Berry phase in the phase space worldline representation: the axial anomaly and classical kinetic theory

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2022 TQCD 1st meeting

Jan. 18, 2022

PC and S. Pu, *to appear on arXiv soon*

Outline

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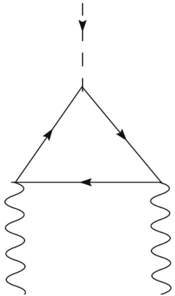
- Unlike the vector current the **axial (chiral) vector current**, j_5^μ, *is not conserved*.[1] Massless case:

$$\partial_\mu j_5^\mu = \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B}$$

- It was first thought that *classically* $\partial_\mu j_5^\mu = 0$. But, due to *quantum* effects, chiral symmetry broken.

$$\frac{d(N_{5R} - N_{5L})}{dt} = \int d^3x \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B}$$

$N_{5(R/L)}$ num of right or left handed (spin and momentum aligned) fermions

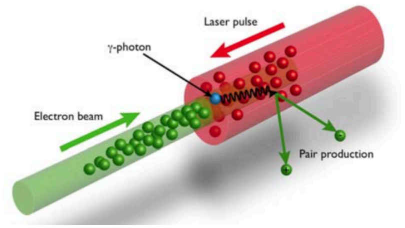


D. E. Kharzeev, *Ann. of Phys.* 325, 205218 (2010)

[1] S. L. Adler, *Phys. Rev.* 177, 2426 (1969); J. S. Bell, R. Jackiw, *Il Nuovo Cimento A* 60, 47.

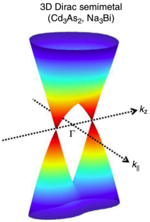
- Relativistic fermionic dispersion relation and anomaly observed directly in Weyl Semimetals.
- Need to probe the anomaly in QED and QCD.

QED - High Powered Lasers ?



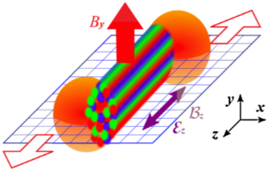
M. Marklund and J. Lundin, *Eur.Phys.J.D* 55, 319326 (2009)

3-D Semimetals ✓



M. Neupane, et. al., *Nat. Comm.* 10.1038 (2014)

QCD - Heavy Ion Collisions ?



K.Fukushima, D.Kharzeev, H.Warringa,*PRL* 104,212001(2010)

- Can we see the anomaly at a *classical* level?
  With the **Berry phase**[2] we can!
- If the system evolves slowly *(adiabatically)* enough it will stay fixed in the same eigenstate.
- Consider for Weyl fermions in quantum mechanics with a positive helicity eigenstate[3]

$$\mathbf{p}(t) \cdot \boldsymbol{\sigma}\, u^{+}(t) = |\mathbf{p}(t)|\, u^{+}(t)$$

  In addition to a dynamic factor $\int_{t_i}^{t_f} dt|\mathbf{p}|$ if we take a closed path such that $\mathbf{p}(t_i) = \mathbf{p}(t_f)$ we will acquire an additional phase

$$i\hbar u^{+} \nabla_{\mathbf{p}} u^{+} \cdot \frac{d\mathbf{p}}{dt}$$

  the Berry phase!

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[2] M. V. Berry, *Royal Soc. London. A.* 392, 45 (1984).

[3] I will show this later in the worldline formalism.

