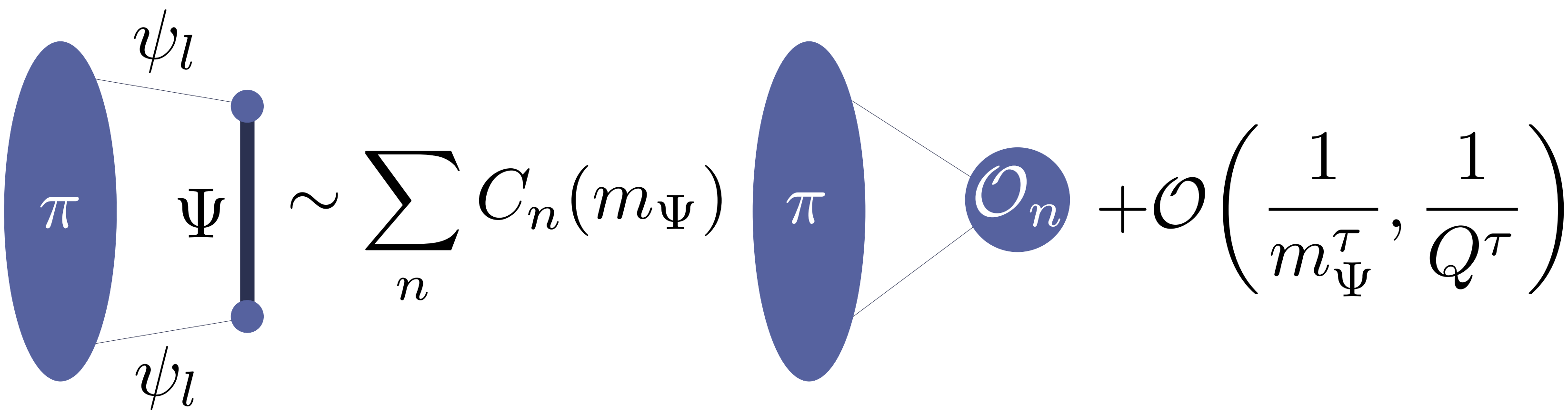
Fourth Mellin moment of pion LCDA using the HOPE method



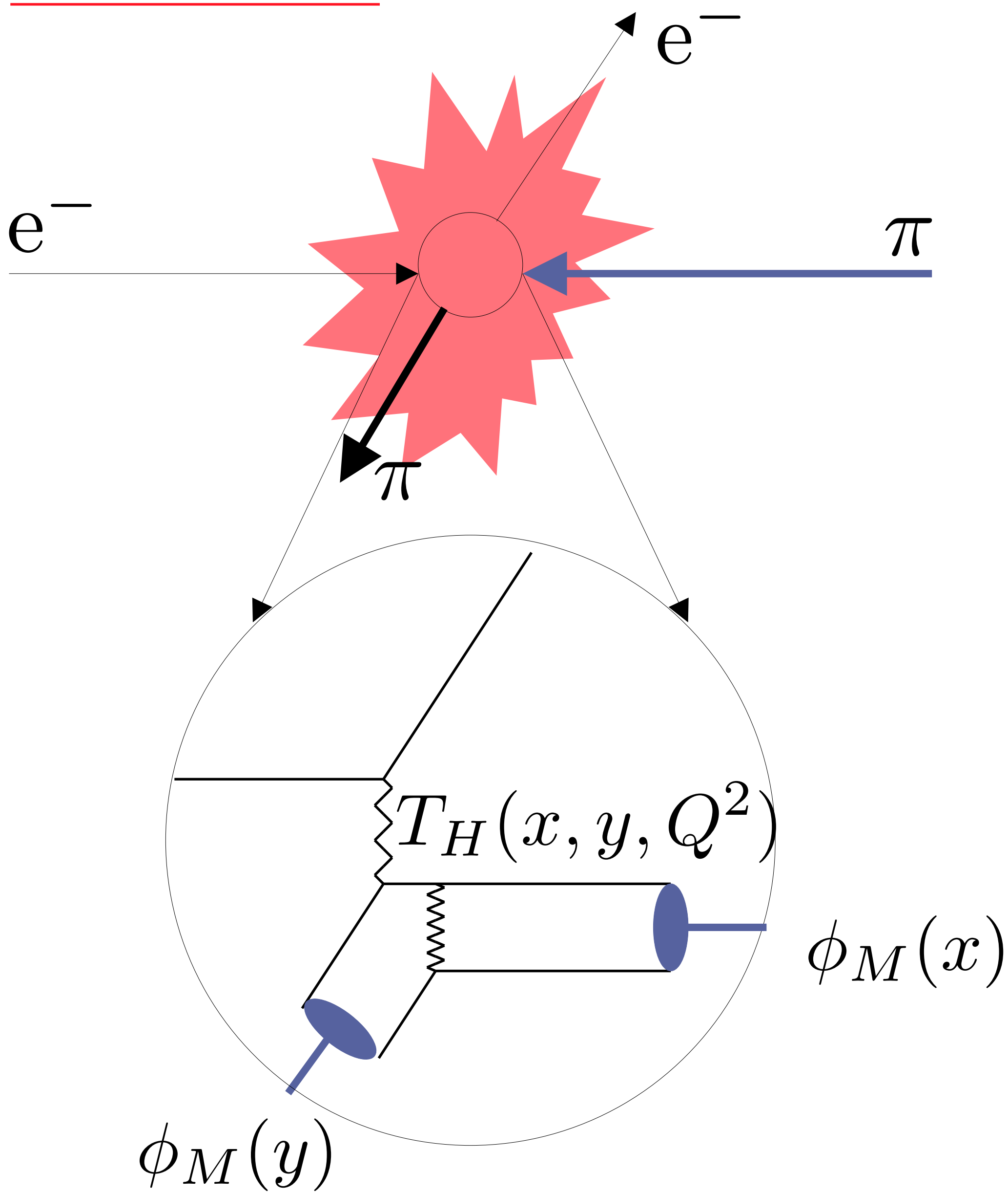
Robert Perry

Phys.Rev.D 113 (2026) 1, 014510, 2509.04799 [hep-lat]

Alex Chang, William Detmold, Matias Gutierrez Escobari, Anthony V. Grebe, Issaku Kanamori, C.-J. David Lin Yong Zhao

10 October, 2025

Exclusive processes in QCD



Exclusive processes may be factorized Phys.Rev.Lett. 43 (1979) 246, Phys.Rev.D 22 (1980) 2157, Phys.Lett.B 94 (1980) 245-250

Convolution of pion wave function and short-distance hard kernel, $T_H(x, y, Q^2)$

Large-Q^2 limit is

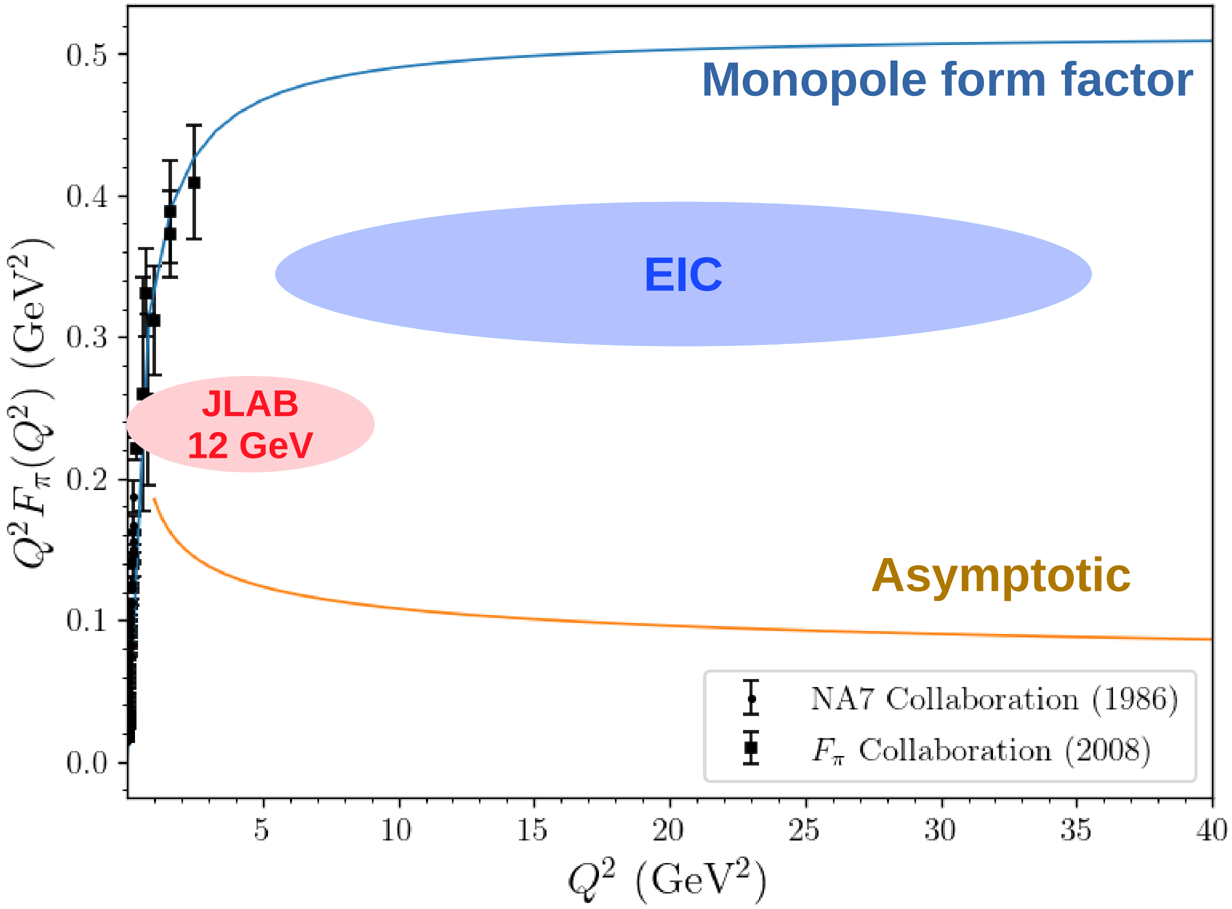
$$F_\pi(Q^2) = \frac{16\pi\,\alpha_S(Q^2)}{Q^2} f_\pi^2 \omega_\phi^2(Q^2)$$

where

$$\omega_\phi = \frac{1}{3}\int_0^1 dx \frac{\phi(x, \mu^2)}{x} \to 1$$

Experimental status

Data from Nucl.Phys.B 277 (1986) 168, Phys.Rev.C 78 (2008) 045203



Need to know LCDA at appropriate scale. Chang et al, Phys.Rev.Lett. 111 (2013) 14, 141802

Requires **non-perturbative** calculation

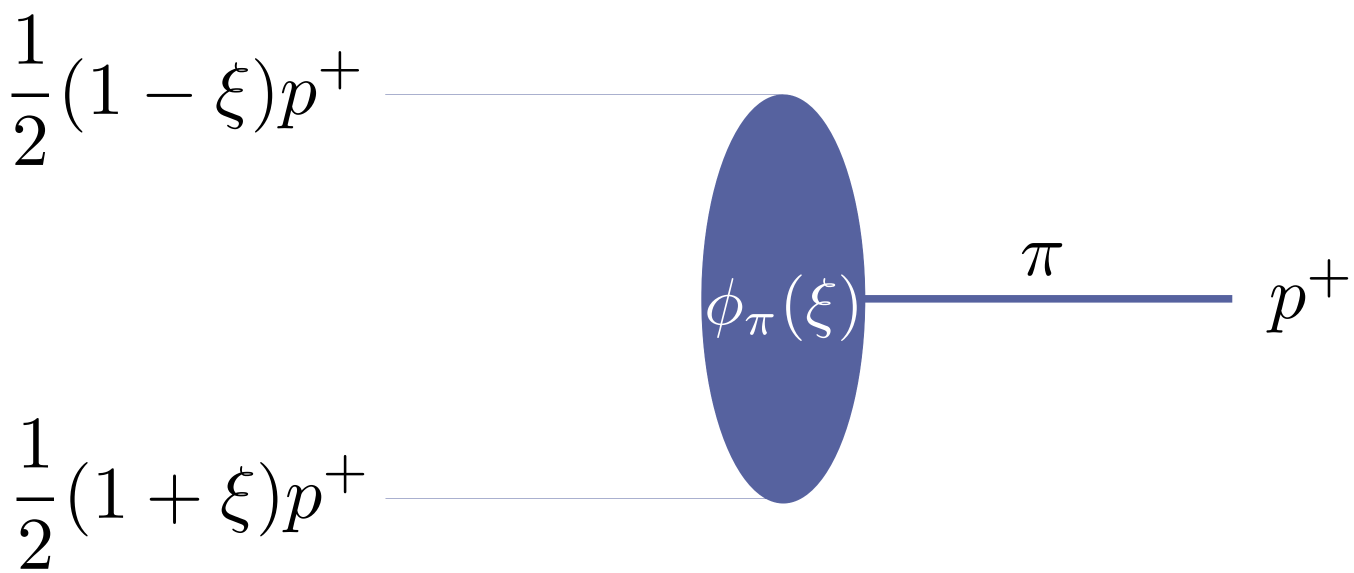
Outline

Introduction to the light cone distribution amplitude (LQCD)

How does the HOPE method work?

