.c.	Yes
120. $\mathrm{Str}(D_\mu \tilde{F}_L D_\mu \Sigma^\dagger \tilde{F}_L \Sigma^\dagger)$ + p.c.	Sometimes
121. $\mathrm{Str}(D_\mu \tilde{F}_L \Sigma^\dagger)\mathrm{Str}(\tilde{F}_L D_\mu \Sigma^\dagger)$ + p.c.	Yes
122. $\mathrm{Str}(D_\mu \tilde{F}_L D_\mu \Sigma^\dagger)\mathrm{Str}(\tilde{F}_L \Sigma^\dagger)$ + p.c.	Sometimes
123. $\mathrm{Str}(D_\mu \tilde{F}_L \tilde{F}_R \Sigma D_\mu \Sigma^\dagger)$ + p.c.	Yes
124. $\mathrm{Str}(D_\mu \tilde{F}_L D_\mu \Sigma^\dagger \Sigma \tilde{F}_R)$ + p.c.	Sometimes
125. $\mathrm{Str}(D_\mu \tilde{F}_L \Sigma^\dagger)\mathrm{Str}(\Sigma \tilde{F}_R \Sigma D_\mu \Sigma^\dagger)$ + p.c.	Yes
126. $\mathrm{Str}(D_\mu \tilde{F}_L D_\mu \Sigma^\dagger)\mathrm{Str}(\Sigma \tilde{F}_R)$ + p.c.	Sometimes

mentals with one adjoint. There are four such singlets, and eight such operators, given in Table XVI.

We can now eliminate operators using integration by parts and EoM. Operator pairs 119 & 120, 121 & 122, 123 & 124, and 124 & 125 are each related in this way, and we choose to keep only the first operator in each pair. Thus we are left with four additional operators.

As in the previous sections, four-fermion operators with spin T do not lead to any additional mesonic operators, and simply modify some of the unknown coefficients of the spin S and P operators.

Inserting the appropriate taste matrices is now straightforward. We simplify expressions using the fact that $\partial_\mu F = 0$. We also write operators in terms of ℓ_μ and r_μ because the covariant derivative acts differently on the two types of taste spurions. The final "source term" operators are listed in Table XXIV.

b. Single insertion of FF(B) operators

In Appendix A 3 we discussed in detail how to map onto $\mathcal{O}(a^2 p^2)$ mesonic operators arising from a single insertion of a four-fermion operator in $S_6^{FF(B)}$, so we just summarize the result here: the *chiral* structures are the same as those of single $FF(A)$ insertions, but the *index* structures are different, breaking $SO(4)$-taste and Lorentz symmetry. Simply put, each $FF(B)$ operator is identical to the corresponding $FF(A)$ operator, but with the derivative indices contracted with a pair of taste indices. However, there is one important difference: because the indices on the covariant derivative and the taste matrices are *correlated*, the number of operators can *no longer be reduced* using the EoM. Thus there are twice as many a^2 source operators corresponding to a single insertion of $[V_\mu \times T_\mu]$ or $[A_\mu \times T_\mu]$ as those from a single insertion of $[V, A \times T]$, and the same is true of $[T_\mu \times V_\mu, A_\mu]$ versus $[T \times V, A]$.[15] Table XXV lists all of the resulting $FF(B)$ "source term" operators.

Note that $[V_\mu \times T_\mu]$ only generates single supertrace operators, so there are no "source term" hairpins with tensor taste. This has interesting consequences for $SO(4)$-taste symmetry breaking in PGB decay constants.

APPENDIX B: $\mathcal{O}(a^2p^2)$, $\mathcal{O}(a^2m)$, AND $\mathcal{O}(a^4)$ OPERATORS IN THE S$\chi\mathcal{L}$

TABLE XVII. Operators in the staggered chiral Lagrangian arising from a single insertion of an $S_6^{FF(A)}$ operator with spin V or A. Implicit sums over repeated indices follow the summation convention defined in (1), and p.c. indicates parity conjugate. Each chiral operator has an independent undetermined coefficient of $\mathcal{O}(a^2)$. Partial derivatives act by convention only on the object immediately to their right. To include left- and right-handed sources, derivatives should be replaced with covariant derivatives. M and $M^\dagger$ serve as (pseudo)scalar sources; if such sources are absent then $M = M^\dagger = \mathcal{M}$.