- Phase space evolution of a gas of Fermi particles (but with chiral effects) → **Chiral Kinetic Theory**[4]
- Liouville equation for distribution f for L/R handed particles w/o collisions:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial \mathbf{x}} \cdot f \dot{\mathbf{x}} + \frac{\partial}{\partial \mathbf{p}} \cdot f \dot{\mathbf{p}} = 0$$

- Introduce Berry phase for chiral fermions

$$\frac{\partial}{\partial t} f' + \frac{\partial}{\partial \mathbf{x}} \cdot f' \dot{\mathbf{x}} + \frac{\partial}{\partial \mathbf{p}} \cdot f' \dot{\mathbf{p}} = \mathbf{E} \cdot \mathbf{B} \; i\hbar \boldsymbol{\nabla}_{\mathbf{p}} \times u^{+} \boldsymbol{\nabla}_{\mathbf{p}} u^{+}$$

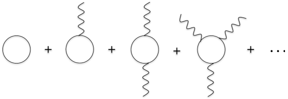
  - Aquire a collision-like term with **Berry curvature**
  - Makes possible a *non-convervation* of particle number into the phase space!
  - Can write down a current agreeing with anomaly.

[4] M. A. Stephanov, Y. Yin, *Phys. Rev. Lett.* 109, 162001 (2012),

The Worldline Formalism
Background

- Goal: To write a chiral kinetic theory at the classical level but with **Lorentz covariance**!
- How to go from 3 dimensions in QM to 3+1 dimensions? $\rightarrow$**The Worldline Formalism**
- For example let's look at the QED effective action in a background field, A^μ.[5]

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int d^4x\bar{\psi}[i\not{D}-m]\psi} = \det[i\not{D} - m] = e^{i\Gamma[A]}$$

$\Gamma[A] =$ 

1. Calculations to *all* orders in the gauge field!
2. Wealth of physics in compact expressions, e.g.:
   - Schwinger effect (a vacuum instability) in $\mathrm{Im}\Gamma[A]$
   - Light-by-light scattering in $\mathrm{Re}\Gamma[A]$
   - Can also calculate observables like the chiral current $\bar{\psi}\gamma^\mu\gamma_5\psi$

[5] C. Schubert, *Phys. Rept.* 10.101 (2001).

Outline

- We can write the worldline formalism in **phase space** representation[6] (to find Berry phase in momentum space)

$$G(A,x,y) = \langle x| \frac{-\hbar}{i\hbar\partial\!\!\!/ - \frac{e}{c}A\!\!\!/ - mc + i\epsilon} |y\rangle$$
$$= \int_0^\infty dT\, i\, \langle x|\, e^{-\frac{i}{\hbar}(-i\hbar\partial\!\!\!/ + \frac{e}{c}A\!\!\!/(x) + mc)T} |y\rangle$$

- The path integral form of the Green's function is

$$G(A,x,y) = i\int_0^\infty dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi\hbar}\, e^{\frac{i}{\hbar}S_{\mathrm{A}}}\mathcal{W}_{\mathrm{D}}$$

$$S_{\mathrm{A}} := \int_0^T d\tau[-mc - p_\mu\dot{x}^\mu - \frac{e}{c}A_\mu\dot{x}^\mu]\,,\ \mathcal{W}_{\mathrm{D}} := \mathcal{P}\exp\Big\{\frac{i}{\hbar}\int_0^T d\tau\, p\!\!\!/\Big\}$$

- All of the Berry phase contained in $\mathcal{W}_{\mathrm{D}}$!
- Can also write a **worldline phase space effective action**

[6] A. Migdal, *Nuclear Physics B* 265, 594 (1986).

Outline

Worldline Chiral Kinetic Theory for Weyl Fermions

- To begin, let's look at the simpler *massless* Weyl fermion case, just the left-handed case, i.e., stemming from the Lagrangian:

$$i\hbar \int d^4x\, \psi_{\mathrm{L}}^{\dagger} \bar{\sigma}_{\mu} D^{\mu} \psi_{\mathrm{L}} \qquad \bar{\sigma}^{\mu} = (\mathrm{I}_2, -\sigma^i)$$

- Same Green's function as before but with

$$\mathcal{W}_{\mathrm{D}} \to \mathcal{W}_{\mathrm{W}} := \mathcal{P} \exp\left\{ \frac{i}{\hbar} \int_0^T d\tau\, p_{\mu} \bar{\sigma}^{\mu} \right\}$$