Calculation of 4th Mellin moment of pion LCDA using LQCD.

The light-cone distribution amplitude



- LCDA matrix element:

$$\left\langle\Omega\left|\bar{\psi}(z_-)\gamma_\mu\gamma_5 W[z_-,-z_-]\psi(-z_-)\right|\pi(\mathbf{p})\right\rangle = ip_\mu f_\pi \int_{-1}^{1} d\xi \mathrm{e}^{i\xi p^+ z_-}\phi_\pi(\xi,\mu^2)$$

- Probability amplitude for converting a meson into a collinear quark anti-quark pair with momentum fraction x and 1-x, respectively.

Local Operators

“Traditional” approach: OPE

$$\mathcal{O}_1(z)\mathcal{O}_2(0) \sim \sum_n C_n(z^2)\mathcal{O}_n(0)$$

For LCDA matrix element, twist-2 operators lead to:

$$\langle\Omega|\mathcal{O}_n|\pi(\vec{p})\rangle = \langle\Omega|\mathcal{O}_n|\pi(\vec{p})\rangle = f_\pi\langle\xi^{n-1}\rangle[p^{\mu_1\cdots\mu_n} - \mathrm{traces}]$$

Coefficients related to integral (Mellin) moments of LCDA:

$$\langle\xi^n\rangle(\mu^2) = \frac{1}{2}\int_{-1}^{1} d\xi\,\xi^n\phi_\pi(\xi,\mu^2)$$

BUT: lattice breaks Lorentz symmetry: non-perturbative mixing of operators!

Continuum limit challenging*

*Phys.Rev.D 86 (2012) 054505, Phys.Rev.D 110 (2024) 5, L051503

Introduction to HOPE

Idea due to Detmold & Lin Phys.Rev.D 73 (2006) 014501

Compute hadronic matrix element using lattice QCD:

$$V^{\mu\nu}(p,q) = \int d^4z e^{iq\cdot z} \langle \Omega | T\{J_\Psi^\mu(z/2) J_\Psi^\nu(-z/2)\} | \pi(\mathbf{p}) \rangle$$

Employ current-current operator with fictitious heavy-quark species:

$$J^\mu \to J_\Psi^\mu = \overline{\Psi}\Gamma^\mu \psi_l + \overline{\psi}_l \Gamma^\mu \Psi$$

✔Hadronic matrix element has well-defined continuum limit

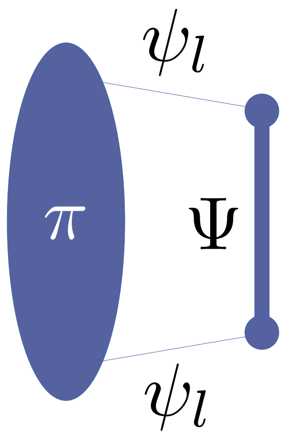
✔Can compute directly using LQCD

✔Heavy quark limit: Wilson line

Introduction to HOPE

THE HOPE COLLABORATION

Compute **H**eavy quark **O**perator **P**roduct **E**xpansion (**HOPE**):



$$V^{\mu\nu}(p,q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2}\left[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots\right]$$

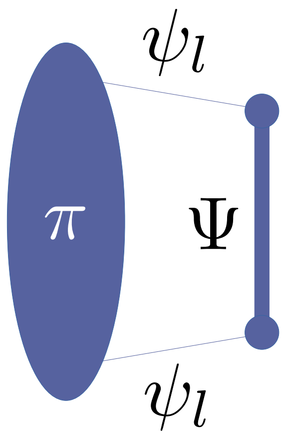
$$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \tilde{\omega} = \frac{2p\cdot q}{\tilde{Q}^2} = p_\mu \frac{2q^\mu}{\tilde{Q}^2}$$

Hard scale

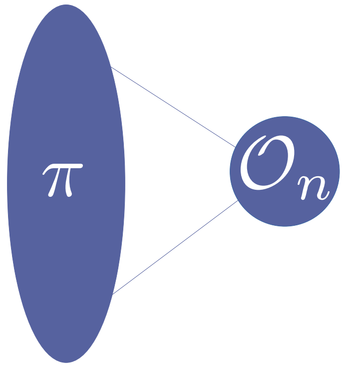
Small expansion parameter

Introduction to HOPE

Compute **H**eavy quark **O**perator **P**roduct **E**xpansion (**HOPE**):



$$\sim \sum_n C_n(m_\Psi)$$



$$+\mathcal{O}\left(\frac{1}{m_\Psi^\tau}, \frac{1}{Q^\tau}\right)$$

$$V^{\mu\nu}(p,q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2}\left[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots\right]$$

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Hard scale — Small expansion parameter

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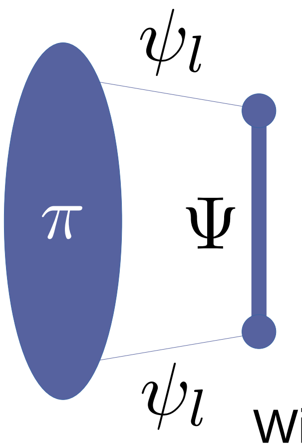

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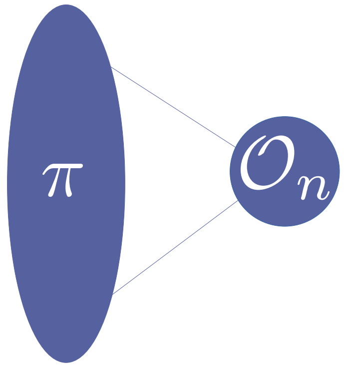

$$+\mathcal{O}\left(\frac{1}{m_\Psi^\tau}, \frac{1}{Q^\tau}\right)$$

Wilson coefficients
(Contain dep on
heavy quark)

$$V^{\mu\nu}(p,q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2}\left[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots\right]$$

$$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2} = p_\mu \frac{2q^\mu}{\tilde{Q}^2}$$

Hard scale

Small expansion parameter

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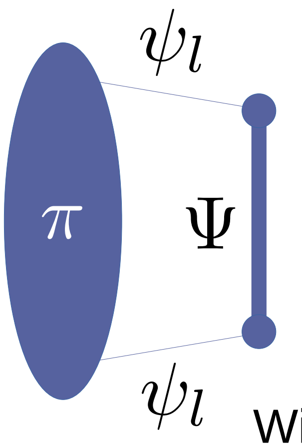

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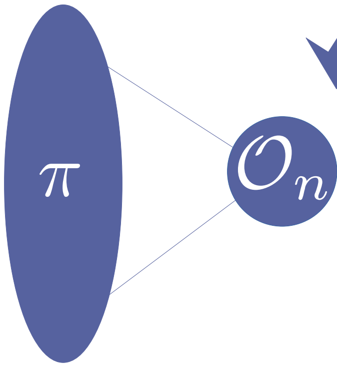

$$+\mathcal{O}\left(\frac{1}{m_\Psi^\tau}, \frac{1}{Q^\tau}\right)$$

Local operators

Wilson coefficients
(Contain dep on
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$$V^{\mu\nu}(p,q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2}\left[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots\right]$$

$$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2} = p_\mu \frac{2q^\mu}{\tilde{Q}^2}$$

Hard scale

Small expansion parameter

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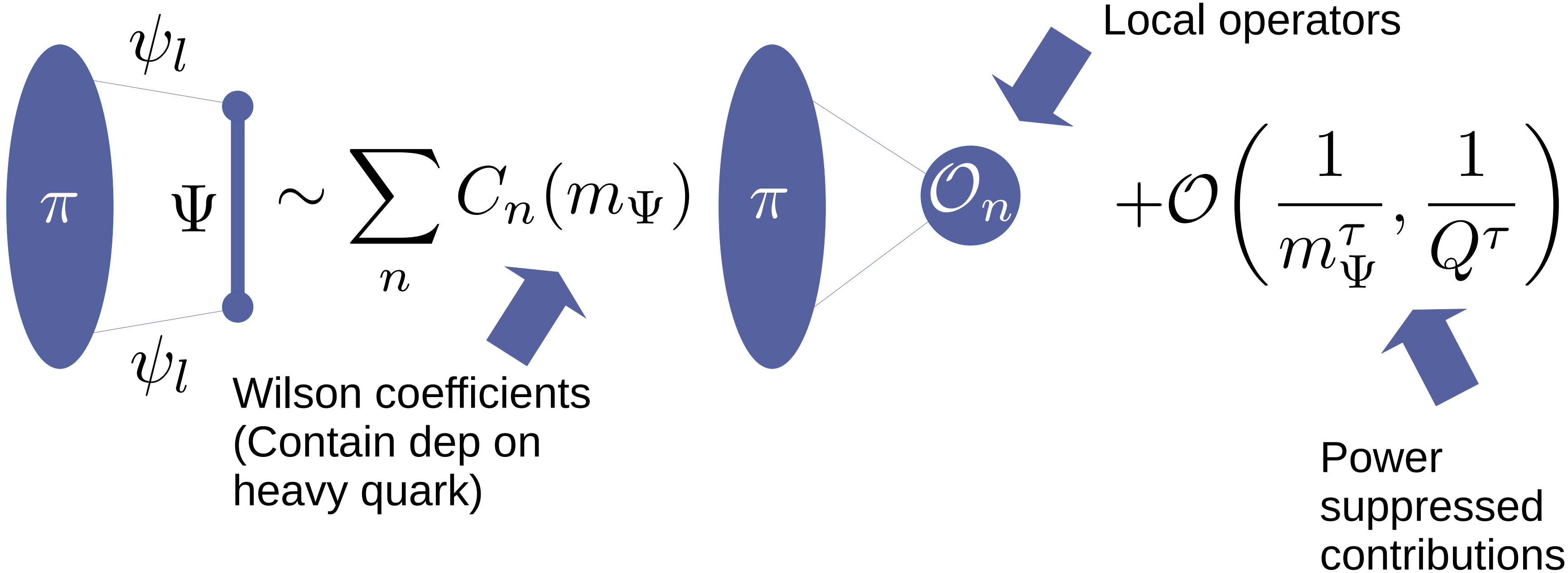


$$V^{\mu\nu}(p,q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2}\left[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots\right]$$

$$\tilde{Q}^2 = -q^2 + m_\Psi^2 \qquad \tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2} = p_\mu \frac{2q^\mu}{\tilde{Q}^2}$$