Generic operator	Specific $\mathcal{O}(a^2p^2)$ operator $[V, A \times P]$	$[V, A \times T]$
1 + 3	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma\Sigma^\dagger\xi_5\Sigma\xi_5)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma\Sigma^\dagger\xi_{\nu\rho}\Sigma\xi_{\rho\nu})$ + p.c.
2	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_5\partial_\mu\Sigma\xi_5)$	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\nu\rho}\partial_\mu\Sigma\xi_{\rho\nu})$
6	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)$	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\nu\rho}\Sigma\xi_{\rho\nu}\Sigma^\dagger)$
7	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_5)\mathrm{Str}(\Sigma^\dagger\partial_\mu\Sigma\xi_5)$	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu\rho})\mathrm{Str}(\Sigma^\dagger\partial_\mu\Sigma\xi_{\rho\nu})$
10	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_5\Sigma\partial_\mu\Sigma^\dagger\xi_5)$ + p.c.	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu\rho}\Sigma\partial_\mu\Sigma^\dagger\xi_{\rho\nu})$ + p.c.
13	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_5)\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_5)$ + p.c.	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu\rho})\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\rho\nu})$ + p.c.

Generic operator	Specific $\mathcal{O}(a^2m)$ operator $[V, A \times P]$	$[V, A \times T]$
15 + 22	$\mathrm{Str}(\xi_5\Sigma\xi_5M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}M^\dagger)$ + p.c.
16 + 21	$\mathrm{Str}(\xi_5\Sigma M^\dagger\Sigma\xi_5\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu\nu}\Sigma M^\dagger\Sigma\xi_{\nu\mu}\Sigma^\dagger)$ + p.c.
17 + 23	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.

[15] Recall that $[T \times V]$ has fewer operators than $[S \times V]$ since there are no $L - R$ cross terms. Note also that the operators resulting from generic forms 115 and 116 are the same up to a sign, so the doubling of operators does not apply in this case.

TABLE XVIII. Operators in the staggered chiral Lagrangian arising from a single insertion of an $S_6^{FF(A)}$ operator with spin S or P. Notation as in Table XVII.

Generic operator	Specific $\mathcal{O}(a^2p^2)$ operator $[S, P \times V]$	$[S, P \times A]$
36	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger\partial_\mu\Sigma\xi_\nu)$	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger\partial_\mu\Sigma\xi_{5\nu})$
38 + 39	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma\Sigma^\dagger\xi_\nu)\mathrm{Str}(\Sigma\xi_\nu)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma\Sigma^\dagger\xi_{\nu5})\mathrm{Str}(\Sigma\xi_{5\nu})$ + p.c.
41	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_\nu)\mathrm{Str}(\partial_\mu\Sigma\xi_\nu)$	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\nu5})\mathrm{Str}(\partial_\mu\Sigma\xi_{5\nu})$
42	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu)$	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu})$
43	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger\xi_\nu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger)$ + p.c.
44	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_\nu\partial_\mu\Sigma^\dagger\xi_\nu)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\nu5}\partial_\mu\Sigma^\dagger\xi_{5\nu})$ + p.c.
45	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_\nu\Sigma^\dagger\xi_\nu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger)$ + p.c.
46	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger)$ + p.c.
47	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_\nu)^2$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\nu5})\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{5\nu})$ + p.c.
48	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_\nu\Sigma^\dagger)^2$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger)$ + p.c.

