Bayesian Inference of Dense Matter Equation of State from Heavy-Ion Reactions and Neutron Stars

Bao-An Li

East Texas A&M University

Formerly known as Texas A&M University-Commerce (2024-1996), East Texas State University/College (1996-1957), East Texas State/Normal College (1957-1889)

Office of Science
U.S. DEPARTMENT OF ENERGY

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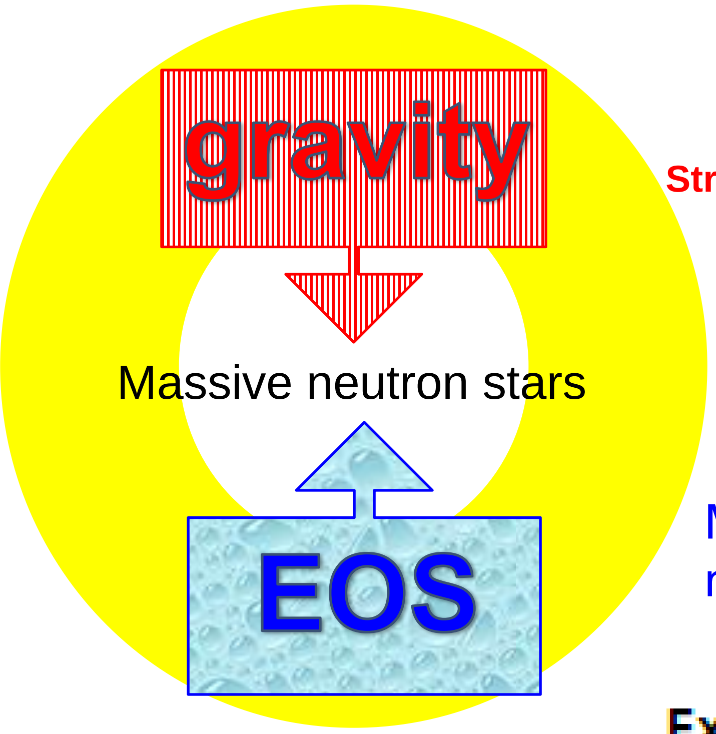
Gravity-EOS degeneracy

Even perfect data cannot fully disentangle gravity and EOS



2020 Astronomy
& Astrophysics
Decadal Survey

arXiv:1903.09221v3

Hamilton's variational principle

Strong-field gravity: Einstein's General Relativity (GR) or Modified Gravity

+ 5th force?

Total Action $S = S_{gravity} + S_{matter}$ + their couplings

Matter [nucleons, baryon resonances, hyperons, mesons, leptons,…] or +[Dark Matter] + [X17]?

Extreme Gravity and Fundamental Physics

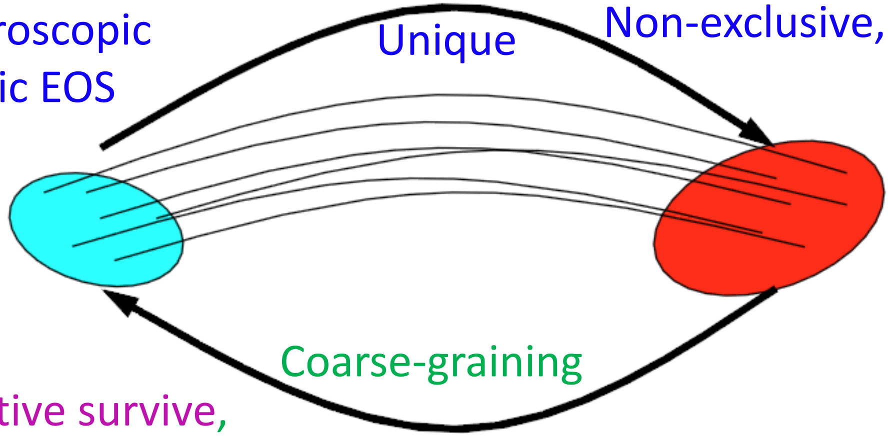
- **The nature of gravity.** Can we prove Einstein wrong? What building-block principles and symmetries in nature invoked in the description of gravity can be challenged?
- **The nature of dark matter.** Is dark matter composed of particles, dark objects or modifications of gravitational interactions?
- **The nature of compact objects.** Are black holes and neutron stars the only astrophysical extreme compact objects in the Universe? What is the equation of state of densest matter?

Forward-modeling versus backward inference

A chosen microscopic or macroscopic EOS

Non-exclusive, possible realizations consistent with experimental data

Parameter space



Experimental observation space

Masses, radii, tidal deformations of neutron stars

P(ϵ) and its derivative survive,
NOT a unique underlying Lagrangian or EOS model

$$
\begin{aligned}
\mathcal{L} =& \sum_{\alpha=n,p} \bar{\psi}_\alpha \Big(\gamma^\mu \Big(i\partial_\mu - g\omega\omega_\mu - \frac{1}{2} g\rho \tau_\alpha \cdot \rho_\mu\Big) - (M - g\sigma\sigma) - g\delta\tau_\alpha \cdot \delta\Big)\psi_\alpha \\
&+\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}m_\sigma^2\sigma^2 + \frac{\zeta_0}{4!}g^2\omega(\omega_\mu\omega^\mu)^2 - g\sigma m_\sigma^2 M\frac{k_3}{3!} + \frac{k_4}{4!}\frac{g\sigma}{M}\sigma^3 \\
&+\frac{1}{2}m_\omega^2\omega_\mu\omega^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\frac{g\sigma\sigma}{M}\Big(\eta_1 + \frac{\eta_2}{2}\frac{g\sigma\sigma}{M}\Big)m_\omega^2\omega_\mu\omega^\mu \\
&+\frac{1}{2}\eta_\rho m_\rho^2\frac{g\sigma\sigma}{M}(\rho_\mu\cdot\rho^\mu) + \frac{1}{2}m_\rho^2(\rho_\mu\cdot\rho^\mu) - \frac{1}{4}R_{\mu\nu}R^{\mu\nu} - \Lambda_\omega g^2\omega g^2\rho(\omega_\mu\omega^\mu)(\rho_\mu\cdot\rho^\mu) \\
&+\frac{1}{2}\partial_\mu\delta\partial^\mu\delta - \frac{1}{2}m_\delta^2\delta^2
\end{aligned}
$$

Tolman-Oppenheimer-Volkoff (TOV) equations

$$
\frac{dP}{dr} = \underbrace{-\frac{GM\varepsilon}{r^2}}_{\text{Newtonian}} \underbrace{\overbrace{\left(1+\frac{P}{\varepsilon c^2}\right)}^{\text{matter correction}} \overbrace{\left(1+\frac{4\pi r^3 P}{Mc^2}\right)}^{\text{matter-geometry coupling}} \underbrace{\left(1-\frac{2GM}{rc^2}\right)^{-1}}_{\text{geometry correction}}}_{\text{matter+geometry corrections: } \gg 1}; \quad \overbrace{\frac{dM}{dr} = 4\pi r^2\varepsilon/c^2}^{\text{same as Newtonian}}
$$

Composition blindness/degeneracy of TOV equations: Regardless how and what compositions are used, as long as the same EOS $P(\varepsilon)$ is used, the same mass-radius sequence is obtained→ EOS $P(\varepsilon)$ is necessary and sufficient to get the M and R but NOT sufficient to understand NS physics unambiguously.

Degeneracy → Universality or consistent patterns under vastly different conditions

Outline

P(M): Some new, possibly crazy, but educated thoughts on the EOS P(ϵ) of cold dense matter

Bayes' theorem: $P(M|D) \propto P(D|M)P(M)$

A mathematically consistent way to update our believe about model parameters given new data

Burn-in steps: why EOS matters

What is EOS? Why is it important? What have we learned about cold dense matter EOS?

Why are we still working on it? What else can be learned? Do I have anything new to say?

P(D|M): χ^2-based Likelihood to convince you by trying millions of times (MCMC)

Your expert opinion or data D

Data

My thought or model prediction

$$P[D|\mathcal{M}(K_0, J_0, Z_0)] = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_{D,j}} \exp\left[-\frac{(P_{\mathrm{th},j} - P_{D,j})^2}{2\sigma_{D,j}^2}\right]$$

EOS Model parameters

Degree of democracy or combined errors

Using physics principles, theoretical arguments, model simulations, emulations and experimental data

P(M|D): Posterior thoughts about EOS with quantified uncertainties

What is the EOS?

- **Relation among P, ε, T, δ, μ**
- **Symmetry energy is a key uncertainty**

Empirical parabolic law of the EOS of cold, neutron-rich nucleonic matter

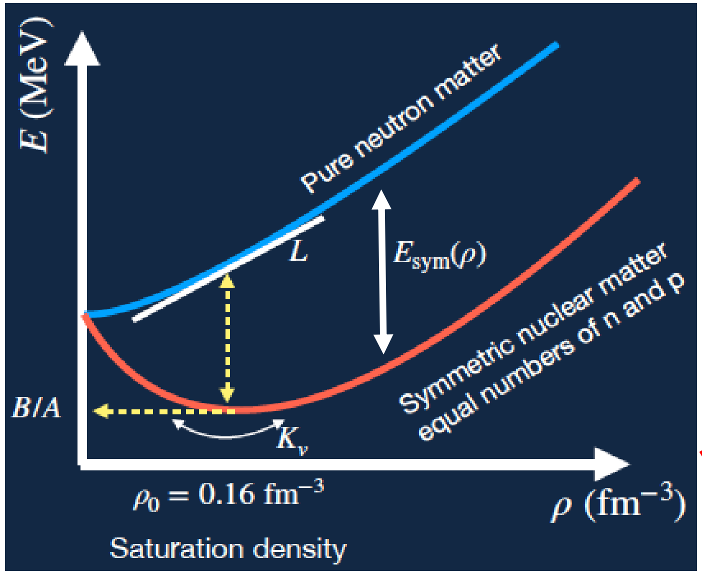
symmetry energy

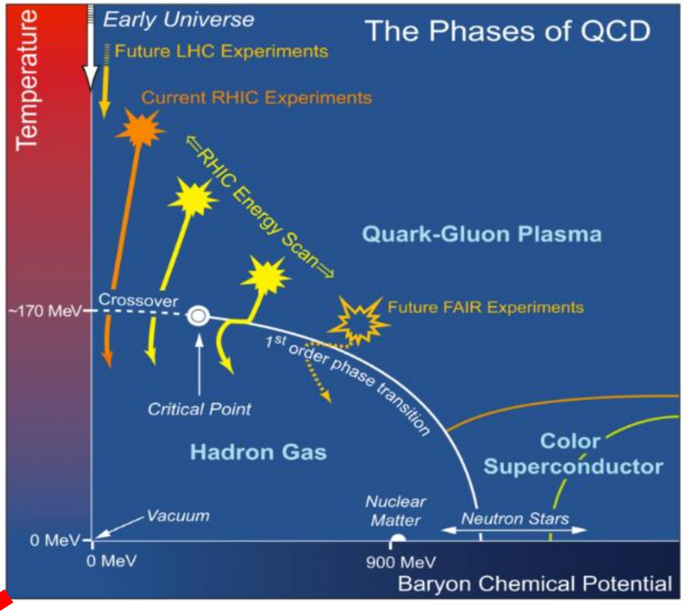
Isospin asymmetry δ

$$E(\rho_n,\rho_p) = E_0(\rho_n=\rho_p) + E_{sym}(\rho)\left(\frac{\rho_n-\rho_p}{\rho}\right)^2 + o(\delta^4)$$