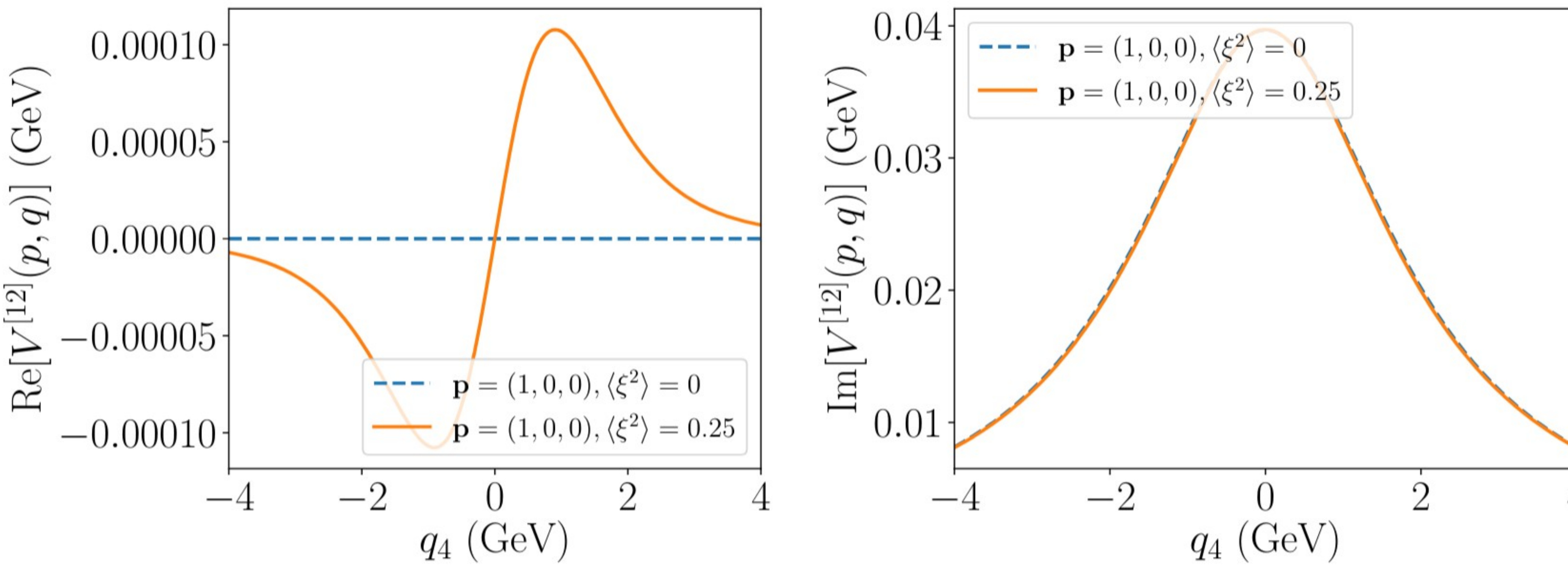
Hard scale — Small expansion parameter

LaMET without ET

$$V^{\mu\nu}(p,q) = K^{\mu\nu}[1 + \langle \xi^2 \rangle \tilde{\omega}^2 + \langle \xi^4 \rangle \tilde{\omega}^4 + \dots]$$

Choose “special kinematics”

$K^{\mu\nu}$ purely imaginary. If $\tilde{\omega} = \dfrac{2p \cdot q}{\tilde{Q}^2}$ is complex, can isolate $\langle \xi^2 \rangle$



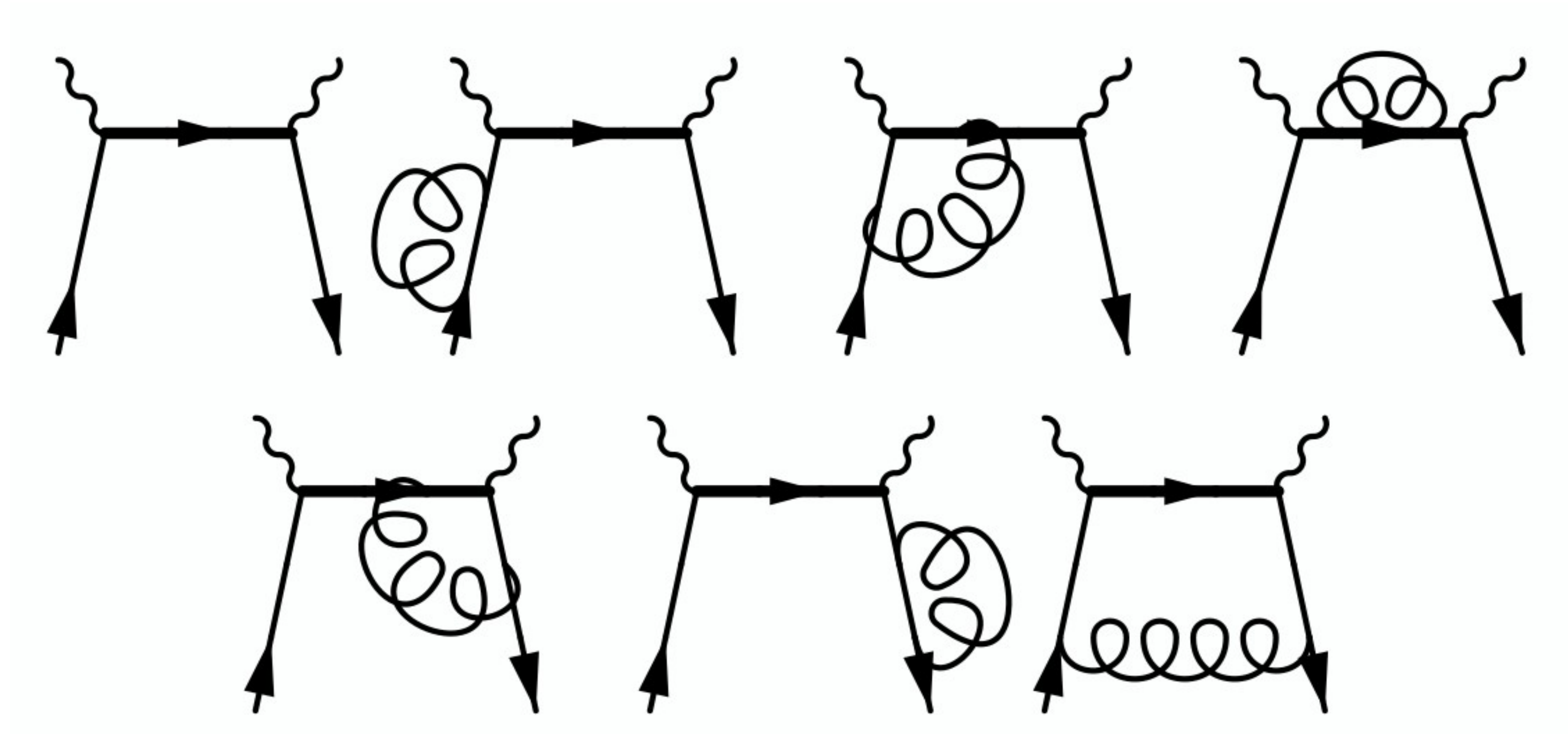
Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, Phys.Rev.D 104 (2021) 7, 074511

Perturbative Matching

Compute parton-level matrix element and isolate Wilson coefficients.

One-loop improved HOPE formula is

$$V^{\mu\nu}(p,q) = -\frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{\tilde{Q}^2} \sum_n C_n(\tilde{Q}^2, m_\Psi^2) \tilde{\omega}^n \langle \xi^n \rangle$$



The HOPE Method

Compute hadronic matrix element using LQCD

$$\int d^4z e^{iq\cdot z}\langle\Omega|T\{J^{\mu}_{\Psi}(z/2)J^{\nu}_{\Psi}(-z/2)\}|\pi(\mathbf{p})\rangle$$

Compute HOPE for hadronic matrix element

$$V^{\mu\nu}(p,q) = K^{\mu\nu}\sum_n C_n(\tilde{Q}^2, m_{\Psi}^2)\tilde{\omega}^n\langle\xi^n\rangle$$

Fit HOPE to data to extract Mellin moments (up to corrections from higher-twist, lattice artifacts etc).

Perform twist-2, continuum limit to isolate Mellin moments

The HOPE for controlled parton physics

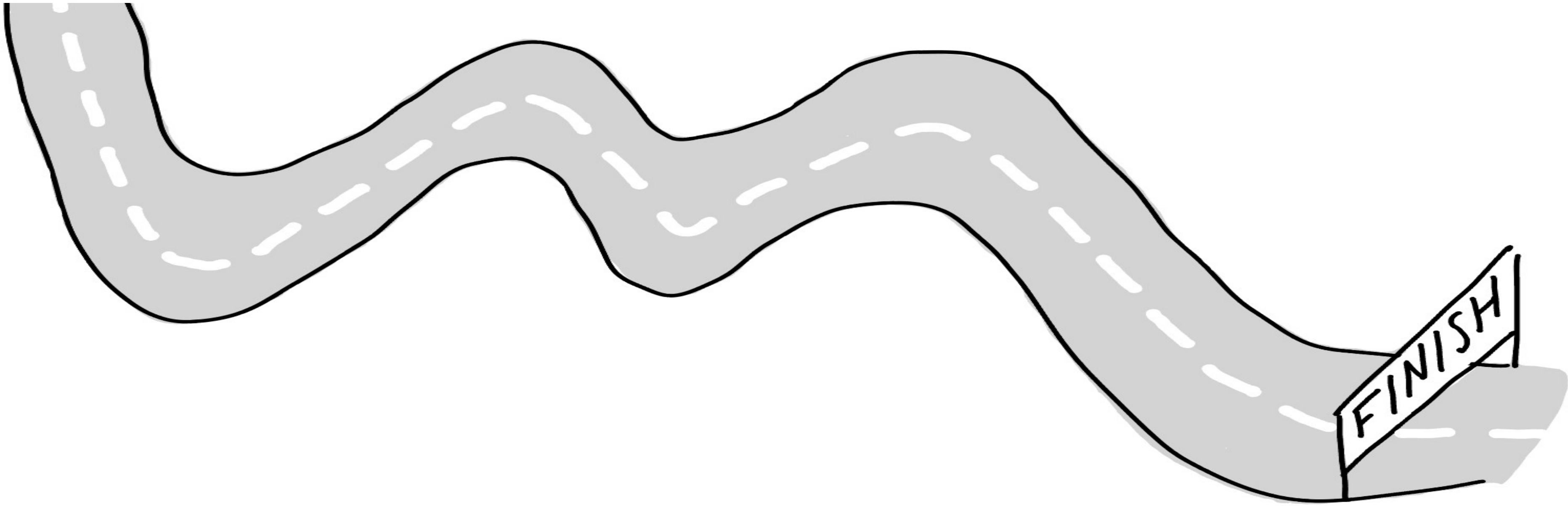
Phys.Rev.D 73 (2006) 014501

2006: First idea

PHYSICAL REVIEW D **73,** 014501 (2006)

Deep-inelastic scattering and the operator product expansion in lattice QCD

William Detmold[1] and C.-J. David Lin[1,2]
[1]*Department of Physics, University of Washington, Box 351560, Seattle, Washington 98195, USA*
[2]*Institute for Nuclear Theory, University of Washington, Box 351550, Seattle, Washington 98195, USA*
(Received 21 July 2005; published 9 January 2006)



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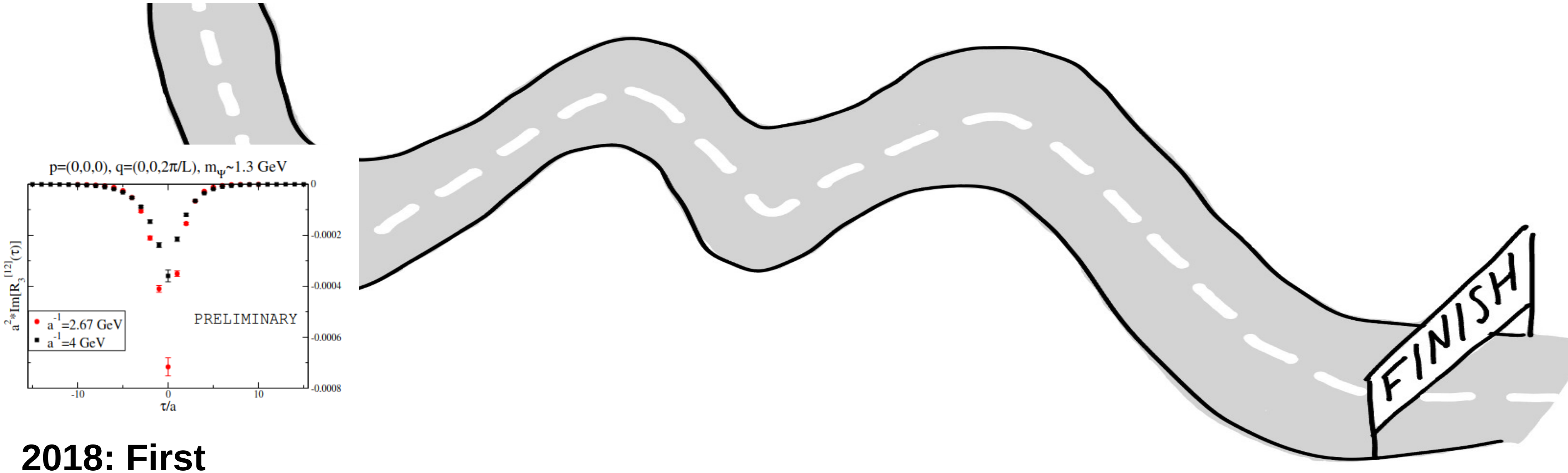