Generic operator	Specific $\mathcal{O}(a^2m)$ operator $[S, P \times V]$	$[S, P \times A]$
52 + 59	$\mathrm{Str}(\xi_\mu M^\dagger)\mathrm{Str}(\Sigma\xi_\mu)$	$\mathrm{Str}(\xi_{\mu5}M^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu})$
53 + 58	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu\Sigma M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu}\Sigma M^\dagger)$ + p.c.
54 + 60	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu})\mathrm{Str}(\Sigma M^\dagger)$ + p.c.
61	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}M^\dagger)$ + p.c.
62	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.
63	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}M^\dagger)$ + p.c.
64	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma M^\dagger)$ + p.c.
65	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger M\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger M\Sigma^\dagger)$ + p.c.
66	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(M\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(M\Sigma^\dagger)$ + p.c.
67	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger M\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger M\Sigma^\dagger)$ + p.c.
68	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(M\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(M\Sigma^\dagger)$ + p.c.

TABLE XIX. Rotationally noninvariant operators in the staggered chiral Lagrangian arising from a single insertion of $S_6^{FF(B)}$ operators. There is an implicit summation over both μ and ν with the constraint $\nu \neq \mu$.

Generic operator	Specific $\mathcal{O}(a^2p^2)$ operator $[V_\mu \times T_\mu]$ and $[A_\mu \times T_\mu]$
1 + 3	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu})$ + p.c.
2	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\mu\nu}\partial_\mu\Sigma\xi_{\nu\mu})$
6	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)$
7	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\mu\nu})\mathrm{Str}(\Sigma^\dagger\partial_\mu\Sigma\xi_{\nu\mu})$
10	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\mu\nu}\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu\mu})$ + p.c.
13	$\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\mu\nu})\mathrm{Str}(\Sigma\partial_\mu\Sigma^\dagger\xi_{\nu\mu})$ + p.c.

Generic operator	Specific $\mathcal{O}(a^2p^2)$ operator $[T_\mu \times V_\mu]$	$[T_\mu \times A_\mu]$
43	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)$ + p.c.
44	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_\mu\partial_\mu\Sigma^\dagger\xi_\mu)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\mu5}\partial_\mu\Sigma^\dagger\xi_{5\mu})$ + p.c.
45	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)$ + p.c.
46	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)$ + p.c.
47	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_\mu)\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_\mu)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{\mu5})\mathrm{Str}(\partial_\mu\Sigma^\dagger\xi_{5\mu})$ + p.c.
48	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)$ + p.c.	$\mathrm{Str}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)$ + p.c.

Staggered chiral perturbation theory

TABLE XX. $\mathcal{O}(a^4)$ operators in the staggered chiral Lagrangian arising from two insertions of an $S_6^{FF(A)}$ operator with spin V or A. Notation as in Table XVII.

Generic operator	$[V, A \times P]$ with $[V, A \times P]$
69 + 74	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger\xi_5\Sigma\xi_5\Sigma^\dagger)$
75	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)$

Generic operator	$[V, A \times P]$ with $[V, A \times T]$
69 + 74	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$
75	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)$
77	$\mathrm{Str}(\xi_5\Sigma\xi_{\mu\nu}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu\mu}\Sigma\xi_5\Sigma^\dagger)$
79	$\mathrm{Str}(\xi_5\Sigma\xi_{\mu\nu}\Sigma^\dagger\xi_5\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$
81	$\mathrm{Str}(\xi_5\Sigma\xi_{\mu\nu}\Sigma^\dagger)\mathrm{Str}(\xi_5\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$

Generic operator	$[V, A \times T]$ with $[V, A \times T]$
69 + 74	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger\xi_{\rho\sigma}\Sigma\xi_{\sigma\rho}\Sigma^\dagger) + \text{p.c.}$
75	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\rho\sigma}\Sigma\xi_{\sigma\rho}\Sigma^\dagger)$
77	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\rho\sigma}\Sigma^\dagger)\mathrm{Str}(\xi_{\sigma\rho}\Sigma\xi_{\nu\mu}\Sigma^\dagger)$
79	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\rho\sigma}\Sigma^\dagger\xi_{\nu\mu}\Sigma\xi_{\sigma\rho}\Sigma^\dagger) + \text{p.c.}$
81	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\rho\sigma}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu\mu}\Sigma\xi_{\sigma\rho}\Sigma^\dagger) + \text{p.c.}$

TABLE XXI. $\mathcal{O}(a^4)$ operators in the staggered chiral Lagrangian arising from two insertions of an $S_6^{FF(A)}$ operator with spin S or P. Notation as in Table XVII.