Energy per nucleon in symmetric matter

Energy in asymmetric nucleonic matter





New opportunities

Isospin chemical potential
$\mu_I = E_{sym}(\rho) \cdot \delta$ in n-rich matter

Structures and collisions of heavy nuclei
Structures and mergers of neutron stars

Empirical parameterizations useful for meta-modeling (template) of EOS

incompressibility

$$E_0(\rho) = E_0(\rho_0) + \frac{K_0}{2}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^2 + \frac{J_0}{6}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^3 + \frac{Z_0}{24}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^4,$$

Relatively well-known

Poorly known

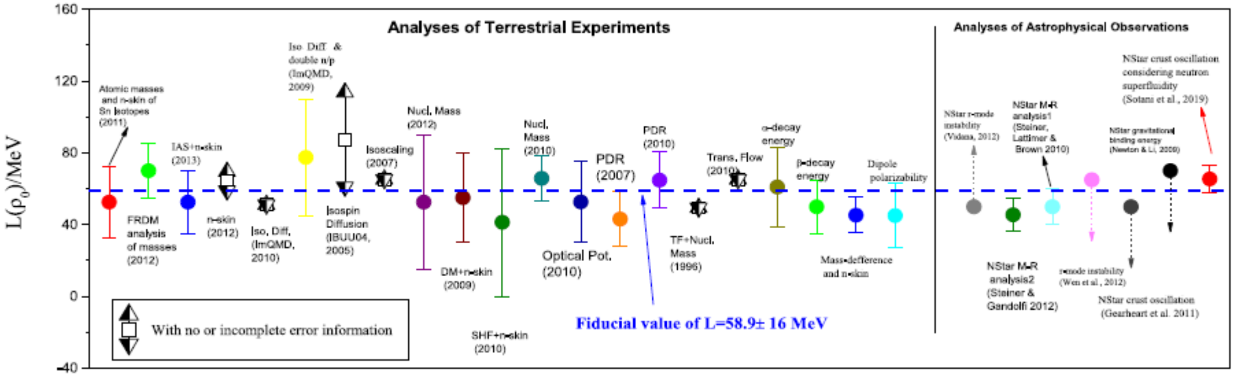
$$E_{sym}(\rho) = E_{sym}(\rho_0) + \frac{L}{3}\left(\frac{\rho}{\rho_0}-1\right) + \frac{K_{sym}}{18}\left(\frac{\rho}{\rho_0}-1\right)^2 + \frac{J_{sym}}{162}\left(\frac{\rho}{\rho_0}-1\right)^3 + \mathcal{O}\left[\left(\frac{\rho}{\rho_0}-1\right)^4\right]$$

slope | curvature | skewness | kurtosis

- Expansion near saturation density
- Uncertainties grow at high density

Prior on L: Broad theoretical range L ≈ 60 ± 30 MeV with a few out-standing points

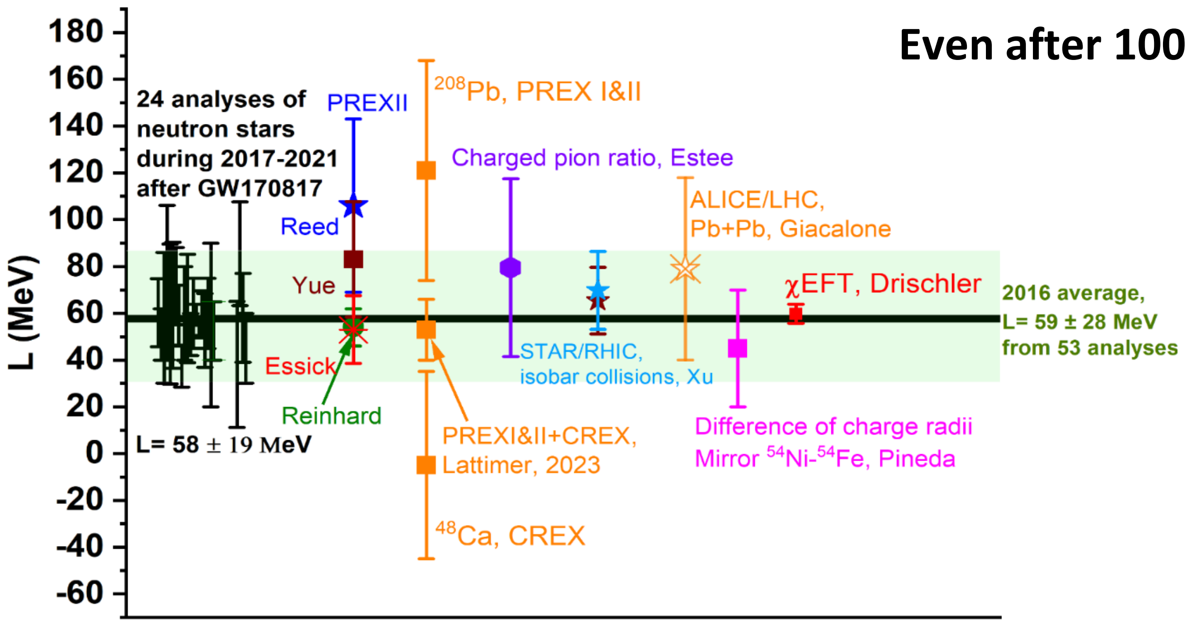
As of 2013, ~30 analyses



Bao-An Li and Xiao Han, Phys. Lett. B727 (2013) 276

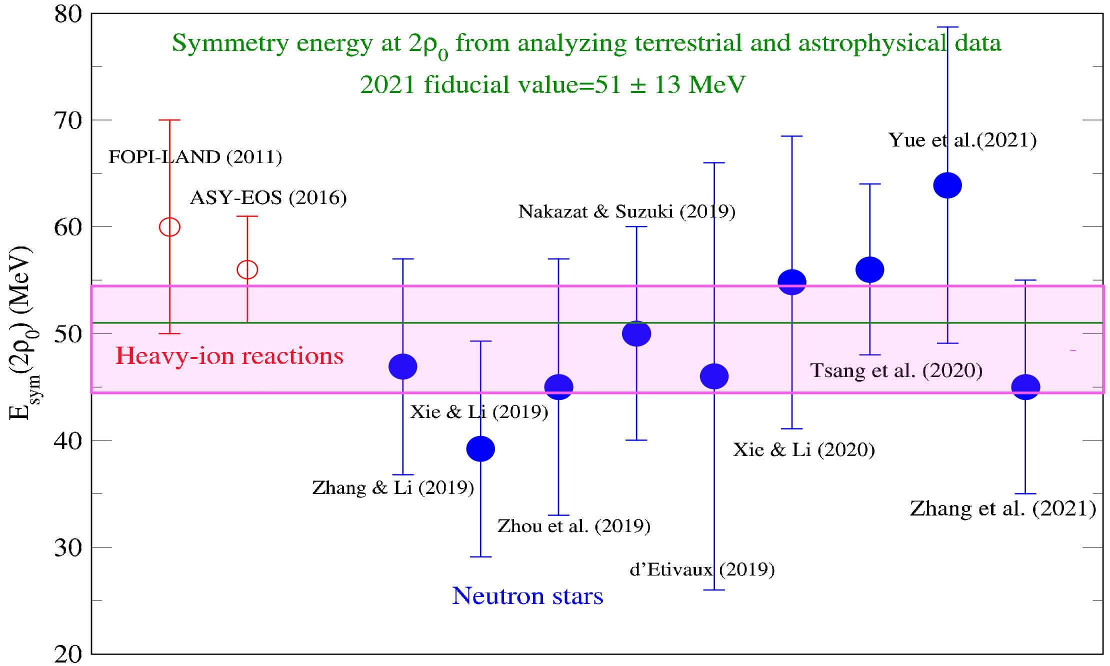
Slope L of nuclear symmetry energy as of 2023

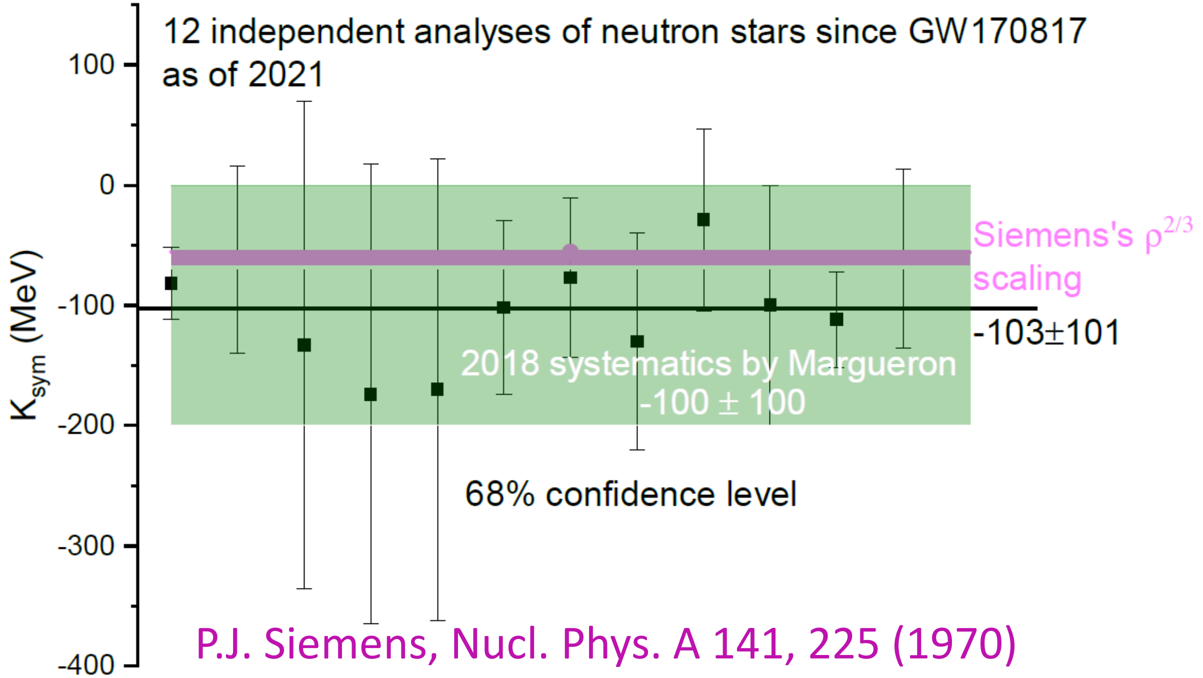
Even after 100 analyses, L remains broadly uncertain



Nai-Bo Zhang and Bao-An Li, EPJA 59, 86 (2023)

Prior on $E_{sym}(2\rho_0)$ and K_{sym}: Despite many studies, constraints remain broad





Examples of recent theoretical predictions for $E_{sym}(2\rho_0)$:

(1) Chiral EFT, $E_{\rm sym}(2\rho_0) \approx 45 \pm 3\ {\rm MeV}$

C. Drischler, R. J. Furnstahl, J. A. Melendez, and D. R. Phillips, PRL**125**, 202702 (2020)

(2) Quantum Monte Carlo, $E_{\rm sym}(2\rho_0) \approx 46 \pm 4$

D. Lonardoni, I. Tews, S. Gandolfi, and J. Carlson, Phys. Rev. Research **2**, 022033(R) (2020)

(3) Relativistic BHF in full Dirac space: 51.6 MeV

Sibo Wang, Hui Tong, Qiang Zhao, Chencan Wang, Peter Ring, Jie Meng, PRC 106 (2022) 2, L021305

(4) Relativistic BHF: ~ 53 MeV

Chencan Wang, Jinniu Hu, Ying Zhang, Hong Shen, *Chin. Phys. C* 46 (2022) 6, 064108

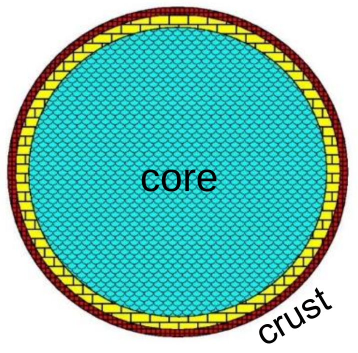
Bao-An Li, Euro Phys. Journal Special Topic (2026)

Symmetry energy controls composition and affects pressure in neutron stars

(1) The proton (electron) fraction $x(Y_e)$ is determined by the $E_{sym}(\rho)$ through charge neutrality and beta-equilibrium conditions:

$$x = 0.048[E_{sym}(\rho)/E_{sym}(\rho_0)]^3(\rho/\rho_0)(1-2x)^3$$

Critical for the cooling mechanism of protoneutron stars and associated neutrino emissions, appearance of hyperons, kaon condensation, baryon resonances…..