2018: First numerical study

The HOPE for controlled parton physics

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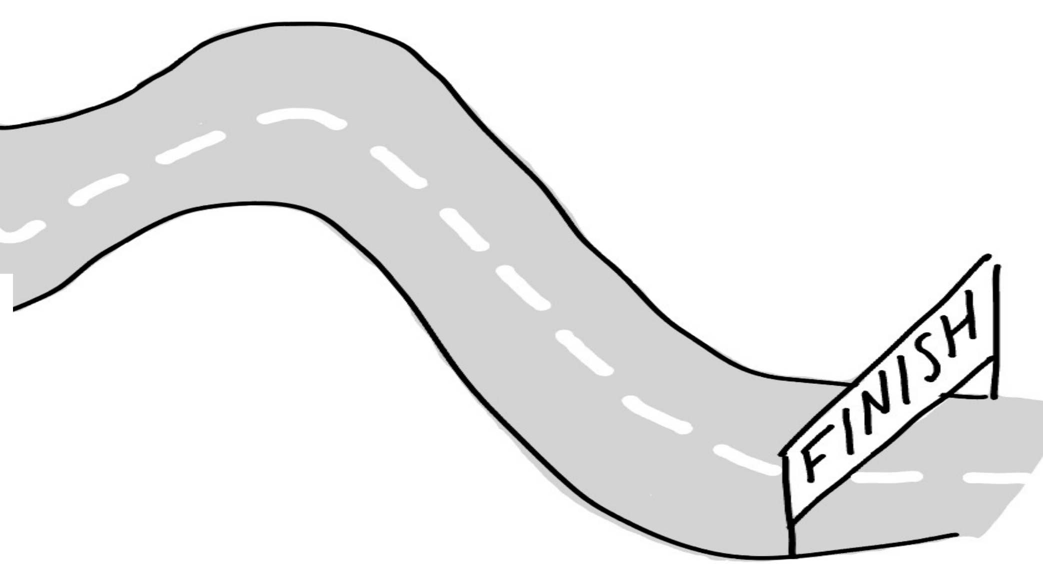
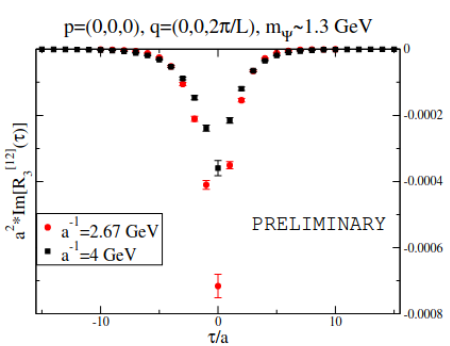
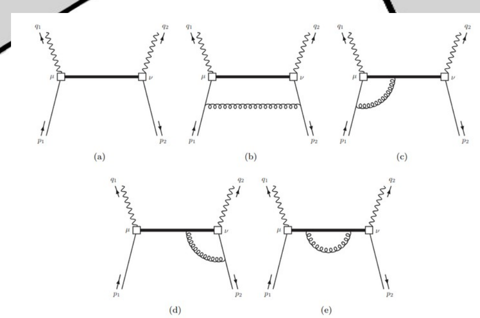




2018: First numerical study



2021: Formal developments

Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, Phys.Rev.D 104 (2021) 7, 074511

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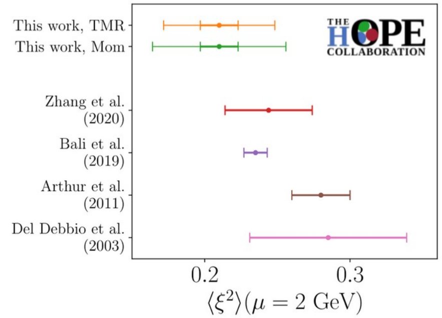
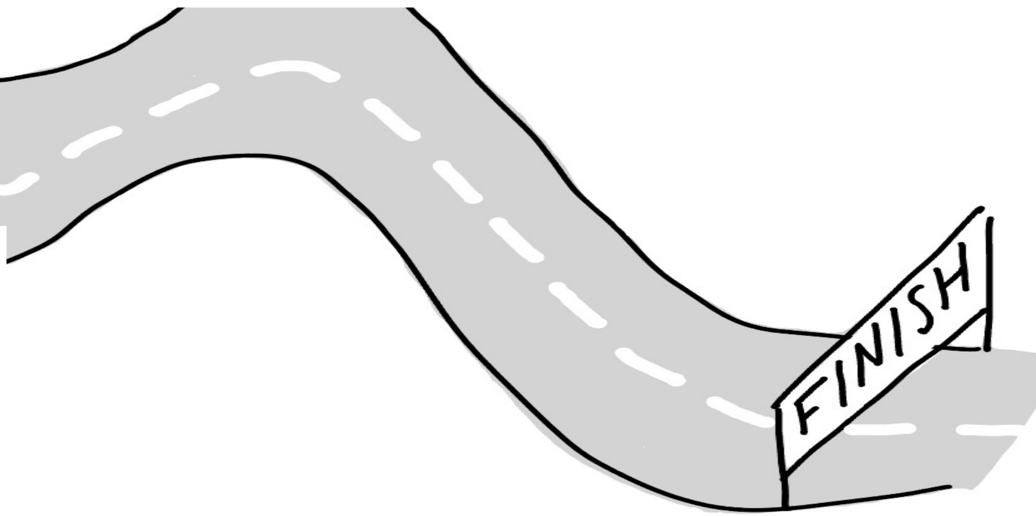
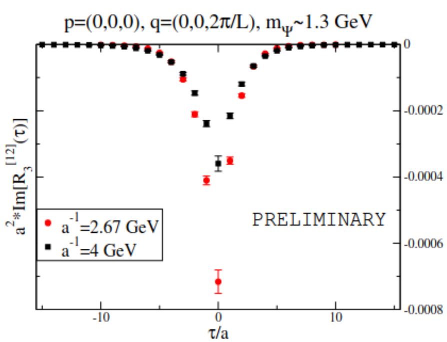
Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, Phys.Rev.D 105 (2022) 3, 034506

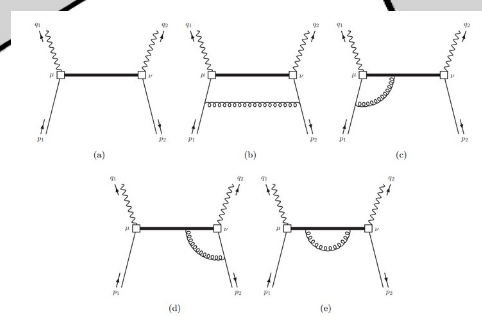
2021: First numerical result: Second Mellin moment







2018: First numerical study

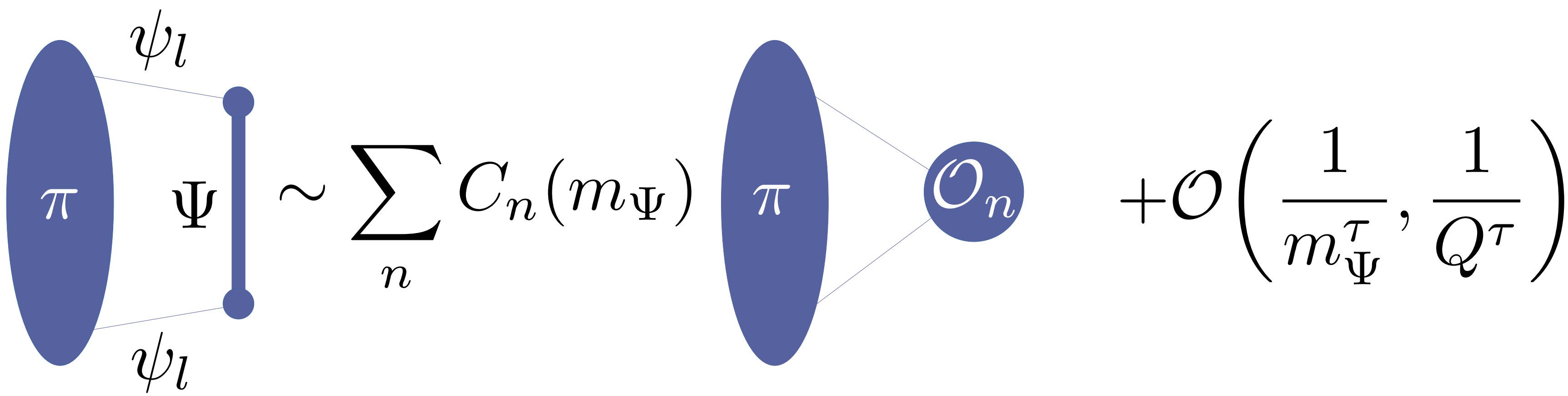


2021: Formal developments

Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, Phys.Rev.D 104 (2021) 7, 074511

Can we go beyond second moment?

Lattice calculation of fourth Mellin Moment of pion LCDA



Lattice Details

Quenched calculation: study continuum limit for higher moment

$m_\pi \approx 550\ \mathrm{MeV}$

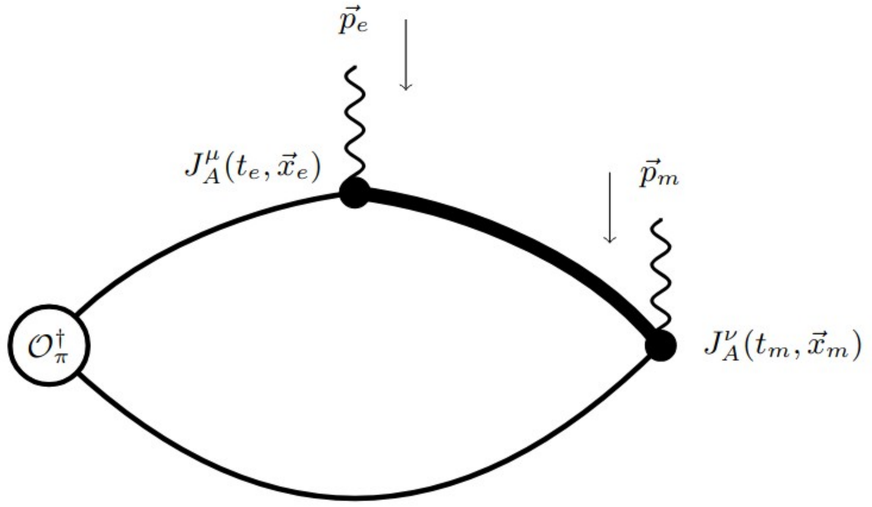
$(L/a)^3 \times T/a$	β	a (fm)	κ_l	κ_{H}	c_{sw}	N_{cfg}	N_{meas}
$24^3 \times 48$	6.10050	0.0813	0.134900	0.1300 0.1250 0.1200 0.1160 0.1100	1.6842	6550	6550
$32^3 \times 64$	6.30168	0.0600	0.135154	0.1320 0.1280 0.1250 0.1184 0.1130 0.1095	1.5792	7000	14000
$40^3 \times 80$	6.43306	0.0502	0.135145	0.1270 0.1217 0.1150	1.5292	250	10800
$48^3 \times 96$	6.59773	0.0407	0.135027	0.1285 0.1244 0.1192 0.1150	1.4797	341	10000

Correlators and Matrix Element

Three-point correlation function:

$$\mathcal{C}_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m) = \sum_{\vec{x}_e \in \Lambda} \sum_{\vec{x}_m \in \Lambda} e^{i\vec{p}_e \cdot \vec{x}_e} e^{i\vec{p}_m \cdot \vec{x}_m}$$

$$\times \langle 0 | T\{ J_A^\mu(t_e, \vec{x}_e) J_A^\nu(t_m, \vec{x}_m) \mathcal{O}_\pi^\dagger(\vec{0}, 0)\} | 0 \rangle$$



$$\lim_{t_+ \to \infty} \mathcal{C}_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m) = R^{\mu\nu}(t_-; \vec{p}, \vec{q}) \frac{Z_\pi(\vec{p})}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p}) t_+ / 2}$$