Generic operator	$[S, P \times V]$ with $[S, P \times V]$	$[S, P \times A]$ with $[S, P \times A]$
82	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu\Sigma\xi_\nu)$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu}\Sigma\xi_{5\nu})$
83	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_\nu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu})\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
84	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_\nu)$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu})\mathrm{Str}(\Sigma\xi_{5\nu})$
85	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
87	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
89	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
91	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
93	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
94	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
95	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
97	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
98	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
99	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
100	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
102	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger)\mathrm{Str}(\xi_\nu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
103	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu\Sigma\xi_\nu)$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\nu5}\Sigma\xi_{5\nu})$
104	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu)\mathrm{Str}(\Sigma\xi_\nu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\nu5})\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
106	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\nu)\mathrm{Str}(\Sigma\xi_\nu)$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\nu5})\mathrm{Str}(\Sigma\xi_{5\nu})$

Generic operator	$[S, P \times V]$ with $[S, P \times A]$
82	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu\Sigma\xi_{5\nu})$
83	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
84	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_{5\nu})$
85	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + p.c.$
86	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$
87	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
88	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$
89	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
90	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$
91	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
92	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$
93	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
94	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
95	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
96	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$
97	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
98	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
99	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
100	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
101	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger)Str(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$
102	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\nu}\Sigma^\dagger) + \text{p.c.}$
103	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\nu5}\Sigma\xi_{5\nu}) + \text{p.c.}$
104	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\nu5})\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$
105	$\mathrm{Str}(\xi_{\nu5}\Sigma^\dagger\xi_{5\nu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$
106	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\nu5})\mathrm{Str}(\Sigma\xi_{5\nu}) + \text{p.c.}$

Staggered chiral perturbation theory

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TABLE XXII. $\mathcal{O}(a^4)$ operators in the staggered chiral Lagrangian arising from two insertions of $S_6^{FF(A)}$ operators, one with spin V or A, the other with spin S or P. Notation as in Table XVII.

Generic operator	$[V, A \times P]$ with $[S, P \times V]$	$[V, A \times P]$ with $[S, P \times A]$
107	$\mathrm{Str}(\xi_5\Sigma\xi_\mu\Sigma\xi_5\Sigma^\dagger\xi_\mu\Sigma^\dagger)$	$\mathrm{Str}(\xi_5\Sigma\xi_{\mu5}\Sigma\xi_5\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)$
108 + 109	$\mathrm{Str}(\xi_5\Sigma\xi_\mu\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_5\Sigma\xi_{\mu5}\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
110	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)$	$\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\mu})$
111	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_5\Sigma\xi_5\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_5\Sigma\xi_5\Sigma^\dagger) + \text{p.c.}$
112	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger) + \text{p.c.}$
113	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_5\Sigma\xi_5\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
114	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_5\Sigma\xi_5\Sigma^\dagger) + \text{p.c.}$