- Insert complete sets of unitary transform, U, into the path ordered element, where $U^{\dagger} p_{\mu} \bar{\sigma}^{\mu} U = p^0 \mathrm{I}_2 + |\mathbf{p}| \sigma_3$. p^0 is already diagonal and *does not contribute* to Berry's phase
- The transformation matrix is $U = (u^-, u^+)$ with

$$u^- = \begin{pmatrix} e^{-i\omega_p} \cos \frac{\theta_p}{2} \\ \sin \frac{\theta_p}{2} \end{pmatrix} \quad u^+ = \begin{pmatrix} -e^{-i\omega_p} \sin \frac{\theta_p}{2} \\ \cos \frac{\theta_p}{2} \end{pmatrix}$$

  Here the momentum is in spherical coordinates
  $\mathbf{p} = |\mathbf{p}|(\sin\theta_p \cos\omega_p, \sin\theta_p \sin\omega_p, \cos\theta_p)$

Worldline Chiral Kinetic Theory for Weyl Fermions

- **The adiabatic Berry phase and curvature are**

$$\mathbf{B}^{\pm}_{\mathrm{W}} = -i\hbar u^{\pm}\nabla_{\mathbf{p}} u^{\pm} \ , \quad \mathbf{S}^{\pm}_{\mathrm{W}} = \nabla_{\mathbf{p}} \times \mathbf{B}^{\pm}_{\mathrm{W}} = \mp\hbar\frac{\mathbf{p}}{2|\mathbf{p}|^3}$$

- The path ordered element reads

$$\mathrm{tr}\mathcal{W}_{\mathrm{W}} \approx \sum_{\pm} \exp\Big\{\frac{i}{\hbar}\int_0^T d\tau[p^0 \pm |\mathbf{p}| - \mathbf{B}^{\mp}_{W}\cdot\dot{\mathbf{p}}]\Big\}$$

- From which the worldline action (+ helicity) reads

$$\mathcal{S}_{\mathrm{W}} = \int_0^T d\tau\big[-p_\mu \dot{x}^\mu - \frac{e}{c}A_\mu\dot{x}^\mu + p^0 - |\mathbf{p}| - \mathbf{B}^{+}_{\mathrm{W}}\cdot\dot{\mathbf{p}}\big]$$

- Then find the following equations of motion

$$\dot{p}_\mu = \frac{e}{c}F_{\mu\nu}\dot{x}^\nu \qquad \dot{\mathbf{x}} = \hat{\mathbf{p}} + \mathbf{S}^{+}_{\mathrm{W}} \times \dot{\mathbf{p}} \qquad \dot{x}^0 = 1$$

- These are the exact same equations of motion for the **quantum mechanical case**!
- We can find a chiral kinetic theory in the exact same way[7].

[7] M. Stephanov and Y. Yin *Phys. Rev. Lett.* 109, 162001 (2012).

Worldline Chiral Kinetic Theory for Weyl Fermions

- How does this lead to an anomaly? **Incompressible phase space measure!**[8].
- E.O.M. become:

$$(1 + \mathbf{B} \cdot \mathbf{S}_{\mathrm{W}}^{+})\dot{\mathbf{x}} = \hat{\mathbf{p}} + \mathbf{E} \times \mathbf{S}_{\mathrm{W}}^{+} + \mathbf{B}(\mathbf{S}_{\mathrm{W}}^{+} \cdot \hat{\mathbf{p}})$$

$$(1 + \mathbf{B} \cdot \mathbf{S}_{\mathrm{W}}^{+})\dot{\mathbf{p}} = \mathbf{E} + \hat{\mathbf{p}} \times \mathbf{B} + \mathbf{S}_{\mathrm{W}}^{+}(\mathbf{E} \cdot \mathbf{B})$$