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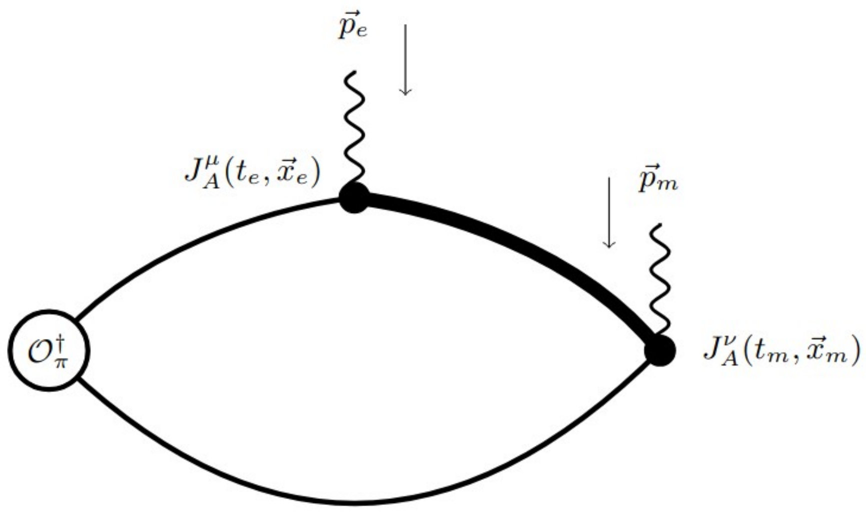

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Time-momentum current-current correlator

Correlators and Matrix Element

Three-point correlation function:

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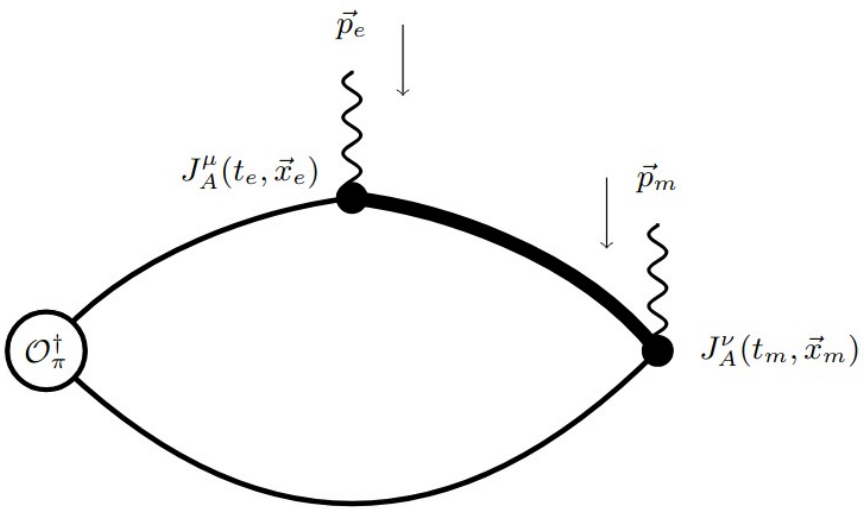

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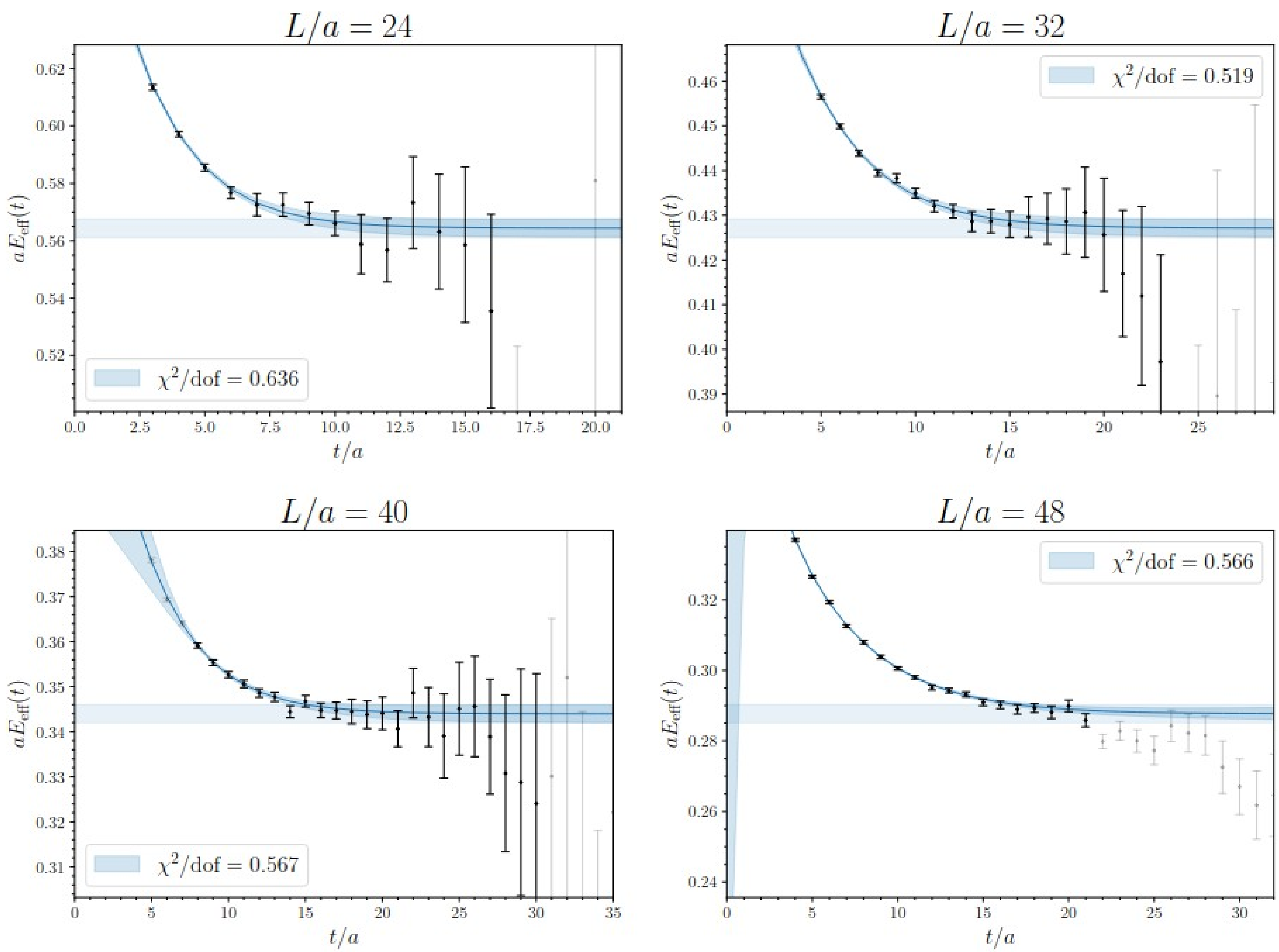
Spectroscopic factors

Spectroscopy

2x2 GEVP: $\mathcal{O}_1 = \overline{\psi}\gamma_5\psi, \qquad \mathcal{O}_2 = \overline{\psi}\gamma_4\gamma_5\psi,$

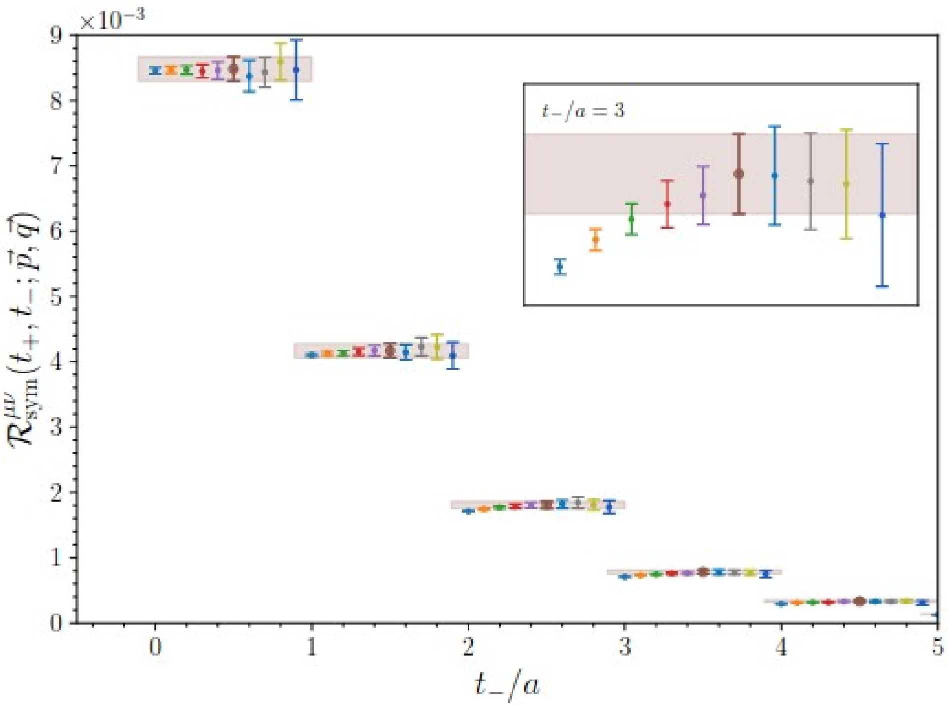


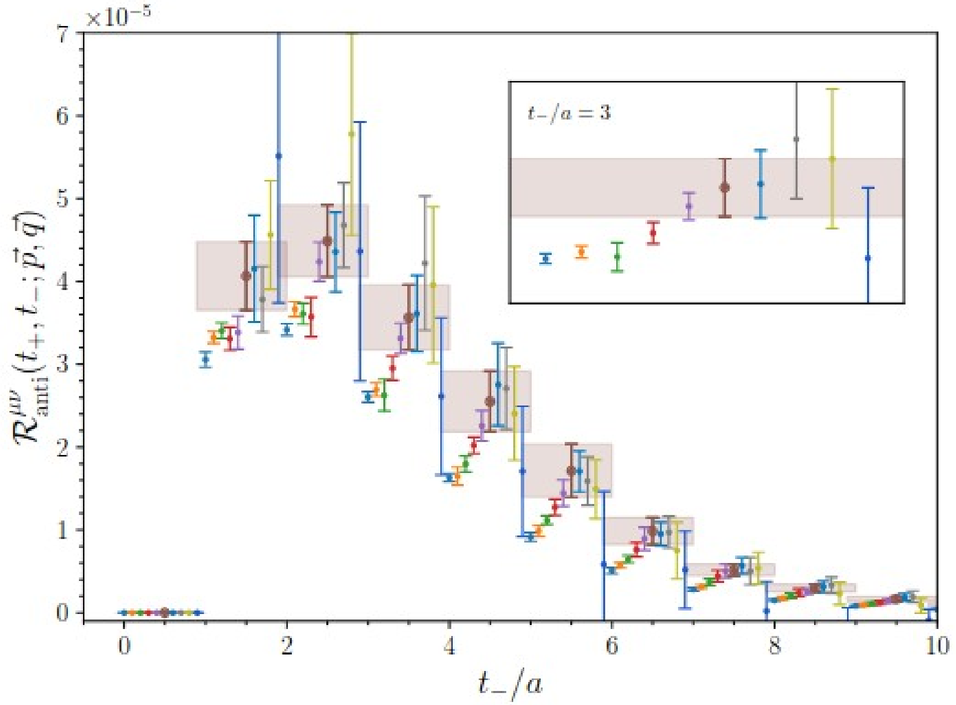
Excited States

Construct ratio of three-point and two-point data

$$\mathcal{R}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q}) \propto \left[\frac{Z_\pi(\vec{p})}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p})(t_e+t_m)/2}\right]^{-1} \times \mathcal{C}_3^{\mu\nu}(t_e, t_m; \vec{p}_e, \vec{p}_m)$$