(2) The pressure in the npe matter at beta equilibrium:

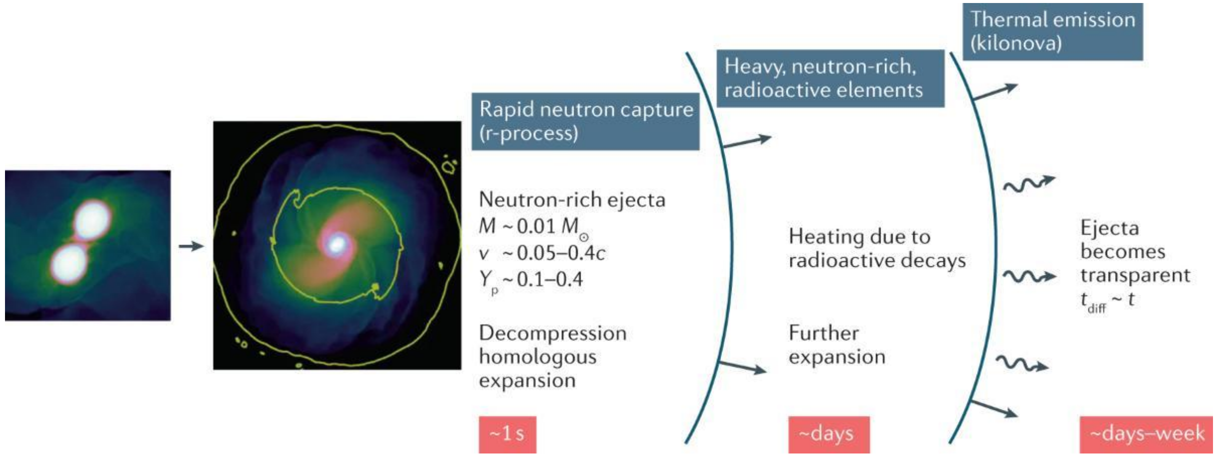
$$P(\rho,\delta) = \rho^2[\frac{dE_0(\rho)}{d\rho} + \frac{dE_{\rm sym}(\rho)}{d\rho}\delta^2] + \frac{1}{2}\delta(1-\delta)\rho E_{\rm sym}(\rho)$$

(3) The crust-core transition density and pressure is determined by setting the incompressibility of neutron star matter =0 (speed of sound becomes imaginary):

$$K_\mu = \rho^2\frac{d^2E_0}{d\rho^2} + 2\rho\frac{dE_0}{d\rho} + \delta^2\left[\rho^2\frac{d^2E_{sym}}{d\rho^2} + 2\rho\frac{dE_{sym}}{d\rho} - 2E_{sym}^{-1}(\rho\frac{dE_{sym}}{d\rho})^2\right] = 0$$

Lattimer & Prakash, Phys. Rep., 442, 109 (2007)

EOS in NS mergers → dynamical electron fraction Y_e +neutrino flux → r-process → Kilonova



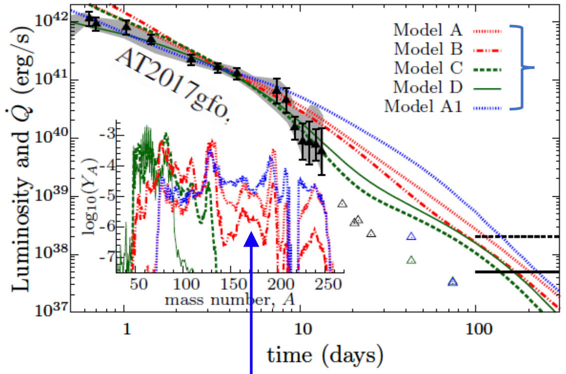
D. Radice et al., APJ. 869, 130 (2018).

B.D. Metzger and R. Fernandez, MNRAS 441, 3444 (2014)

J. Lippuner and L.F. Roberts, APJ 815, 82 (2015)

Fingerprints of Heavy-Element Nucleosynthesis in the Late-Time Lightcurves of Kilonovae PRL 122, 062701 (2019)

Meng-Ru Wu,[1,2,*] J. Barnes,[3,†] G. Martínez-Pinedo,[4,5,‡] and B. D. Metzger[3,§]



With different Y_e means and distributions

Lanthanides are only produced in ejecta with low electron fraction, $Y_e \lesssim 0.25$

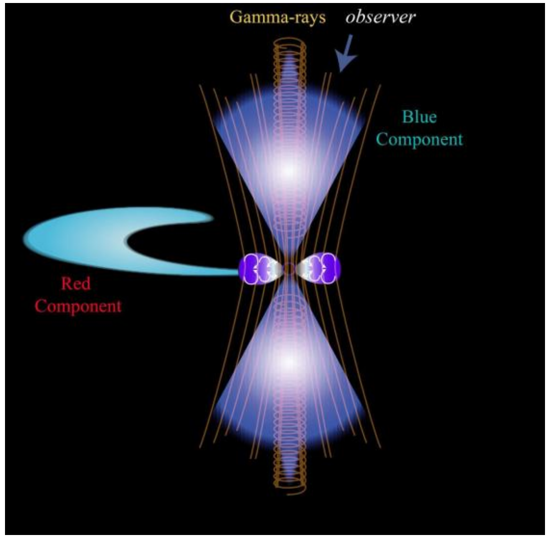
- EOS (E_{sym}) sets neutrino field: L_{ν_e}, $L_{\bar{\nu}_e}$, $\langle E_\nu \rangle$
- Wind is NOT in β-equilibrium

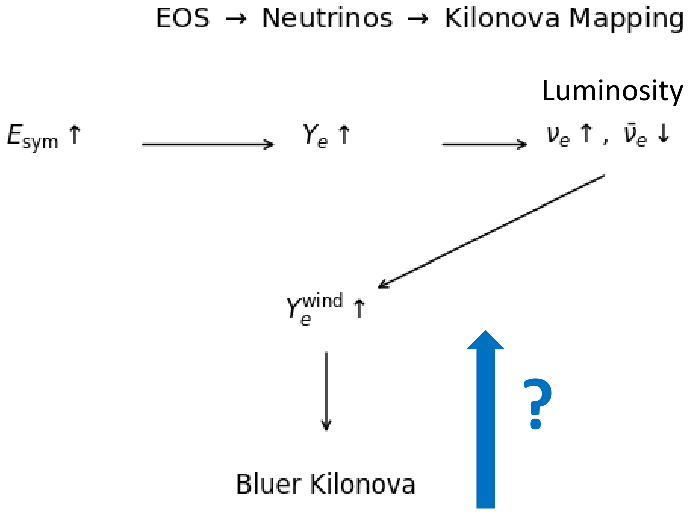
→ Y_e(neutrino-driven wind) set by rate balance:

$$\nu_e + n \rightleftharpoons p + e^-,$$
$$\bar{\nu}_e + p \rightleftharpoons n + e^+.$$

$$Y_e^{\rm wind} \approx \left[1 + \frac{L_{\bar{\nu}_e}\langle E^2_{\bar{\nu}_e}\rangle}{L_{\nu_e}\langle E^2_{\nu_e}\rangle}\right]^{-1}$$

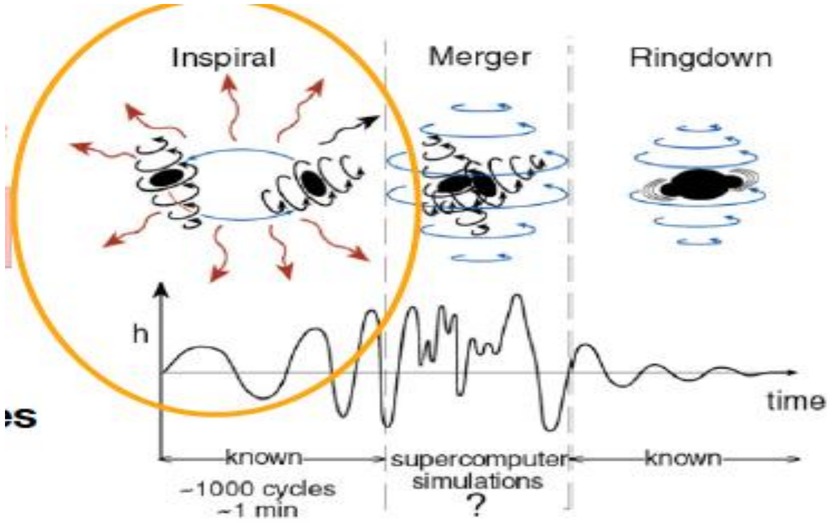
Y.-Z. Qian and S. E. Woosley, ApJ 471, 331 (1996).