- Incompressible phase space measure:

$$(1 + \mathbf{B} \cdot \mathbf{S}_{\mathrm{W}}^{+}) d^3x d^3p/(2\pi)^3$$

- Modified distribution function: $f' = (1 + \mathbf{B} \cdot \mathbf{S}_{\mathrm{W}}^{+})f$
- The phase space current is $j^{\mu} = \int d^3p (f', f'\dot{\mathbf{x}})/(2\pi)^3$, we find

$$\frac{\partial}{\partial t} f' + \frac{\partial}{\partial \mathbf{x}} \cdot f'\dot{\mathbf{x}} + \frac{\partial}{\partial \mathbf{p}} \cdot f'\dot{\mathbf{p}} = \mathbf{E} \cdot \mathbf{B}\, \boldsymbol{\nabla}_{\mathbf{p}} \cdot \mathbf{S}_{\mathrm{W}}^{+}$$

$$\partial_{\mu} j^{\mu} = \frac{\mathbf{E} \cdot \mathbf{B}}{4\pi^2}$$

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[8] M. Stephanov and Y. Yin *Phys. Rev. Lett.* 109, 162001 (2012).

Outline

- Let's extend our scope to **Dirac fermions**!
- "Diagonalize" $\not{p}$ in a Lorentz covariant way using eigenspinors.

$$s^{-1}\not{p}s = p\gamma_0 \quad p := \sqrt{p^\mu p_\mu} \quad (\not{p} - p)u_i = 0 \quad (\not{p} + p)v_i = 0$$

$$s = \frac{1}{\sqrt{2}}[u_1, u_2, v_1, v_2](\gamma_0 - \gamma_5) \qquad \textit{Weyl representation}$$

- For the adiabatic theorem keep only $\bar{u}_i\dot{u}_j$ and $\bar{v}_i\dot{v}_j$ parts.
- The **adiabatic Berry phase** can be found (where $\hat{p}_\mu = p_\mu/p$)

$$B^\mu = i\frac{\hbar}{8}\frac{1}{p_0 + p}\Gamma^{\mu\nu}\hat{p}_\nu \qquad \Gamma_{\mu\nu} := [\gamma_\mu, \gamma_\nu] + \gamma_0[\gamma_\mu, \gamma_\nu]\gamma_0$$

- And the associated curvature is

$$S^{\mu\nu} = \partial^{p\,\mu}B^\nu - \partial^{p\,\nu}B^\mu + i\hbar^{-1}[B^\mu, B^\nu]$$
$$= i\frac{\hbar}{8}\frac{(\hat{p}^{[\mu} + g^{[\mu 0})p_\alpha\Gamma^{\alpha\nu]} - (p + p_0)\Gamma^{\mu\nu}}{p^2(p_0 + p)}$$

where for generic tensor $A^{[\mu\nu]} = A^{\mu\nu} - A^{\nu\mu}$

Barut Zanghi Spinors

- We now have an "action" that is matrix weighted, $\mathcal{S}_{\mathrm{A}} + p\gamma_0 - B_\mu \dot{p}^\mu \ldots$ *How to define equations of motion?*
- With a coherent state! Use a spinor construction of **Barut and Zanghi**.
  Path ordering → path integral over spinors

$$\mathrm{tr}\mathcal{P}e^{\frac{i}{\hbar}\int_0^T d\tau[p\gamma_0 - B_\mu \dot{p}^\mu]} = \int \mathcal{D}z\mathcal{D}\bar{z}e^{\frac{i}{\hbar}\int_0^T d\tau \bar{z}[p\gamma_0 - B_\mu \dot{p}^\mu + i\hbar\frac{d}{d\tau}]z}$$

- z and $\bar{z} = z^\dagger \gamma_0$ are like the QED and QCD fermions, ψ.

$$z \sim \psi$$

- We can see the similarity transform $s \in \mathrm{SO}(1,3)$
- However:
  1. z is just a vector of 4 complex (and commutable!) numbers.
  2. $\int \mathcal{D}z\mathcal{D}\bar{z}$ integrates over all 8 independent variables.