Study excited state contamination





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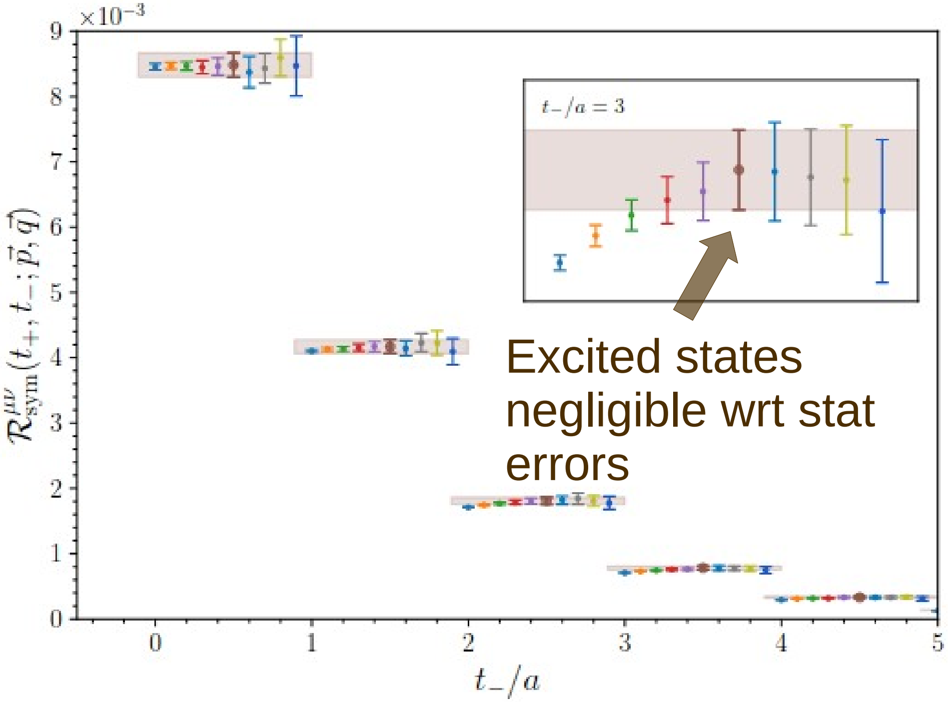
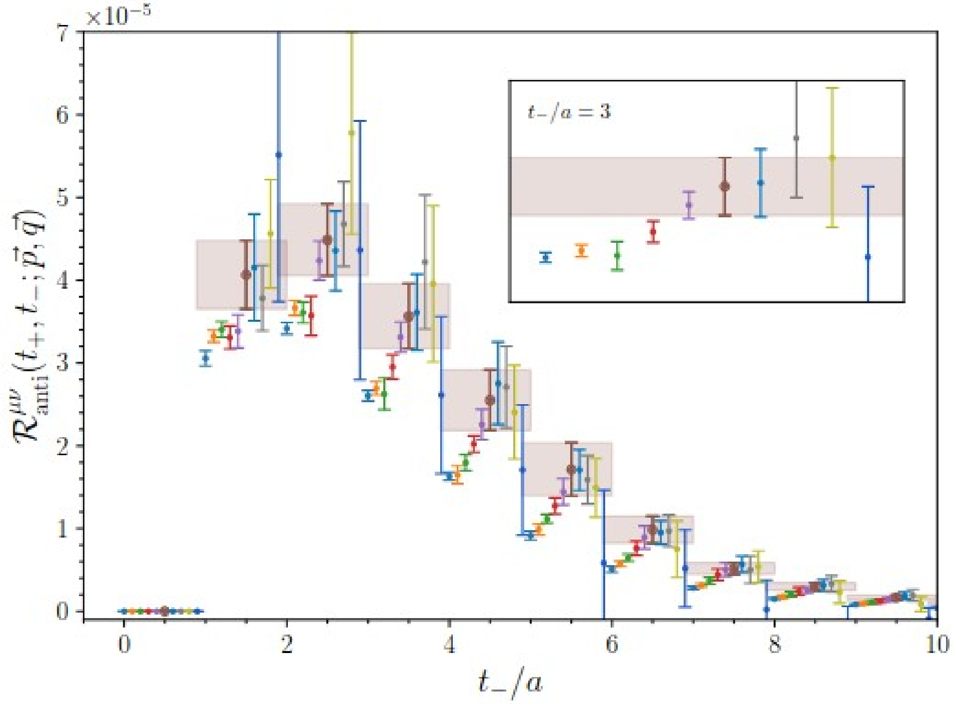
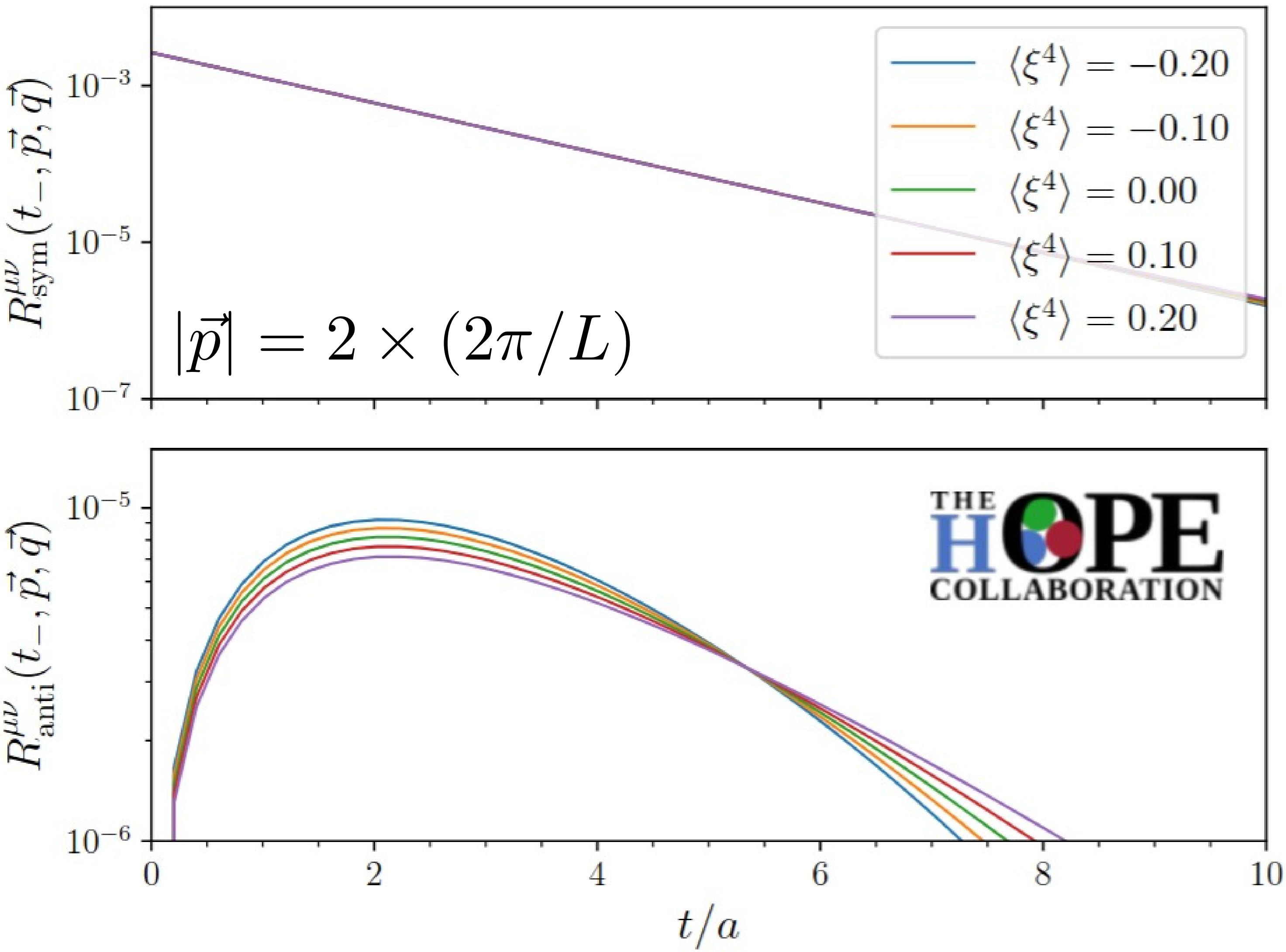




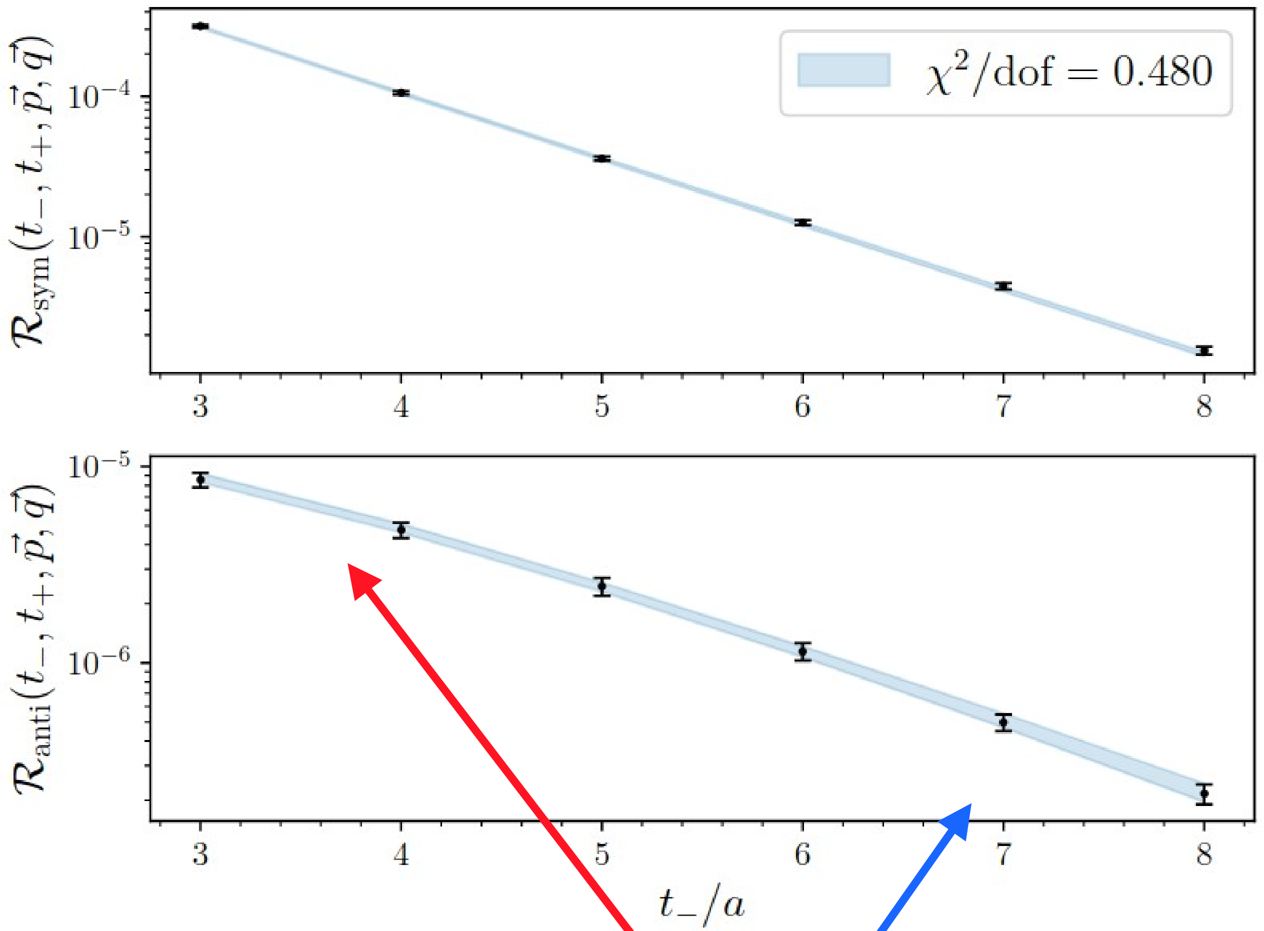
Sensitivity to fourth Mellin Moment

$$R^{\mu\nu}(t_-;\vec{p},\vec{q}) = \int \frac{dq_4}{(2\pi)} e^{-iq_4 t_-} V^{\mu\nu}(q,p).$$



The HOPE Ratio

$$\mathcal{R}^{\mu\nu}(t_+, t_-; \vec{p}, \vec{q}) = \mathcal{R}^{\mu\nu}_{\text{sym}}(t_+, t_-; \vec{p}, \vec{q}) + \mathcal{R}^{\mu\nu}_{\text{anti}}(t_+, t_-; \vec{p}, \vec{q})$$