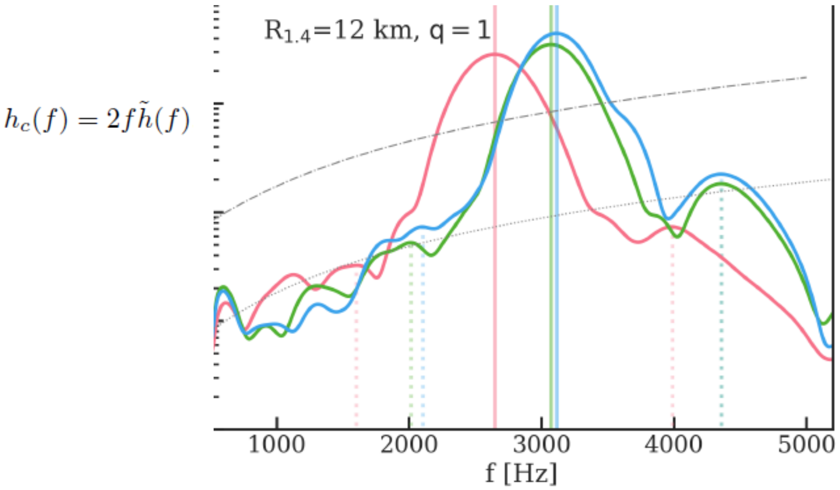


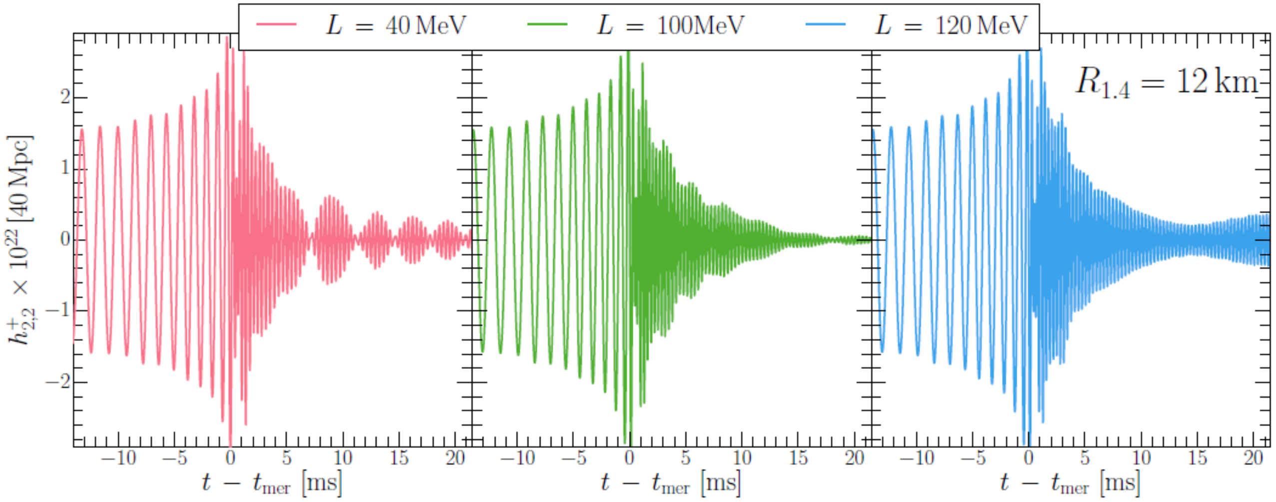


Impact of nuclear symmetry energy on the post-merger phase of a binary neutron star coalescence

Elias R. Most and Carolyn A. Raithel, PRD **104**, 124012 (2021)







The peak positon depends on L

Need to measure high-frequency GWs

nature communications

Article https://doi.org/10.1038/s41467-023-43932-6

An updated nuclear-physics and multi-messenger astrophysics framework for binary neutron star mergers

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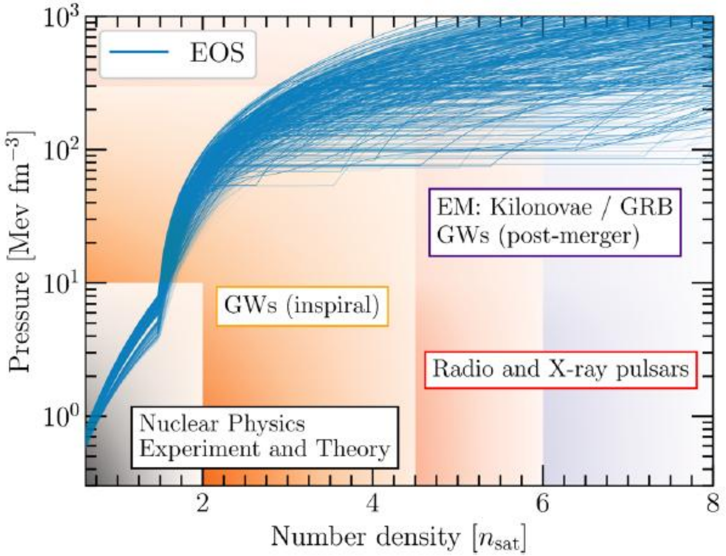
Check for updates

Peter T. H. Pang[1,2], Tim Dietrich[3,4], Michael W. Coughlin[5], Mattia Bulla[6,7,8,9], Ingo Tews[10], Mouza Almualla[11], Tyler Barna[5], Ramodgwendé Weizmann Kiendrebeogo[12,13], Nina Kunert[3], Gargi Mansingh[5,14], Brandon Reed[5,15], Niharika Sravan[16], Andrew Toivonen[5], Sarah Antier[13], Robert O. VandenBerg[5], Jack Heinzel[17], Vsevolod Nedora[4], Pouyan Salehi[3], Ritwik Sharma[18], Rahul Somasundaram[10,19,20] & Chris Van Den Broeck[1,2]



EOS=QMC+multi-segments of constant speed of sound at high densities

M. Bulla's transport code POSSIS for Kilonovae, MNRAS 489, 5037 (2019).

as a first attempt of analyzing the gravitational-wave signal, the kilonova, and the gamma-ray burst afterglow simultaneously. Incorporating all available information, we estimate the radius of a $1.4M_\odot$ neutron star to be $R = 11.98^{+0.35}_{-0.40}$ km.

LIGO/VIRGO; $R_{1.4}$=11.9±0.87 at 68% CFL

Bayesian inference of EOS from neutron star observables

Model → Data → Posterior

Generating randomly within

Prior ranges of hadronic EOS

Parameters	Lower limit	Upper limit (MeV)
K_0	220	260
J_0	-800	400
$K_{\rm sym}$	-400	100
$J_{\rm sym}$	-200	800
L	30	90
$E_{\rm sym}(\rho_0)$	28.5	34.9

$$K_0,\ J_0\ ,L,\ K_{sym},\ J_{sym}\ ,\ E_{sym}(\rho_0)$$

Additional parameters
for phase transition and quark matter

NS model

$P_t \geq 0$

$v_s < c$

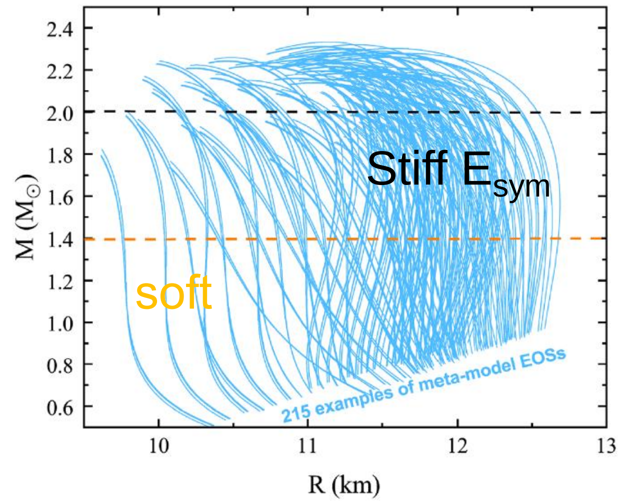
NS EOS

$$\frac{dP}{dr} = -\underbrace{\frac{GM\varepsilon}{r^2}}_{\text{Newtonian}}\overbrace{\left(1+\frac{P}{\varepsilon c^2}\right)}^{\text{matter correction}}\overbrace{\left(1+\frac{4\pi r^3 P}{Mc^2}\right)}^{\text{matter-geometry coupling}}\underbrace{\left(1-\frac{2GM}{rc^2}\right)^{-1}}_{\text{geometry correction}};\quad \overbrace{\frac{dM}{dr}=4\pi r^2\varepsilon/c^2}^{\text{same as Newtonian}}$$

matter+geometry corrections: $\gg 1$

TOV equations

$M_{max} \geq 1.97 M_{sun}$

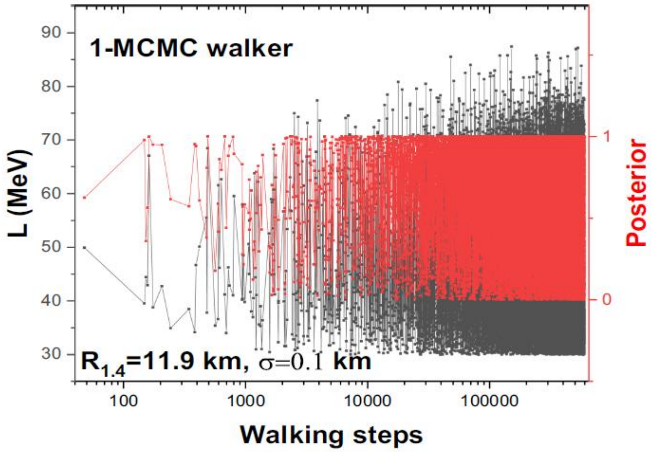


R-M relation

$$\prod_{j=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_{{\rm obs},j}}\exp\left[-\frac{(R_{{\rm th},j}-R_{{\rm obs},j})^2}{2\sigma^2_{{\rm obs},j}}\right]$$

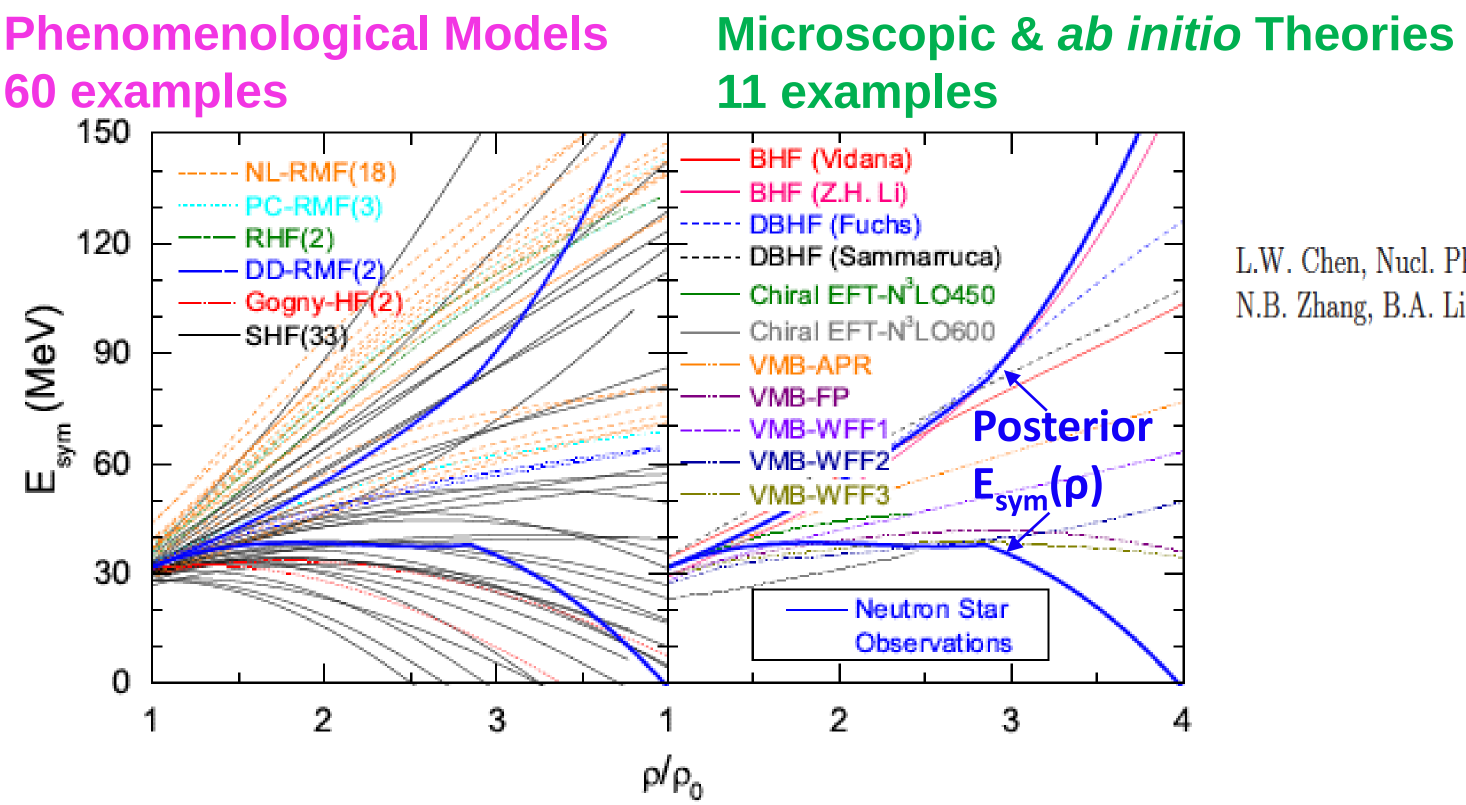
Likelihood function
(prediction vs data)