- After the similarity transform the worldline action with the Berry phase reads:

$$\mathcal{S}_{\mathrm{D}} = \int_0^T d\tau[-mc - p_\mu \dot{x}^\mu - \frac{e}{c} A_\mu \dot{x}^\mu + \bar{z} p \gamma_0 z - \bar{z} B_\mu z \dot{p}^\mu + i\hbar \bar{z}\dot{z}]$$

- The **classical (and covariant!) e.o.m.** are (with 'bars' representing, e.g., $\bar{S}_{\mu\nu} = \bar{z} S_{\mu\nu} z$, and $\dot{x} = d/d\tau$)

$$\dot{x}_\mu = \hat{p}_\mu \bar{\gamma}_0 - \bar{S}_{\mu\nu}\dot{p}^\nu \,, \qquad \dot{p}_\mu = \frac{e}{c} F_{\mu\nu}\dot{x}^\nu \,,$$

$$\dot{\bar{z}} = -\frac{i}{\hbar}\bar{z}(p\gamma_0 - B_\mu \dot{p}^\mu) \,, \quad \dot{z} = \frac{i}{\hbar}(p\gamma_0 - B_\mu \dot{p}^\mu) z$$

(Recall 3-dimensional: $\frac{d\mathbf{x}}{dt} = \hat{\mathbf{p}} + \mathbf{S}_+ \cdot \frac{d\mathbf{p}}{dt}$ and $\frac{dp_\mu}{dt} = \frac{e}{c} F_{\mu\nu} \frac{dx^\nu}{dt}$)

- Must invert $\mathcal{G}_{\mu\nu} := g_{\mu\nu} + \frac{e}{c}\bar{S}_{\mu\sigma}F^\sigma_{\ \nu}$.

$$I_{\tilde{F}F} = -\frac{1}{4}\widetilde{F}_{\mu\nu}F^{\mu\nu}, \quad I_{\tilde{\bar{S}}\bar{S}} = -\frac{1}{4}\tilde{\bar{S}}_{\mu\nu}\bar{S}^{\mu\nu}, \quad I_{\bar{S}F} = \frac{1}{2}\bar{S}_{\mu\nu}F^{\mu\nu}$$

$$\sqrt{\det \mathcal{G}} = 1 - \frac{e}{c} I_{\bar{S}F} - \left(\frac{e}{c}\right)^2 I_{\tilde{F}F} I_{\tilde{\bar{S}}\bar{S}}$$

- We can find for the **Berry phase modified equations:**

$$\sqrt{\det \mathcal{G}}\,\dot{x}_\mu = \left[g_{\mu\nu} + \frac{e}{c}\widetilde{F}_{\mu\sigma}\widetilde{\bar{S}}^{\sigma}{}_{\nu}\right]\hat{p}^\nu \bar{\gamma}_0\,,$$

$$\sqrt{\det \mathcal{G}}\,\dot{p}_\mu = \left[\frac{e}{c}F_{\mu\nu} + \left(\frac{e}{c}\right)^2 I_{\tilde{F}F}\widetilde{\bar{S}}_{\mu\nu}\right]\hat{p}^\nu \bar{\gamma}_0$$

- Conserved phase space measure: $d\mu_{\mathrm{D}} = \sqrt{\det \mathcal{G}}\,\frac{d^4p\,d^4x\,d\Omega_z}{(2\pi)^4}$.
- Phase space distribution function satisfies a Boltzmann equation, $\frac{d}{d\tau}f = \left[\frac{\partial}{\partial\tau} + \dot{x}^\mu\partial_\mu + \dot{p}^\mu\partial^p_\mu + \dot{z}_a\partial^z_a + \dot{\bar{z}}_a\partial^{\bar{z}}_a\right]f = 0$
- Write a general f up to a bilinear term in B.Z. spinors:

$$f(x,p,z,\bar{z}) = \frac{1}{4}\bar{z}\left\{\mathcal{S} + \gamma_5\mathcal{P} + \gamma_\mu\mathcal{V}^\mu + \gamma_\mu\gamma_5\mathcal{A}^\mu + \sigma_{\mu\nu}\mathcal{T}^{\mu\nu}\right\}z$$