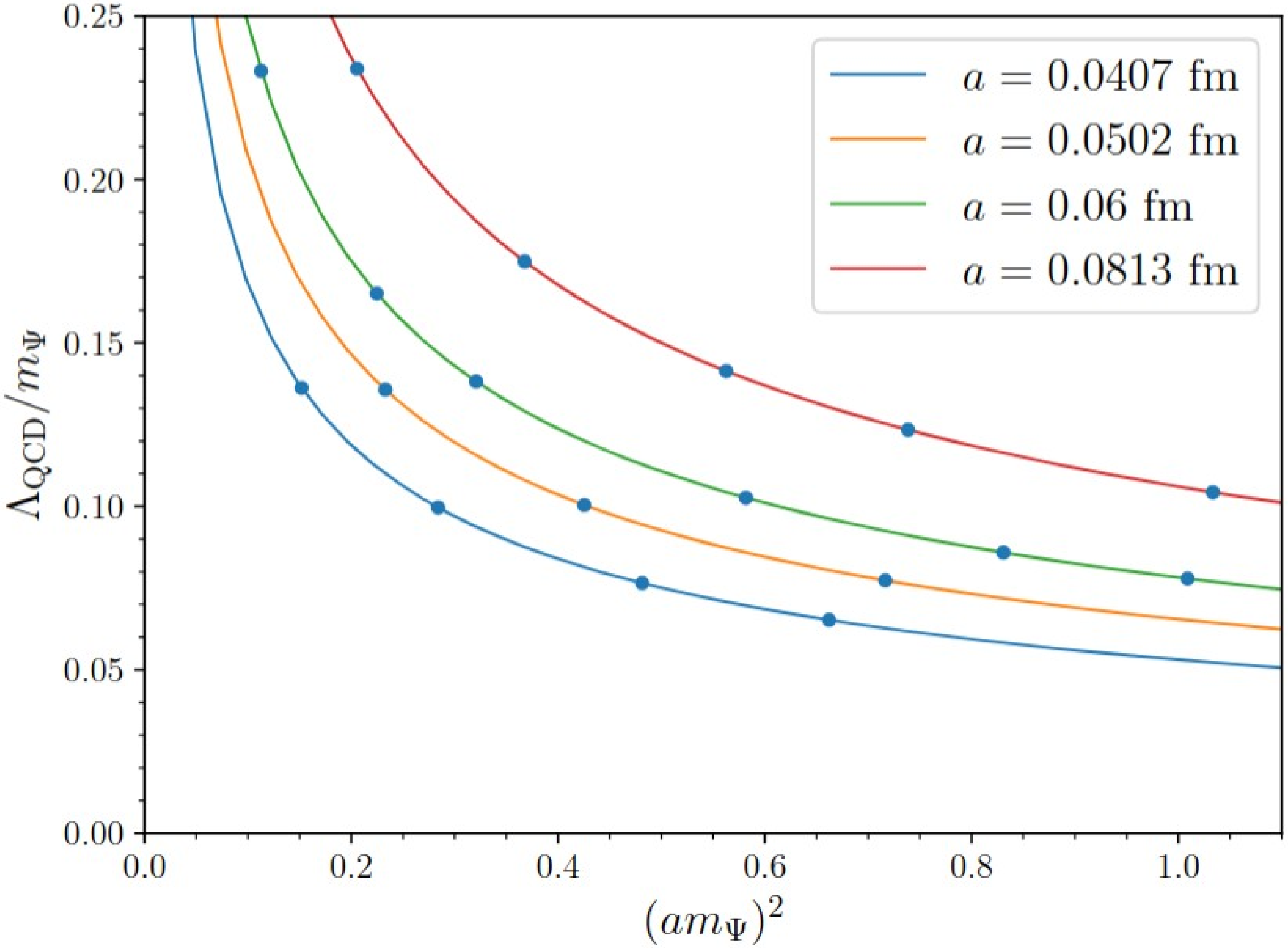


$$V^{\mu\nu}(p,q) = K^{\mu\nu}[1 + \langle \xi^2 \rangle \tilde{\omega}^2 + \langle \xi^4 \rangle \tilde{\omega}^4 + \ldots]$$

Choice of heavy quark mass

Vary heavy quark mass to study systematics

Higher-twist corrections

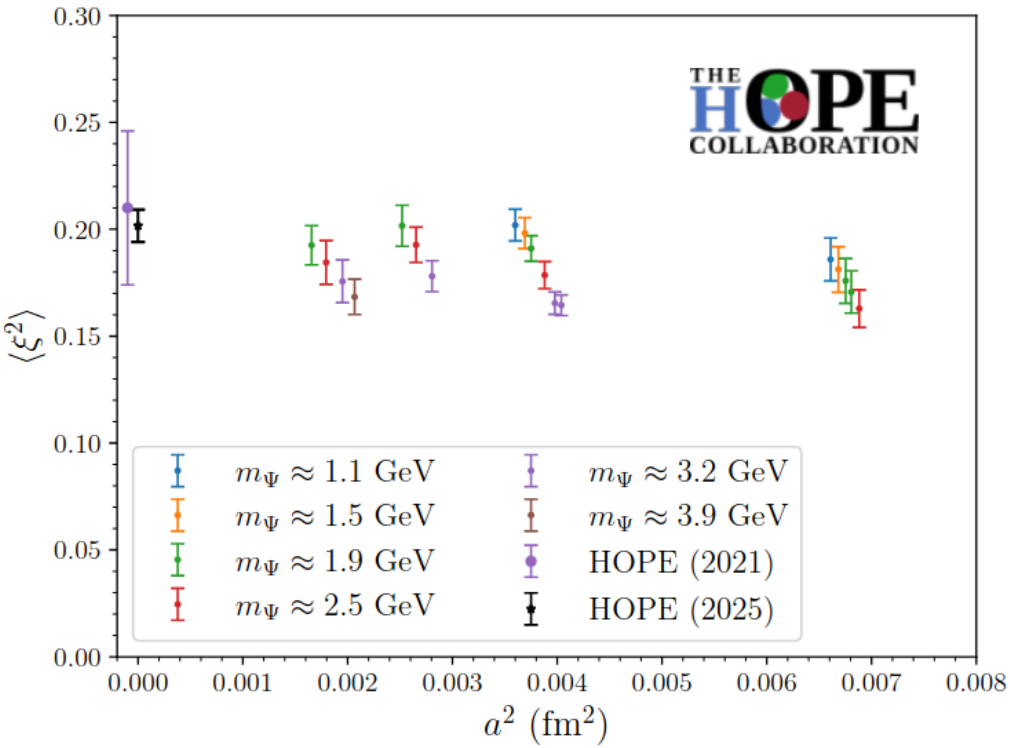


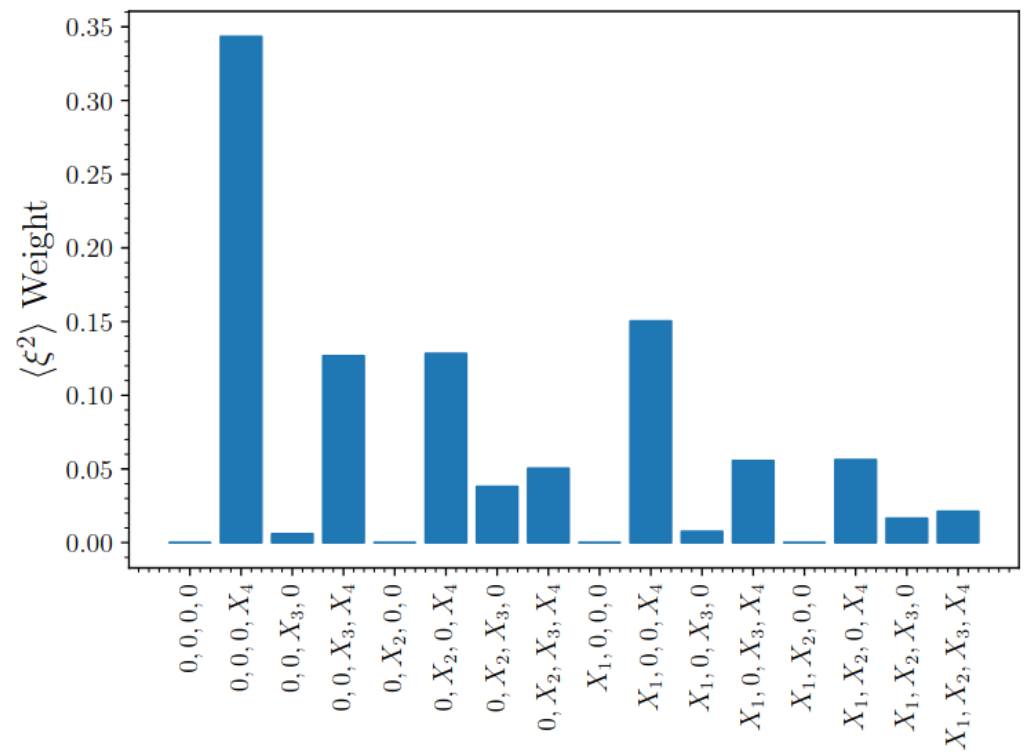
Lattice artifacts

Continuum, Twist-2 Extrapolation

Employ Bayesian Model Averaging Phys.Rev.D 103 (2021) 114502

$$X(a, m_\Psi) = X_0 + \frac{X_1}{m_\Psi} + X_2 a^2 + X_3 a^2 m_\Psi + X_4 a^2 m_\Psi^2$$