Generic operator	$[V, A \times T]$ with $[S, P \times V]$	$[V, A \times T]$ with $[S, P \times A]$
107	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_\rho\Sigma\xi_{\nu\mu}\Sigma^\dagger\xi_\rho\Sigma^\dagger)$	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\rho5}\Sigma\xi_{\nu\mu}\Sigma^\dagger\xi_{5\rho}\Sigma^\dagger)$
108 + 109	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_\rho\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_\rho\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\rho5}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{5\rho}\Sigma^\dagger) + \text{p.c.}$
110	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_\rho\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\rho)$	$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\rho5}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{5\rho})$
111	$\mathrm{Str}(\xi_\rho\Sigma^\dagger\xi_\rho\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\rho5}\Sigma^\dagger\xi_{5\rho}\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$
112	$\mathrm{Str}(\xi_\rho\Sigma^\dagger\xi_\rho\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\rho5}\Sigma^\dagger\xi_{5\rho}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$
113	$\mathrm{Str}(\xi_\rho\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_\rho\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\rho5}\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{5\rho}\Sigma^\dagger) + \text{p.c.}$
114	$\mathrm{Str}(\xi_\rho\Sigma^\dagger)\mathrm{Str}(\xi_\rho\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\rho5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\rho}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$

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TABLE XXIII. $\mathcal{O}(a^4)$ operators in the staggered chiral Lagrangian arising from two insertions of $S_6^{FF(B)}$ operators. The indices μ and ν are separately summed, with the constraint that $\mu \neq \nu$.

Generic operator	$[T_\mu \times V_\mu]$ with $[T_\mu \times V_\mu]$	$[T_\mu \times A_\mu]$ with $[T_\mu \times A_\mu]$
93	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
95	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
97	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
99	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
102	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
103	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu\Sigma\xi_\mu)$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\mu5}\Sigma\xi_{5\mu})$
104	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\mu5})\mathrm{Str}(\Sigma\xi_{5\mu}) + \text{p.c.}$
106	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_\mu)$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\mu5})\mathrm{Str}(\Sigma\xi_{5\mu})$

Generic operator		$[T_\mu \times V_\mu]$ with $[T_\mu \times A_\mu]$
93		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
94		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
95		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
96		$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$
97		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
98		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
99		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
100		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
101		$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$
102		$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
103		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\mu5}\Sigma\xi_{5\mu}) + \text{p.c.}$
104		$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\mu5})\mathrm{Str}(\Sigma\xi_{5\mu}) + \text{p.c.}$
105		$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_\mu)\mathrm{Str}(\Sigma\xi_\mu) + \text{p.c.}$
106		$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\Sigma\xi_{\mu5})\mathrm{Str}(\Sigma\xi_{5\mu}) + \text{p.c.}$

Generic operator	$[V_\mu, A_\mu \times T_\mu]$ with $[T_\mu \times V_\mu]$	$[V_\mu, A_\mu \times T_\mu]$ with $[T_\mu \times A_\mu]$
111	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$
112	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$
113	$\mathrm{Str}(\xi_\mu\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger) + \text{p.c.}$
114	$\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_\mu\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$	$\mathrm{Str}(\xi_{\mu5}\Sigma^\dagger)\mathrm{Str}(\xi_{5\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger) + \text{p.c.}$

Generic operator		$[V_\mu, A_\mu \times T_\mu]$ with $[V_\mu, A_\mu \times T_\mu]$
69 + 74		$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger\xi_{\mu\sigma}\Sigma\xi_{\sigma\mu}\Sigma^\dagger) + \text{p.c.}$
75		$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)\mathrm{Str}(\xi_{\mu\sigma}\Sigma\xi_{\sigma\mu}\Sigma^\dagger)$
77		$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\mu\sigma}\Sigma^\dagger)\mathrm{Str}(\xi_{\sigma\mu}\Sigma\xi_{\nu\mu}\Sigma^\dagger)$
79		$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\mu\sigma}\Sigma^\dagger\xi_{\nu\mu}\Sigma\xi_{\sigma\mu}\Sigma^\dagger) + \text{p.c.}$
81		$\mathrm{Str}(\xi_{\mu\nu}\Sigma\xi_{\mu\sigma}\Sigma^\dagger)\mathrm{Str}(\xi_{\nu\mu}\Sigma\xi_{\sigma\mu}\Sigma^\dagger) + \text{p.c.}$