Markov Chain Monte Carlo (MCMC)



P(D|M)

Predictions diverge at high density



L.W. Chen, Nucl. Phys. Rev. 34, 20 (2017).
N.B. Zhang, B.A. Li, Eur. Phys. J. A 55, 39 (2019).

Why is the high-density symmetry energy still so uncertain?

- Isospin-spin dependent Short-Range Correlations induced by tensor forces, many-body interactions
- New particles and possible phase transitions are important but poorly known

Why did the available neutron star data not help much at high densities?

- R is obtained from P®=0 at the surface, corresponding to an average density of about $2\rho_0$ where P_{sym} is larger than or compatible with $P_{SNM,}$ depending mostly on K_{sym} and L

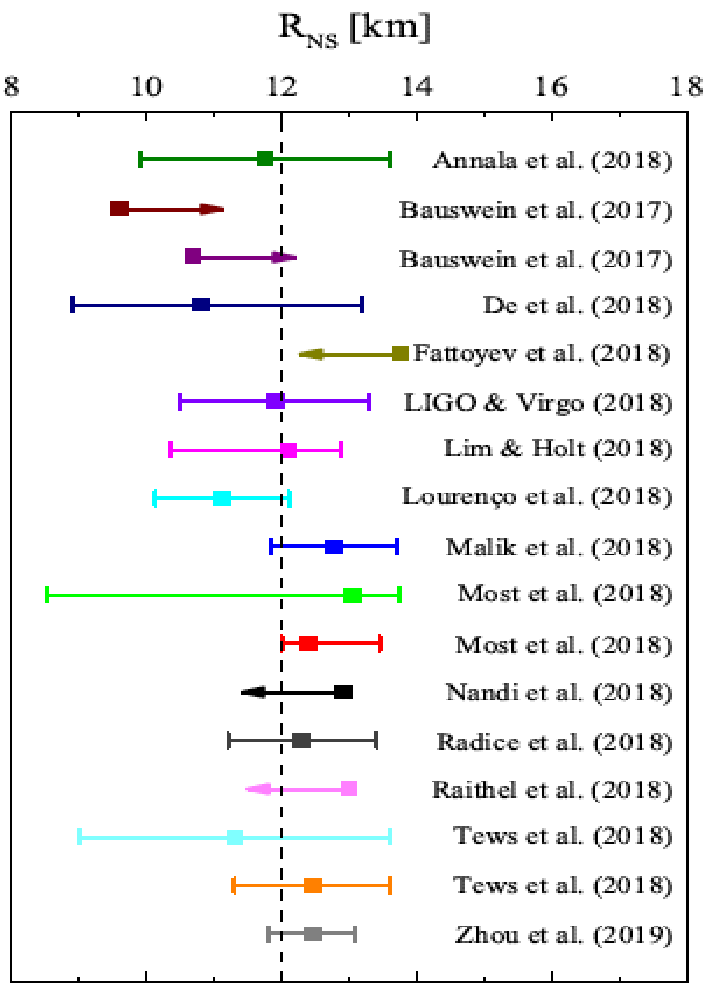
Current precision of neutron star radius measurements σ_R~1.0 km

Gravitational wave results

LIGO/VIRGO for GW170817: $R_{1.4} = 11.9 \pm 0.875$ km

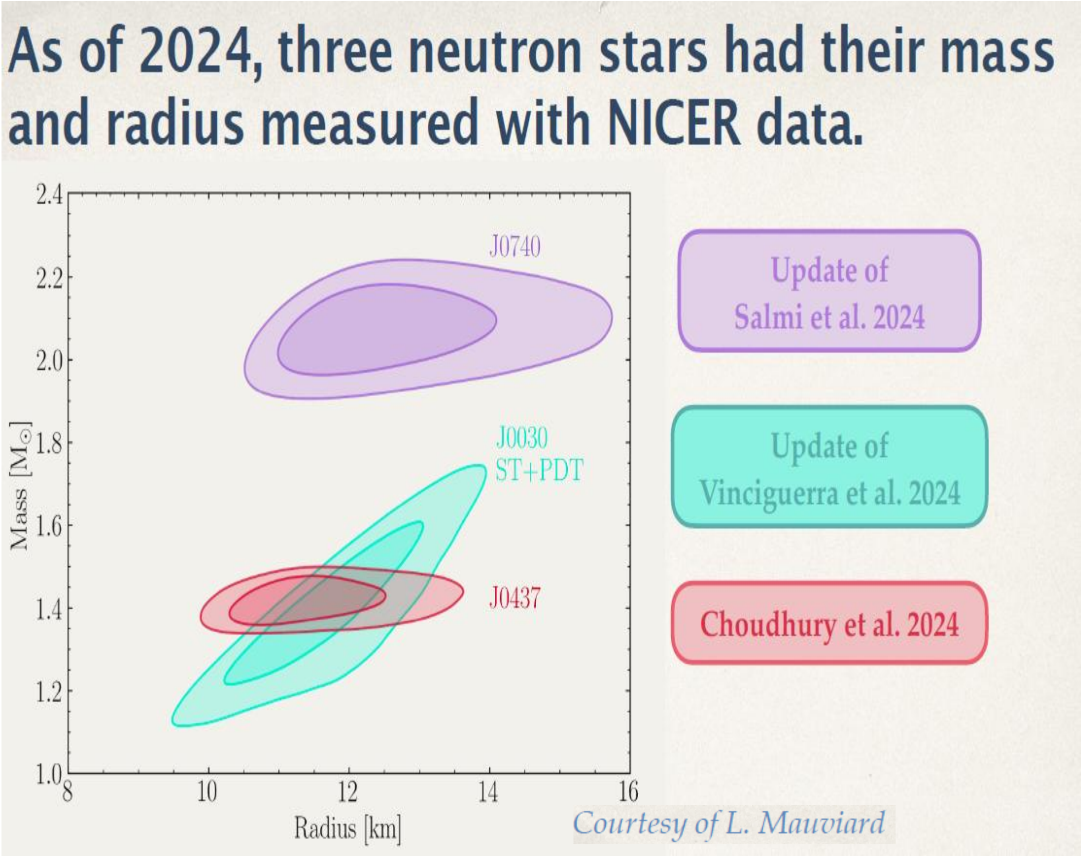
Various analyses during 2-years after GW170817

$R_{1.4} = 12.0 \pm 1.13$



B.A. Li et al., EPJA 55, 117 (2019)

X-ray results

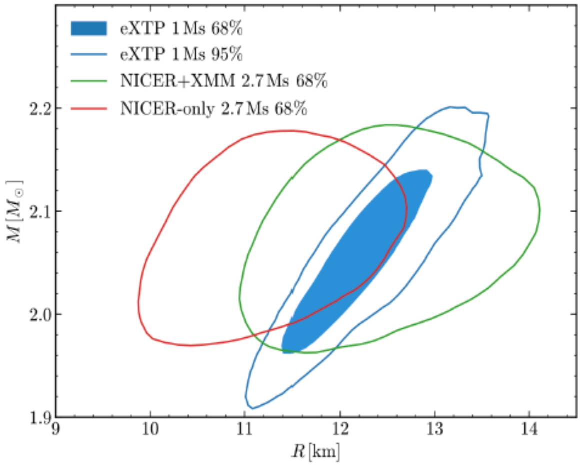


Future promise

eXTP: The enhanced X-ray Timing and Polarimetry mission

+/- 6% precision on radius

PSR J0740+6620



S. N. Zhang et al., arXiv:1812.04020v1

Ang Li et al. 2506.08104 (2025 update)

NewATHENA

Advanced Telescope for High Energy Astrophysics

+/- 3% precision on radius

- **Radius 1-sigma uncertainties**
  - NICER 1600 ksec: ~10%
  - ATHENA 500 ksec: ~3% average (±0.3 km)

The NewAthena mission concept in the context of the next decade of X-ray Astronomy, Nature Astronomy 9, 36 (2025)

Gravitational wave precision forecasts

Precision constraints on the neutron star equation of state with third-generation gravitational-wave observatories

Kris Walker[1,2,*] Rory Smith[3,†] Eric Thrane[2,3] and Daniel J. Reardon[4,5]

PHYSICAL REVIEW D **110**, 043013 (2024)

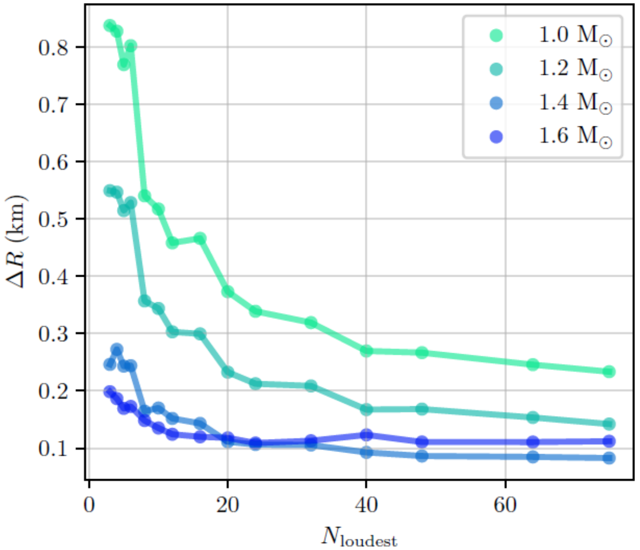
THE ASTROPHYSICAL JOURNAL, 955:45 (8pp), 2023 September 20 https://doi.org/10.3847/1538-4357/acf12f

Prospects for a Precise Equation of State Measurement from Advanced LIGO and Cosmic Explorer

Daniel Finstad[1,2,3], Laurel V. White[4], and Duncan A. Brown[4]