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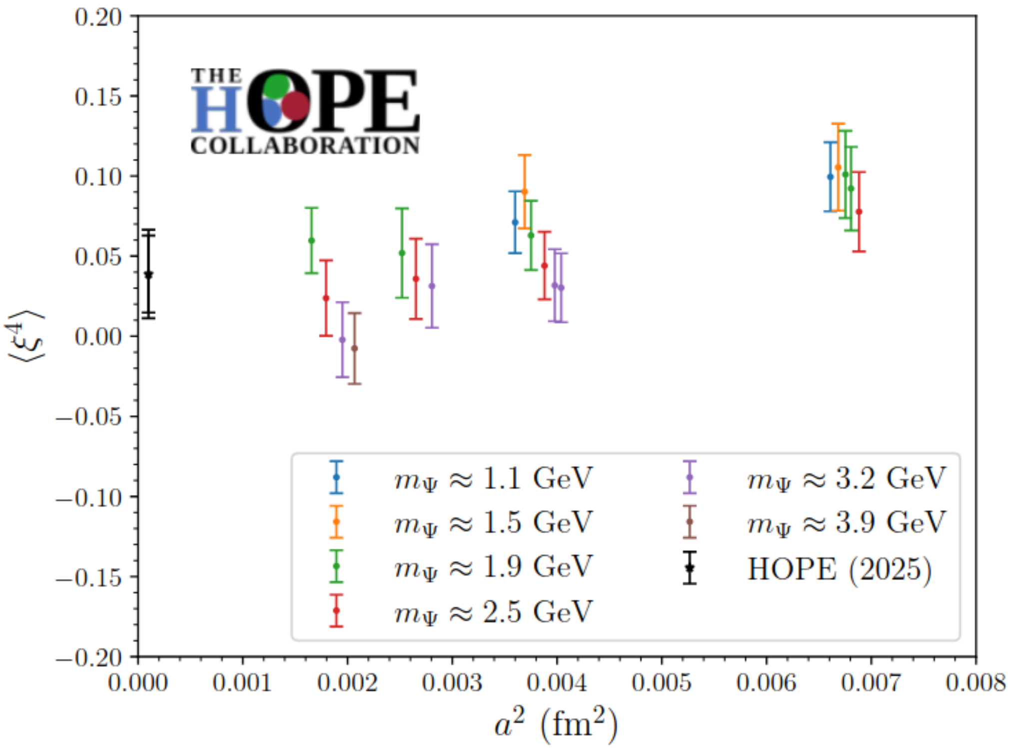
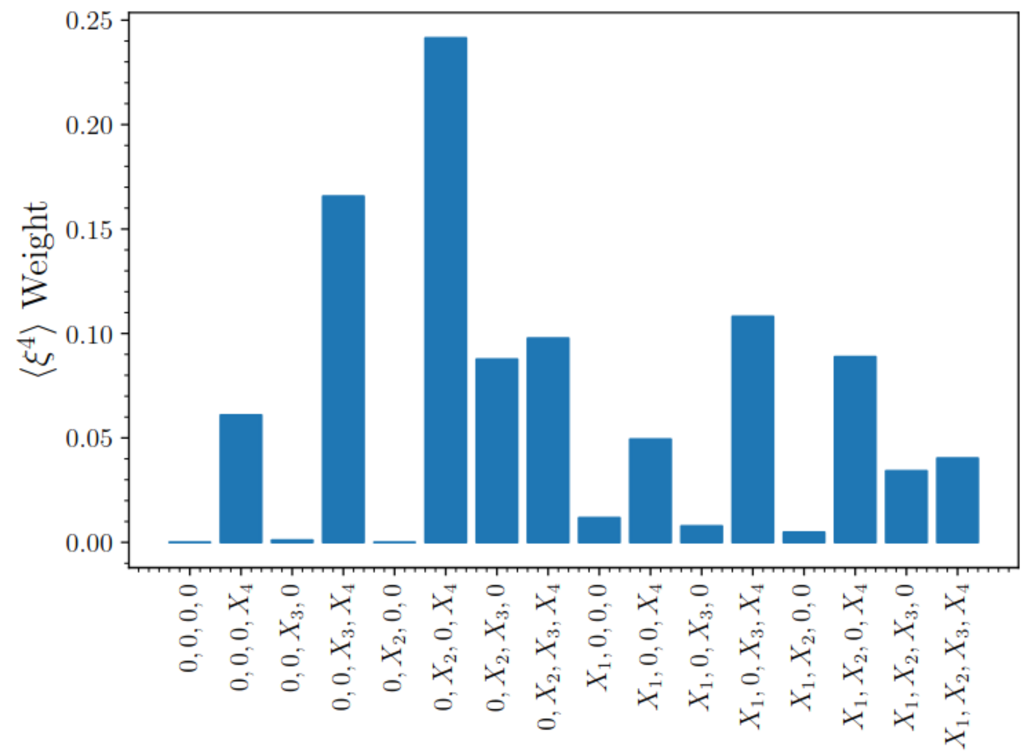
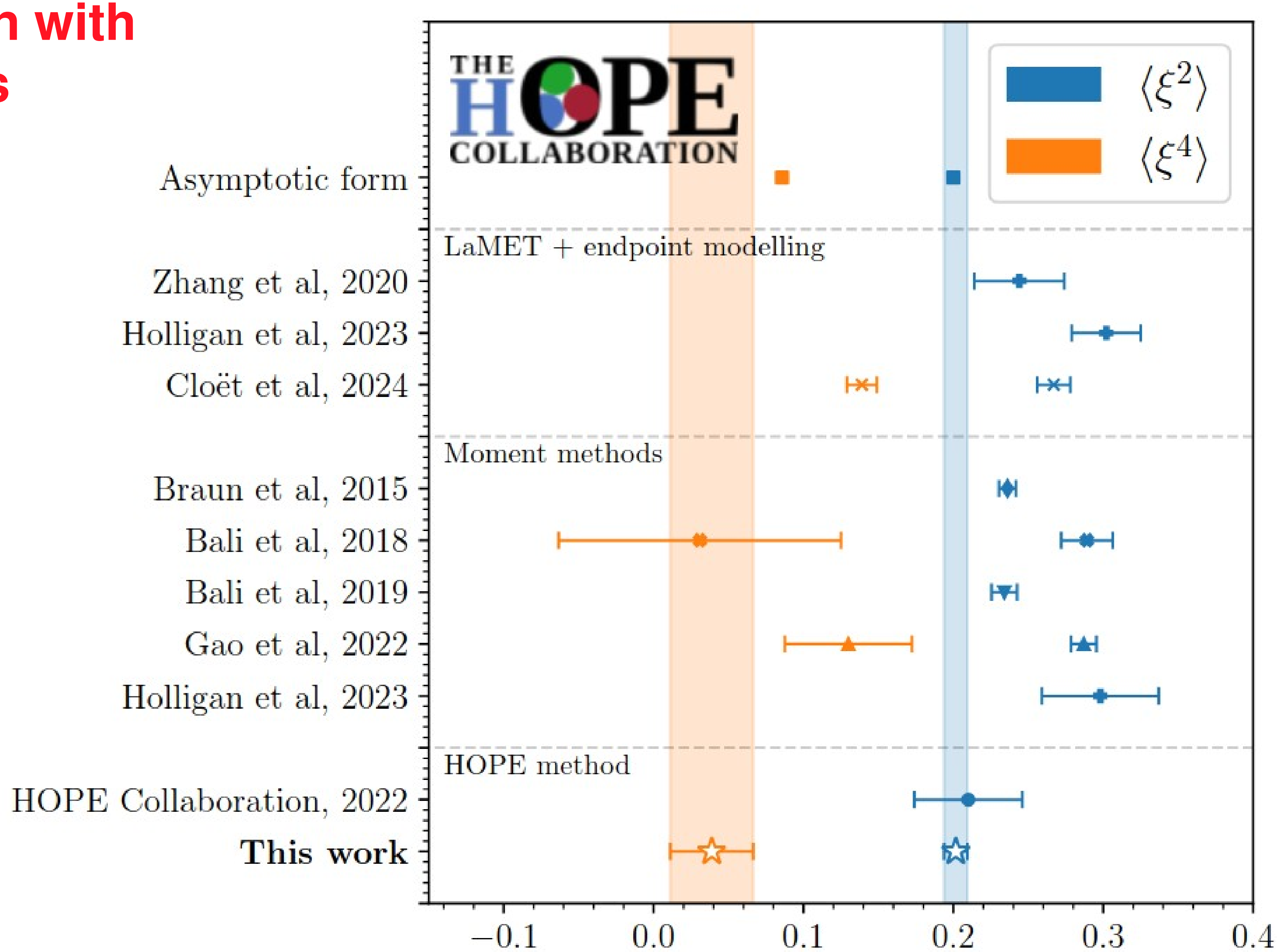




Comparison with other works



First (?) continuum limit determination of fourth Mellin moment of pion

Competitive with other moment methods

Systematic errors: Quenched approximation, pion mass.

The HOPE for controlled parton physics

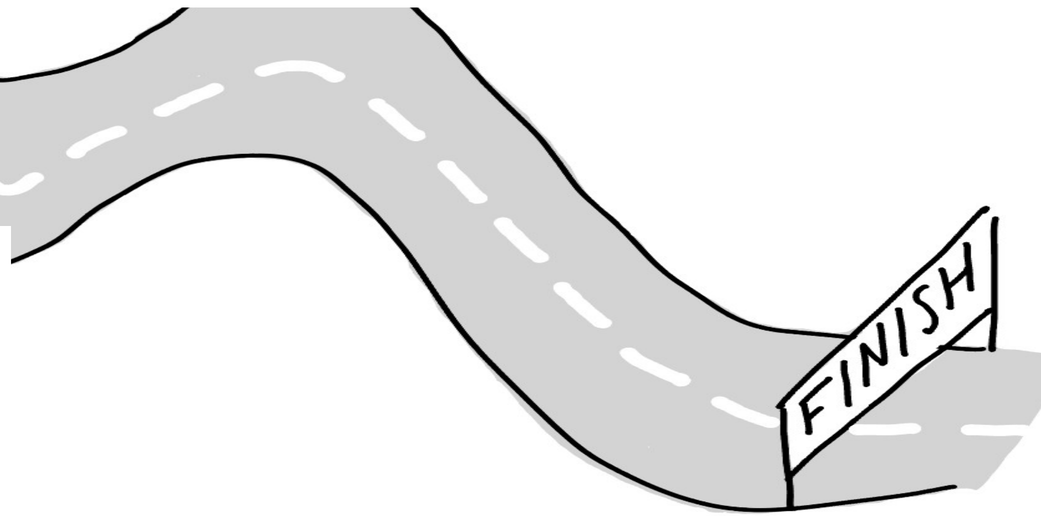
Phys.Rev.D 73 (2006) 014501

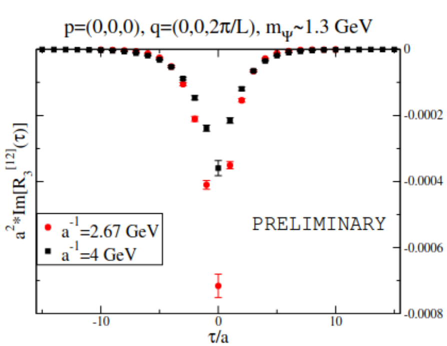
2006: First idea

PHYSICAL REVIEW D 73, 014501 (2006)

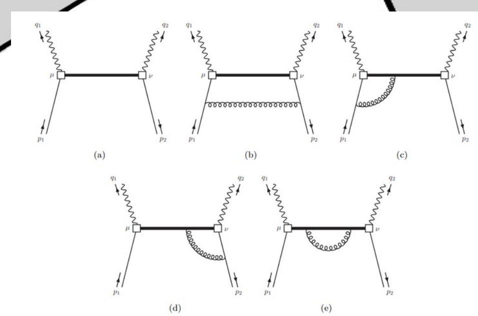
Deep-inelastic scattering and the operator product expansion in lattice QCD

William Detmold[1] and C.-J. David Lin[1,2]
[1]*Department of Physics, University of Washington, Box 351560, Seattle, Washington 98195, USA*
[2]*Institute for Nuclear Theory, University of Washington, Box 351550, Seattle, Washington 98195, USA*
(Received 21 July 2005; published 9 January 2006)





2018: First numerical study

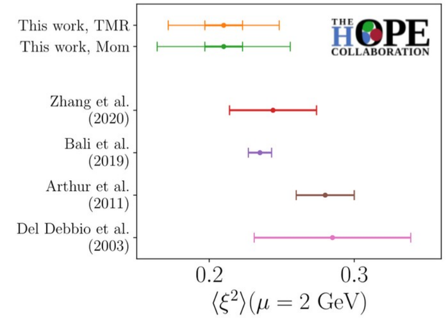


2021: Formal developments

Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, Phys.Rev.D 104 (2021) 7, 074511

Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, Phys.Rev.D 105 (2022) 3, 034506

2021: First numerical result: Second Mellin moment



The HOPE for controlled parton physics

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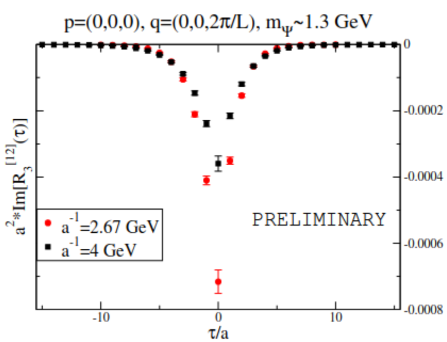

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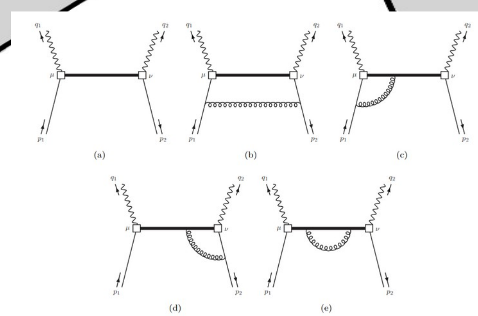

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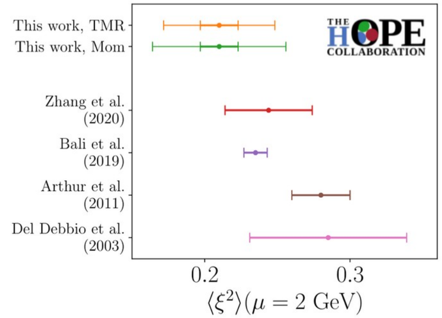

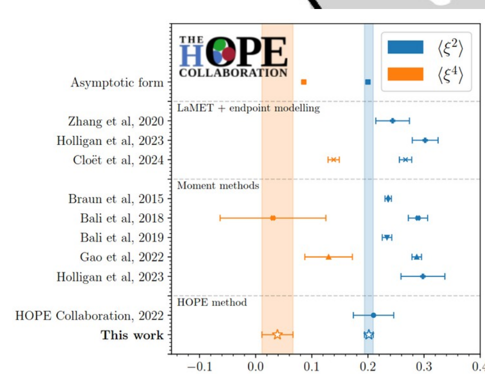

2025: Continuum limit of higher moments

Detmold, Grebe, Kanamori, Lin, **RJP**, Zhao, arXiv:2509.04799 [hep-lat]

Conclusions

HOPE provides an alternate approach to parton physics

Continuum limit of higher moments possible

Determined second and fourth Mellin moment in continuum:

$$\langle \xi^2 \rangle \, (\mu = 2 \text{ GeV}) = 0.202(8)(9),$$
$$\langle \xi^4 \rangle \, (\mu = 2 \text{ GeV}) = 0.039(28)(11).$$

Up next: Dynamical calculations + PDF

Sheng Pin Chang: Thursday 5:10 PM - 5:30 PM: Dynamical calculation of kaon LCDA

EIC: access pion form factor via Sullivan process

Scatter electron against pion cloud of hadron

Analytically continue amplitude to pion pole: residue proportional to $F_\pi(q^2)$