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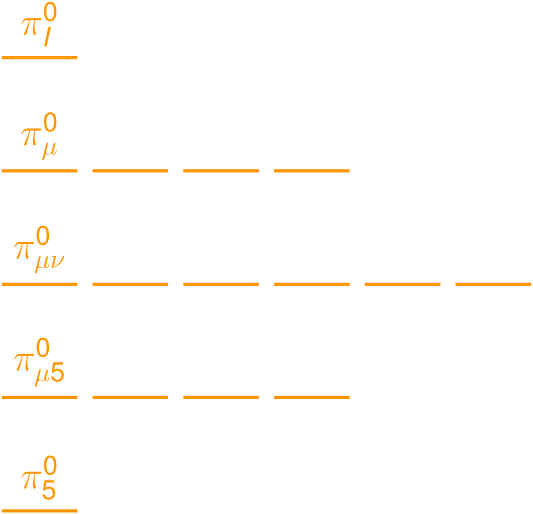
Staggered chiral perturbation theory

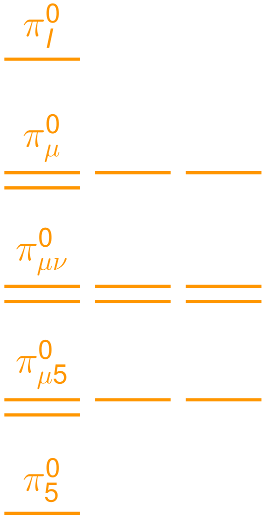
χPT	LO	NLO
Continuum	B	$L_1, \ldots, L_{10}, H_1$
Staggered	$\mathcal{C}_1, \mathcal{C}_{2V}, \mathcal{C}_{2A}, \ldots, \mathcal{C}_6$	*(222 LECs)*

- Calculation seems impossible
- When you address all the wick contractions, the result depends on a much smaller set of linear combinations of the LECs
- These combinations can be accessed through measurements of meson masses on the lattice