- Define a divergence of the velocity, **only the axial vector parts remain!** Anomaly for distribution $\mathcal{A}$.

$$\partial_\mu \int \frac{d^4p\,d\Omega_z}{(2\pi)^4}\sqrt{\det \mathcal{G}}\,\dot{x}_\mu f = -\left(\frac{e}{c}\right)^2 I_{\tilde{F}F}\int \frac{d^4p}{(2\pi)^4}\,\frac{2\hbar}{p^3(p_0+p)}\boldsymbol{\mathcal{A}}\cdot\mathbf{p}\,.$$

Outline

- Perform the **Fujikawa method** [9] on the BZ spinor coherent state to find the axial Ward identity.

$$\Gamma[A] = i\hbar \int_0^\infty \frac{dT}{T} \oint \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi\hbar}\, e^{\frac{i}{\hbar}S_{\mathcal{A}}} \mathrm{tr}\mathcal{W}_{\mathrm{D}}$$

$$\mathrm{tr}\mathcal{W}_{\mathrm{D}} = \oint \mathcal{D}z\mathcal{D}\bar{z} e^{\frac{i}{\hbar}\int_0^T d\tau[\bar{z}\not{p}z + i\hbar\bar{z}\dot{z}]}$$

- Perform the axial rotation $z \to e^{i\theta(\tau)\gamma_5}z$, and absorb phase

$$\mathrm{tr}\mathcal{W}_{\mathrm{D}} = \det\left[e^{2i\theta\gamma_5}\right] \oint \mathcal{D}\Omega_z e^{\frac{i}{\hbar}\int_0^T d\tau[\bar{z}\not{p}z - \hbar\dot{\theta}\bar{z}e^{2i\theta\gamma_5}\gamma_5 z + i\hbar\bar{z}e^{2i\theta\gamma_5}\dot{z}]}$$

- Write the axial rotation as a Hamiltonian transformation: $\not{p} \to e^{-i\theta\gamma_5}\not{p}e^{i\theta\gamma_5} + ie^{-i\theta\gamma_5}\frac{d}{d\tau}e^{i\theta\gamma_5}$ ∼**exact Berry phase**
- Associated functional form of the effective action:

$$\Gamma[A] = -2\hbar\mathrm{Tr}\theta\gamma_5 - i\hbar\mathrm{Tr}\ln\left[\not{p} - \frac{e}{c}\not{A} + \partial_\mu\theta\gamma_5\gamma^\mu - e^{2i\theta\gamma_5}mc\right]$$

This is the form which produces the axial Ward identity.

[9] K. Fujikawa, *Phys. Rev. Lett.* 42, 1195 (1979); *Phys. Rev. D* 21, 2848 (1980).

Noncovariant Berry Phase

Fujikawa Method on the Worldline

- Let's consider another (non-covariant) transform which takes

$$\tilde{s}^{-1}\not{p}\tilde{s} = \gamma_5 p$$

$$\tilde{s} = \frac{1}{\sqrt{2}}(\mathrm{I}_4 - \gamma_5\hat{\not{p}}) = \exp\left[-\frac{8n+1}{4}\pi\gamma_5\hat{\not{p}}\right]$$

- Like with the axial rotation, take the following transformation

$$z \to \tilde{s}z$$

$$\mathrm{tr}\mathcal{W}_{\mathrm{D}} = \det\left[e^{-\frac{(8n+1)}{2}\pi\gamma_5\hat{\not{p}}}\right]\oint \mathcal{D}\Omega_z$$
$$\times \exp\left\{\frac{i}{\hbar}\int_0^T d\tau[\bar{z}\not{p}z + i\hbar\bar{z}\tilde{s}\dot{\tilde{s}}z + i\hbar\bar{z}e^{-\frac{(8n+1)}{2}\gamma_5\hat{\not{p}}}\dot{z}]\right\}$$