Cosmic Explorer
+
Einstein Telescope

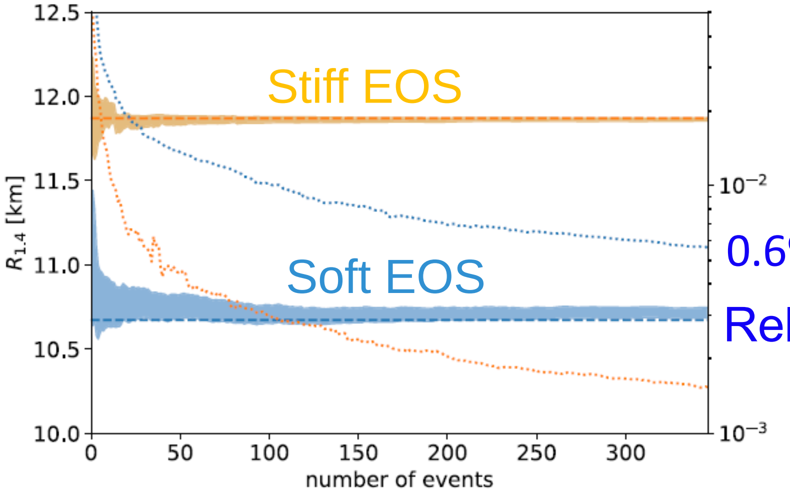
1-year joint operation



0.1 km

Number of loudest NS mergers

1-year Cosmic Explorer operation



0.6%

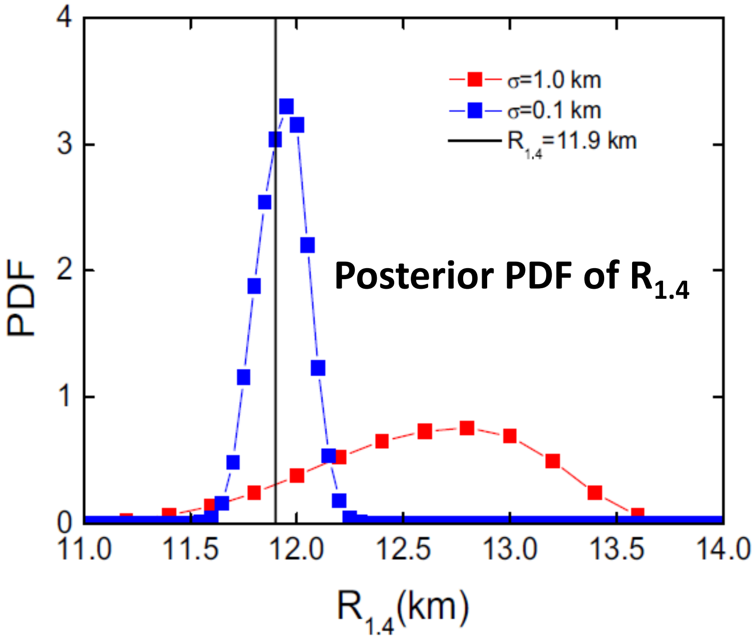
Relative error

The proposed high-precision measurements of neutron star radii are SUPER-expensive in Time & Efforts

What do we expect high-precision R to constrain?

Bao-An Li, Xavier Grundler, Wen-Jie Xie, Nai-Bo Zhang, PRD 110, 103040 (2024)

TOV equations are highly nonlinear
→ Nonlinear filtering bias



Nonlinearity + uncertainty = systematic inference shift

Jensen decision bias (economics, utility theory of money, statistics)

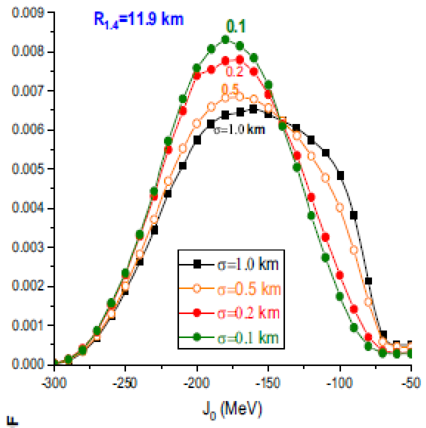
Noise-induced drift (physics)

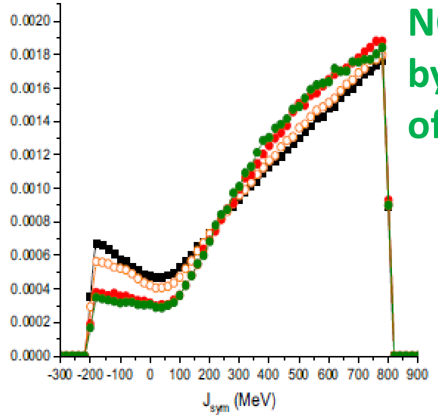
Radius mainly probes EOS ~2ρ_0

Skewness J_0 of symmetric nuclear matter

Skewness J_{sym} of symmetry energy

Constrained by M_{TOV} & casualty

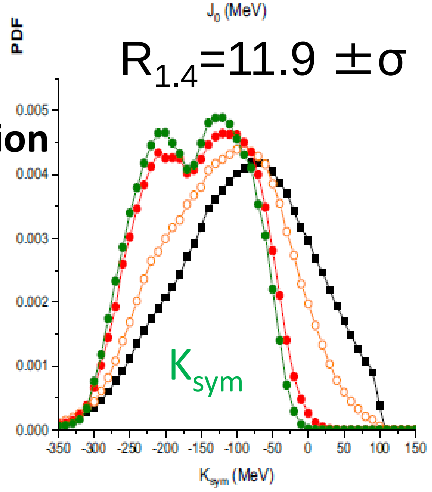


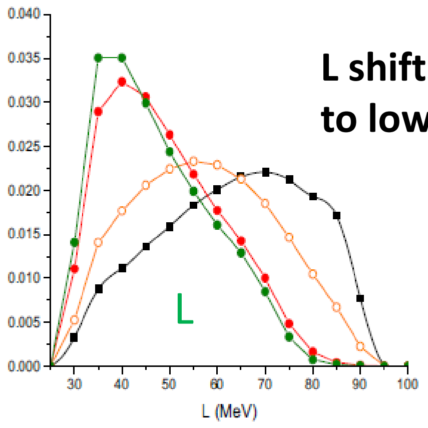


NOT constrained by R data, regardless of its precision

$R_{1.4}$=11.9 ±σ

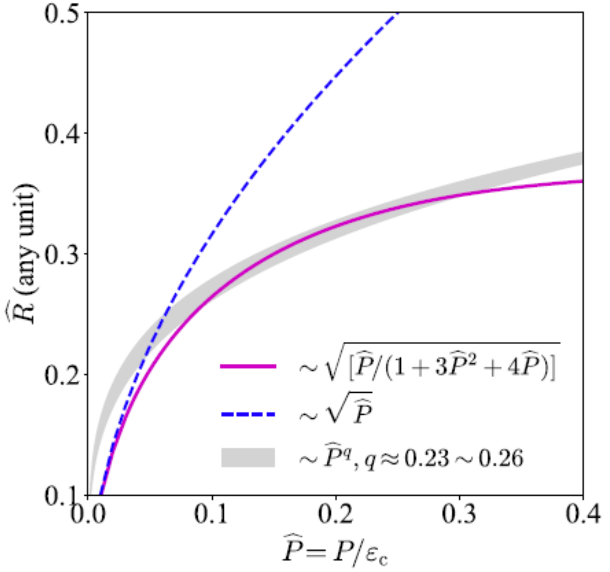
K_{sym} becomes bimodal as σ ↓ duo its correlation with L and J_{sym}





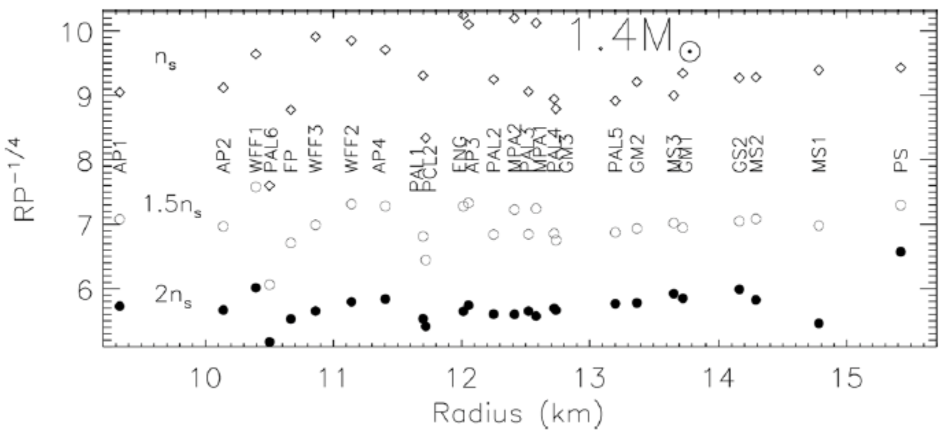
L shifts systematically to lower values as σ ↓

Understanding the nonlinear mapping between radii with varying precision σ and EOS parameters of neutron stars



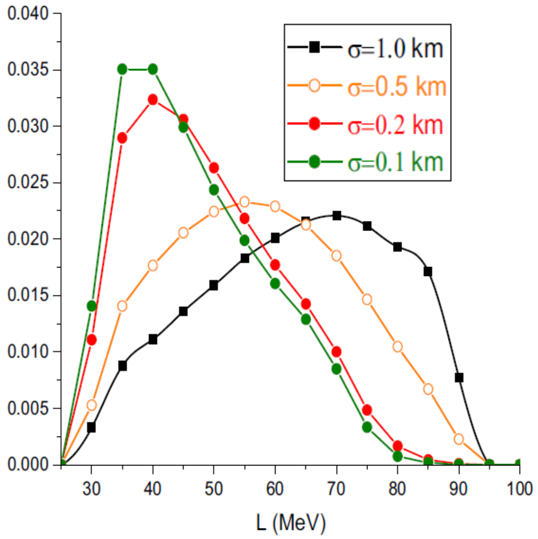
Cai, Li & Zhang, Phys. Rev. D 108 (2023)

$L \propto R^4$



Lattimer & Prakash, Phys. Rep., 442, 109 (2007)

At $n_s=\rho_0$, because p(SNM)=0, P(neutron stars)~ L



Radius measurement with precision σ:

Inferred L probability distribution P(L) is skewed

$$P(R) \propto \exp\left[-\frac{(R-R_0)^2}{2\sigma^2}\right]$$

$$R = R_0 + \delta,$$

$$\langle \delta \rangle = 0, \quad \langle \delta^2 \rangle = \sigma^2.$$

$$\langle R^4 \rangle = R_0^4 + 6R_0^2\sigma^2 + 3\sigma^4.$$

Because L~R^4, the inferred L acquires a positive bias proportional to σ^2

$$P(L) = P(R)\left|\frac{dR}{dL}\right|.$$

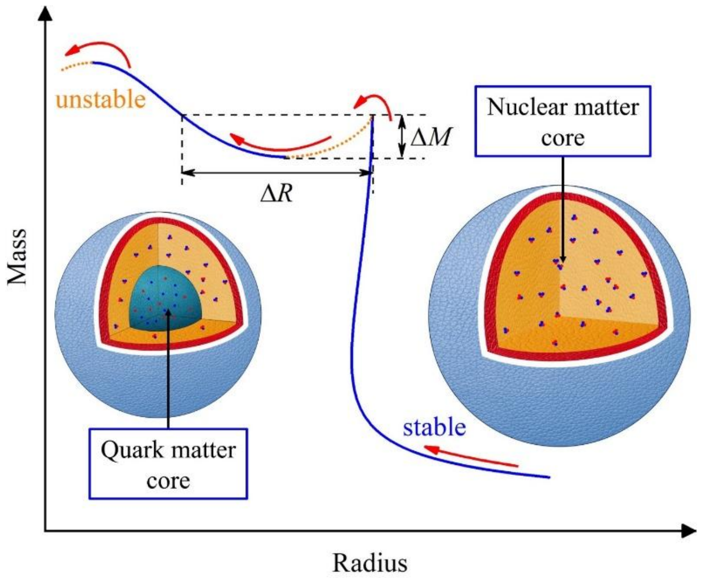
$$R = L^{1/4}, \quad \frac{dR}{dL} = \frac{1}{4}L^{-3/4},$$

$$P(L) \propto \exp\left[-\frac{(L^{1/4}-R_0)^2}{2\sigma^2}\right] L^{-3/4}.$$

- Large $\sigma \Rightarrow$ broad R distribution
- Allows larger R values
- Since $L \sim R^4$, this produces a long high-L tail

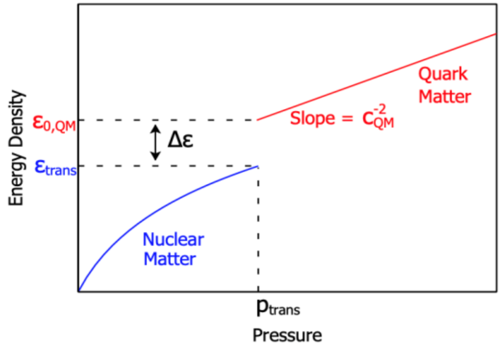
Bayesian quantification of observability of twin stars from future R data

Xavier Grundler and Bao-An Li, PRD 112, 103012 (2025)



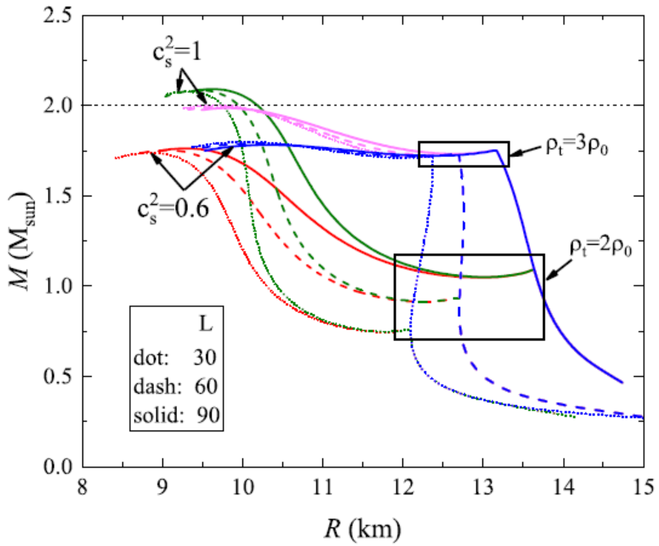
Hybrid EOS with a 1st order hadron-quark PT

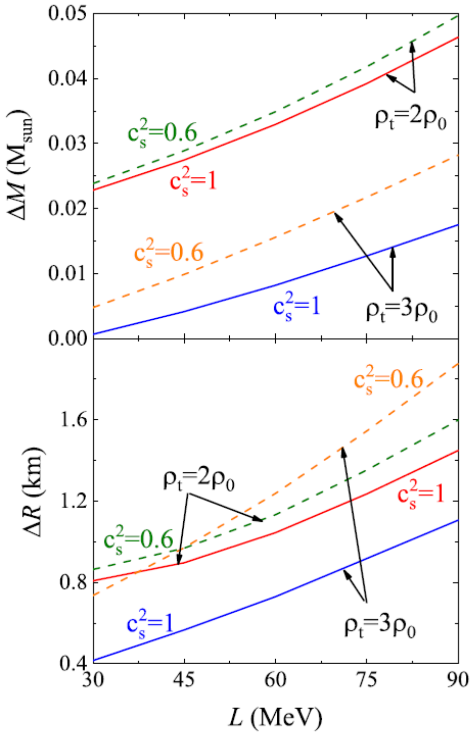
with a constant speed of sound model for QM



M.G. Alford, S. Han, M. Prakash, PRD 88, 083013 (2013)

Effects of symmetry energy slope L, hadron-quark transition density ρ_t, speed of sound in quark matter c_s

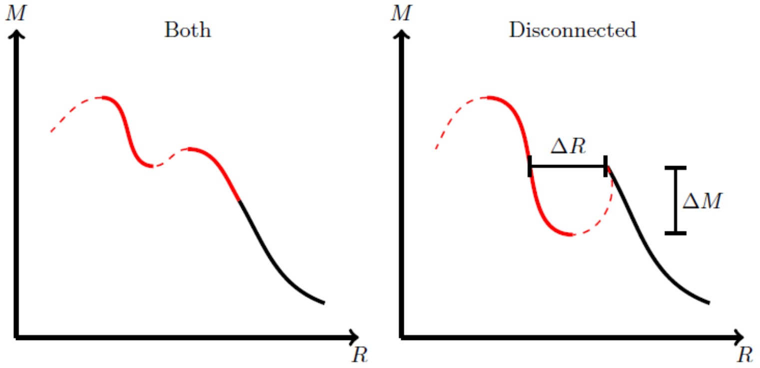




N.B. Zhang and B.A. Li, EPJA 61, 31 (2025).

Classification of M-R curves of hybrid stars

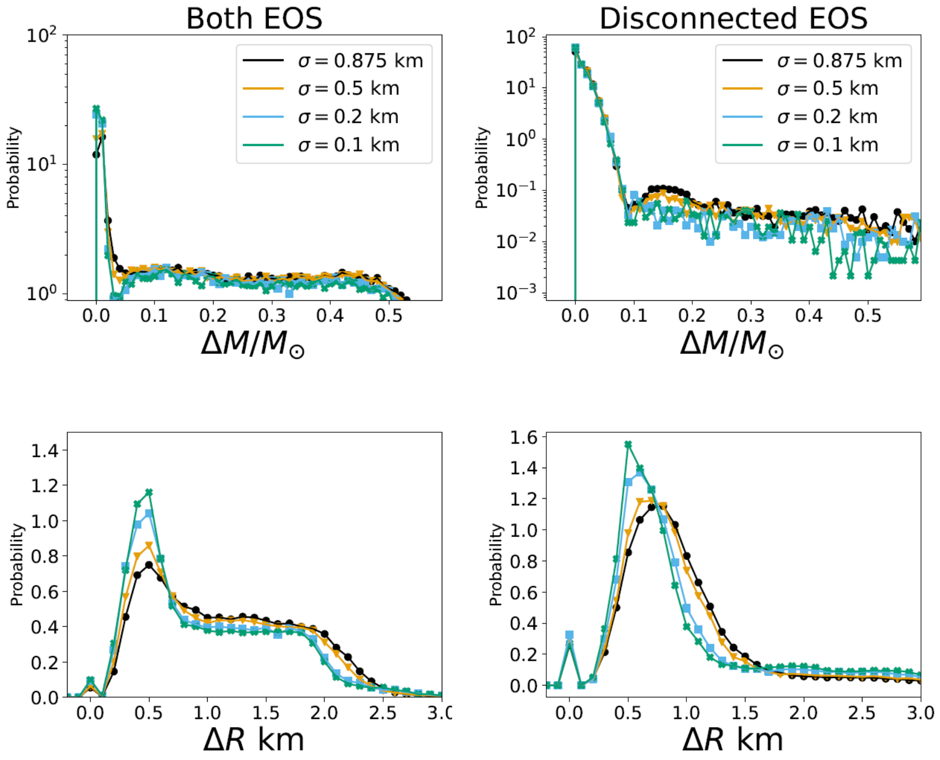
M.G. Alford, S. Han, M. Prakash, PRD 88, 083013 (2013)



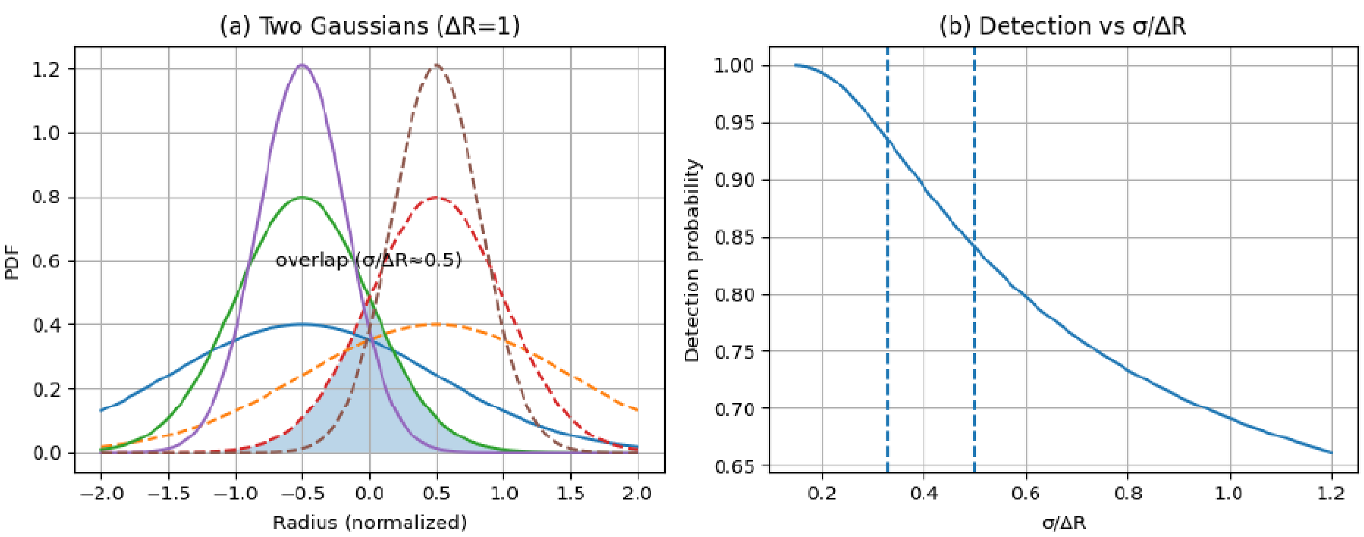
Probability of M and R separations

Xavier Grundler and Bao-An Li, PRD 112, 103012 (2025)

Bayesian inference using mock radius data: $R_{1.4}$=11.9 ±σ



Detectability of twin stars separated by ΔR in radii with precision σ



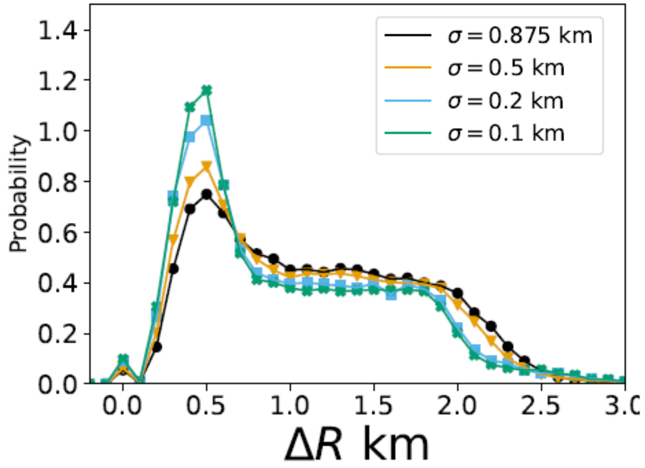
C. M. Bishop, Pattern Recognition and Machine Learning (Springer, 2006).

$$P_{\text{det}} = \Phi\left(\frac{\Delta R}{2\sigma}\right) \qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

The detectability is governed by the ratio $\Delta R/\sigma$:

- $\sigma \sim \Delta R$: strong overlap, indistinguishable populations;
- $\sigma \lesssim \Delta R/2$: onset of resolvability;
- $\sigma \lesssim \Delta R/3$: small overlap, near-saturated detection.