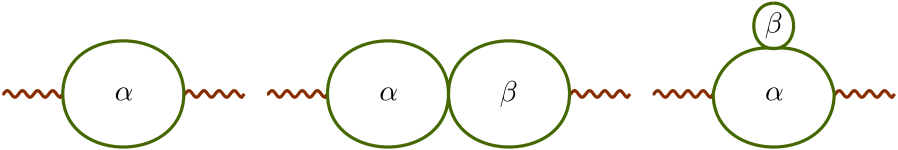
Staggered Mass Splitting



Staggered Mass Splitting



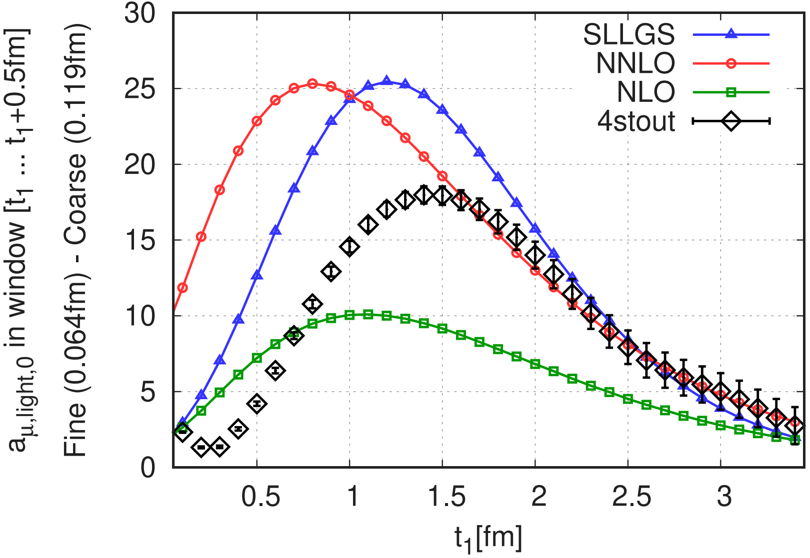
Discretisation Errors



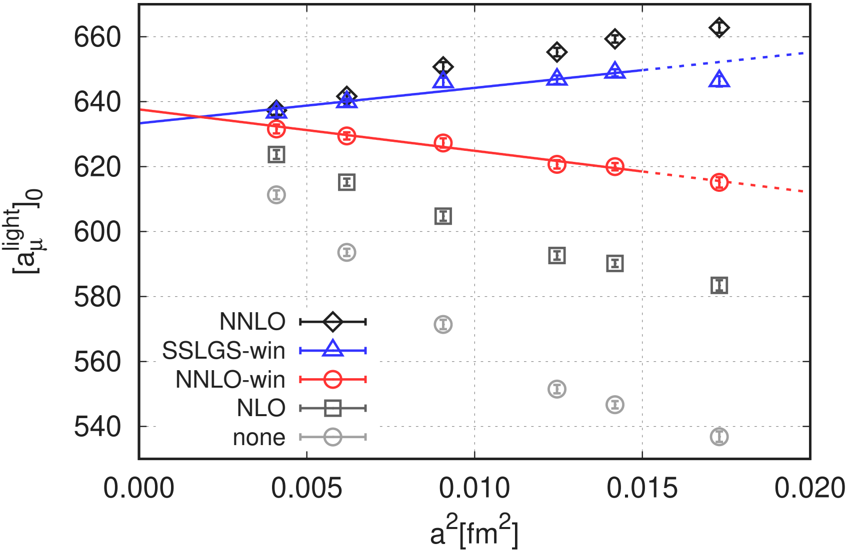
- At finite lattice spacing, results in taste average for each loop

$$\frac{1}{16}\sum_{\alpha}, \qquad \frac{1}{16}\sum_{\beta}$$

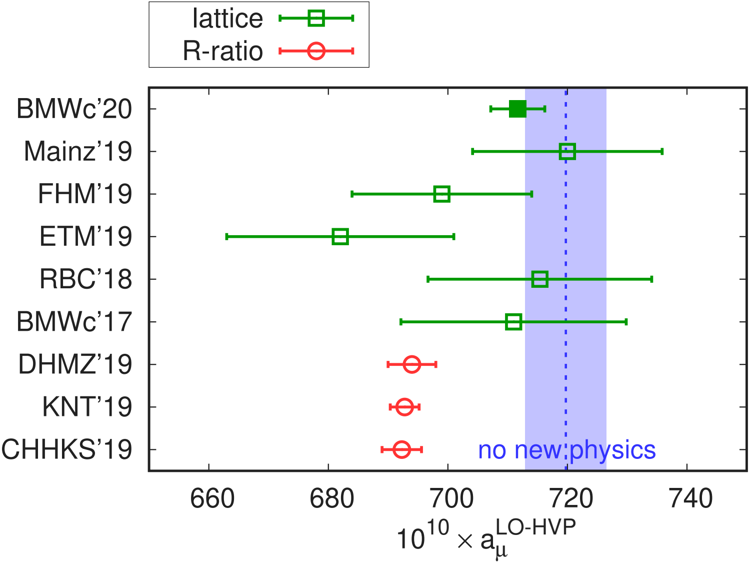
Region of validity



Continuum extrapolation



Result


lattice
R-ratio
BMWc’20
Mainz’19
FHM’19
ETM’19
RBC’18
BMWc’17
DHMZ’19
KNT’19
CHHKS’19
no new physics
660
680
700
720
740
$10^{10} \times a_\mu^{\text{LO-HVP}}$

Conclusion

- Muon anomalous magnetic moment known very precisely
- 3.7σ discrepancy between experiment and theory
- Could be a sign of new physics beyond the standard model
- Cutting-edge lattice results for the HVP contribution agree with experiment
- This suggests that the discrepancy may lie in the theoretical prediction of the HVP contribution
- Suggests a new direction for searching for new physics
- Look at effect within specific energy ranges
  - Narrow in on where the discrepancy is
  - Find the new physics