- *Noncovariant Berry phase transformation has similar structure as axial rotation.*

Conclusions

Berry Phase on the Phase Space Worldline

1. How to develope a classical and **covariant** chiral kinetic theory? $\rightarrow$ **Phase Space Worldline Formalism.**
2. Weyl fermions *reduce in dimension* $\rightarrow$ quantum mechanics
3. Dirac fermions
   - Non-Abelian Berry phase
   - Barut Zanghi spinors
   - **Covariant phase space evolution!**

Thank you for your time and attention!

- Adiabaticity at a classical level permitted the anomaly. How about at the *quantum* level?
- Study the **index theorem** under the same adiabaticity that led to a chiral kinetic theory.
- The index theorem[10] counts the number of zero modes of the fermion operator describing the non-conservation of chiral current in the anomaly.
- While the index is normally defined[11] as $I_n = \lim_{M\to\infty} \mathrm{Tr}\,\gamma_5[M^2/(\not{D}^2 + M^2)]$ we use the equivalent expression

$$I_n = \lim_{M\to\infty} \mathrm{Tr}\,\gamma_5 \frac{-M}{i\not{D} - M}\,,$$

[10] G. t Hooft, *Phys. Rev. D* 14, 3432 (1976); M. Atiyah, et.al., *Phys. Lett. A* 65, 185 (1978).

[11] S. Vandoren and P. van Nieuwenhuizen, *arXiv:0802.1862* (2008).

- Cast the index definition into a phase space path integral:

$$I_n = \lim_{M\to\infty} \mathrm{tr} \int d^4p\, M\gamma_5\, G(A,p,p)$$

- In the momentum representation we have (with m→M)

$$G(A,p,p) = i\int_0^\infty dT \int \mathcal{D}x \int_{p(0)=p}^{p(T)=p} \frac{\mathcal{D}p}{2\pi\hbar}\, e^{\frac{i}{\hbar}S_{\mathrm{A}}} \mathcal{W}_{\mathrm{D}}$$

$$\mathcal{W}_{\mathrm{D}} = \mathcal{P}\exp\Big\{\frac{i}{\hbar}\int_0^T d\tau\, \not{p}\Big\}\,.$$

- Take a gauge transformation to arrive at the Berry phase

$$\mathrm{tr}\gamma_5\mathcal{W}_{\mathrm{D}} = \mathrm{tr}\gamma_5\mathcal{P}\exp\Big\{\frac{i}{\hbar}\int_0^T d\tau\,[\gamma_0 p - B_{\mathrm{Exact}\mu}\dot{p}^\mu]\Big\}$$

- Actually we find independence of propertime T:

$$\frac{\hbar}{i}\frac{d}{dT}\mathrm{tr}\gamma_5\mathcal{W}_{\mathrm{D}} = \mathrm{tr}\gamma_5 \not{p}(T)\mathcal{W}_{\mathrm{D}} = \mathrm{tr}\gamma_5\mathcal{W}_{\mathrm{D}}\not{p}(0)$$
$$= -\mathrm{tr}\gamma_5\not{p}(T)\mathcal{W}_{\mathrm{D}} = 0\,,$$

The adiabatic theorem should be permissible as $T \to \infty$, but

$$\mathrm{tr}\gamma_5\mathcal{P}\exp\Big\{\frac{i}{\hbar}\int_0^T d\tau\,[\gamma_0 p - B_{\mathrm{Ad}\mu}\dot{p}^\mu]\Big\} = 0\,,$$

- And hence the index (and Chern Simons term leading to the non-conservation of axial current)
will vanish under adiabaticity!