The peak detectability saturates with σ<0.2

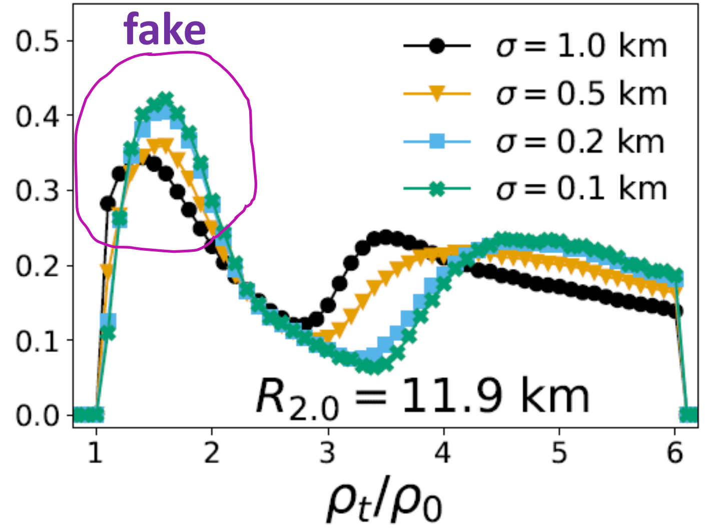


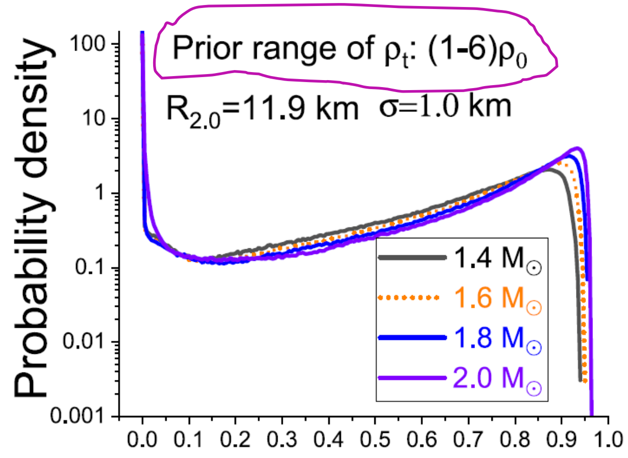
Easier to detect twins with larger ΔR

Can the high-precision R tell us anything new about hadron-quark PT?

Bao-An Li, Xavier Grundler, W.J. Xie, N.B. Zhang, The Astrophysical Journal 998, 262 (2026)

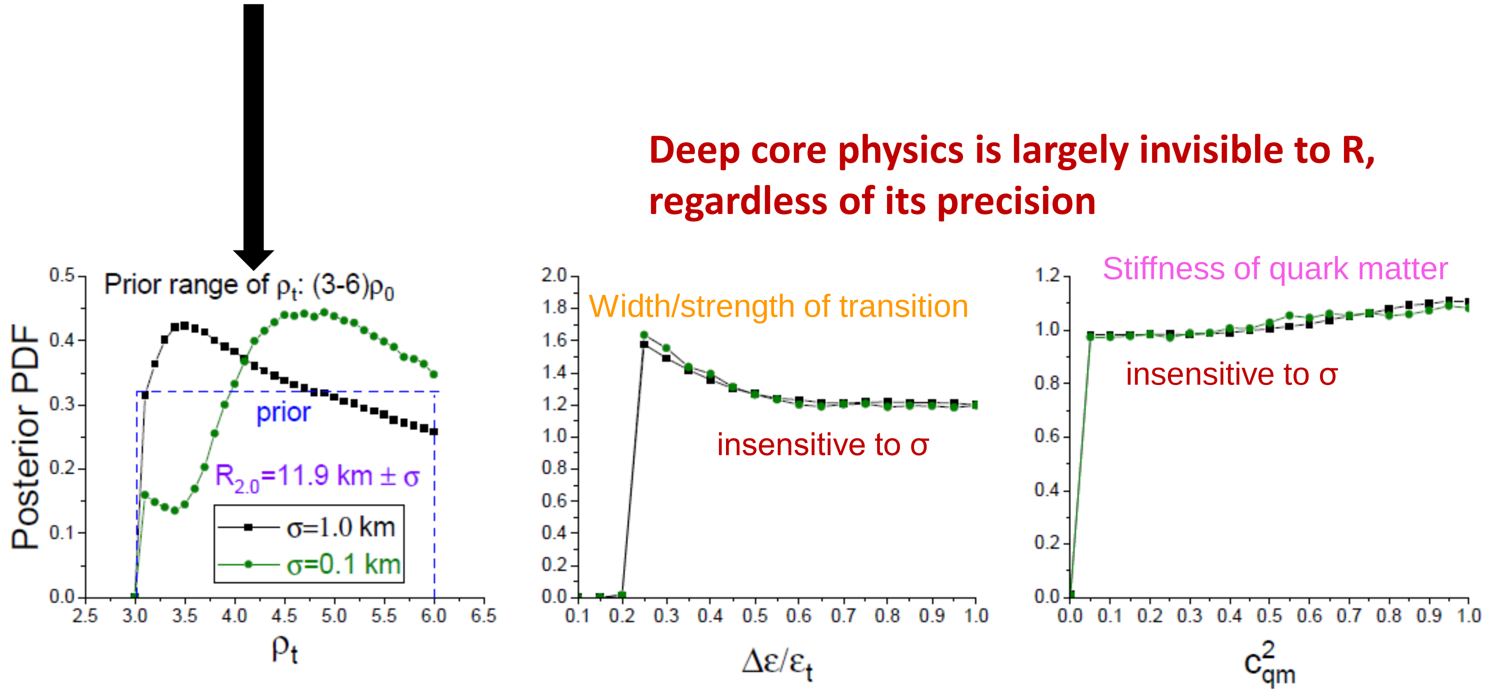
Pure statistics can NOT replace physics insight based on experiments





Consistent with indications of Beam Energy Scan Experiments at RHIC STAR, Phys. Lett. B 827, 137003 (2022), Phys. Rev. Lett. 135, 072301 (2025).

Deep core physics is largely invisible to R, regardless of its precision



Precision ≠ information

What is a robust EOS quantity that can be inferred?

PHYSICAL REVIEW LETTERS **129**, 252702 (2022)

Trace Anomaly as Signature of Conformality in Neutron Stars

Yuki Fujimoto,[1,*] Kenji Fukushima,[2,†] Larry D. McLerran,[1,‡] and Michał Praszałowicz[3,1,§]

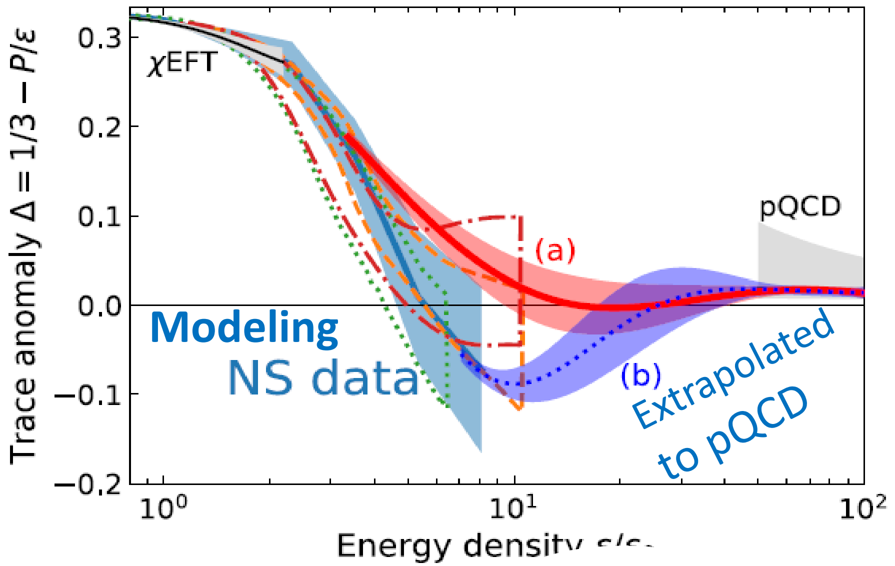
$$\Delta \equiv \frac{\langle\Theta\rangle_{T,\mu_B}}{3\varepsilon} = \frac{1}{3} - \frac{P}{\varepsilon}$$

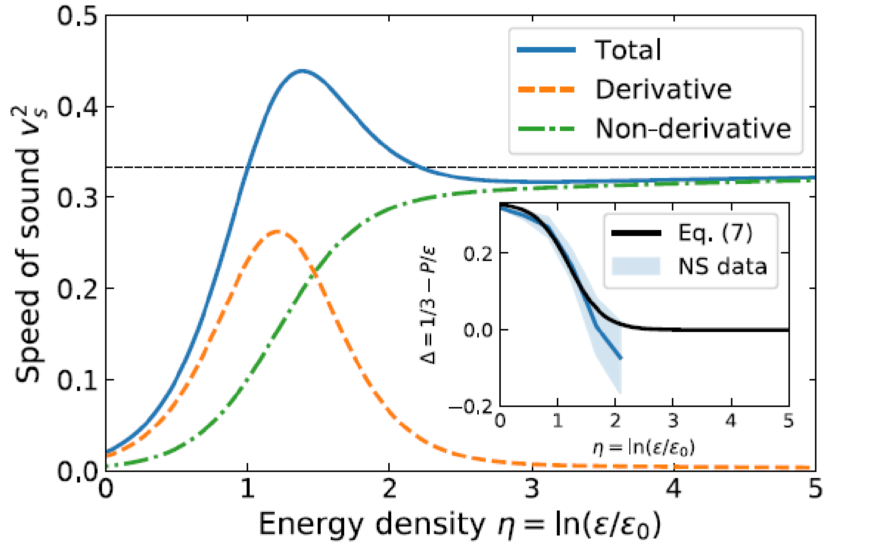
pQCD conformal (scale invariance) limit for ultra-relativistic massless gas $P/\epsilon=1/3$, $\Delta=0$, $v^2_s=1/3$ valid at $\rho > 40\rho_0$

$$v_s^2 = \frac{dP}{d\varepsilon} = v^2_{s,\mathrm{deriv}} + v^2_{s,\mathrm{nonderiv}}, \tag{5}$$

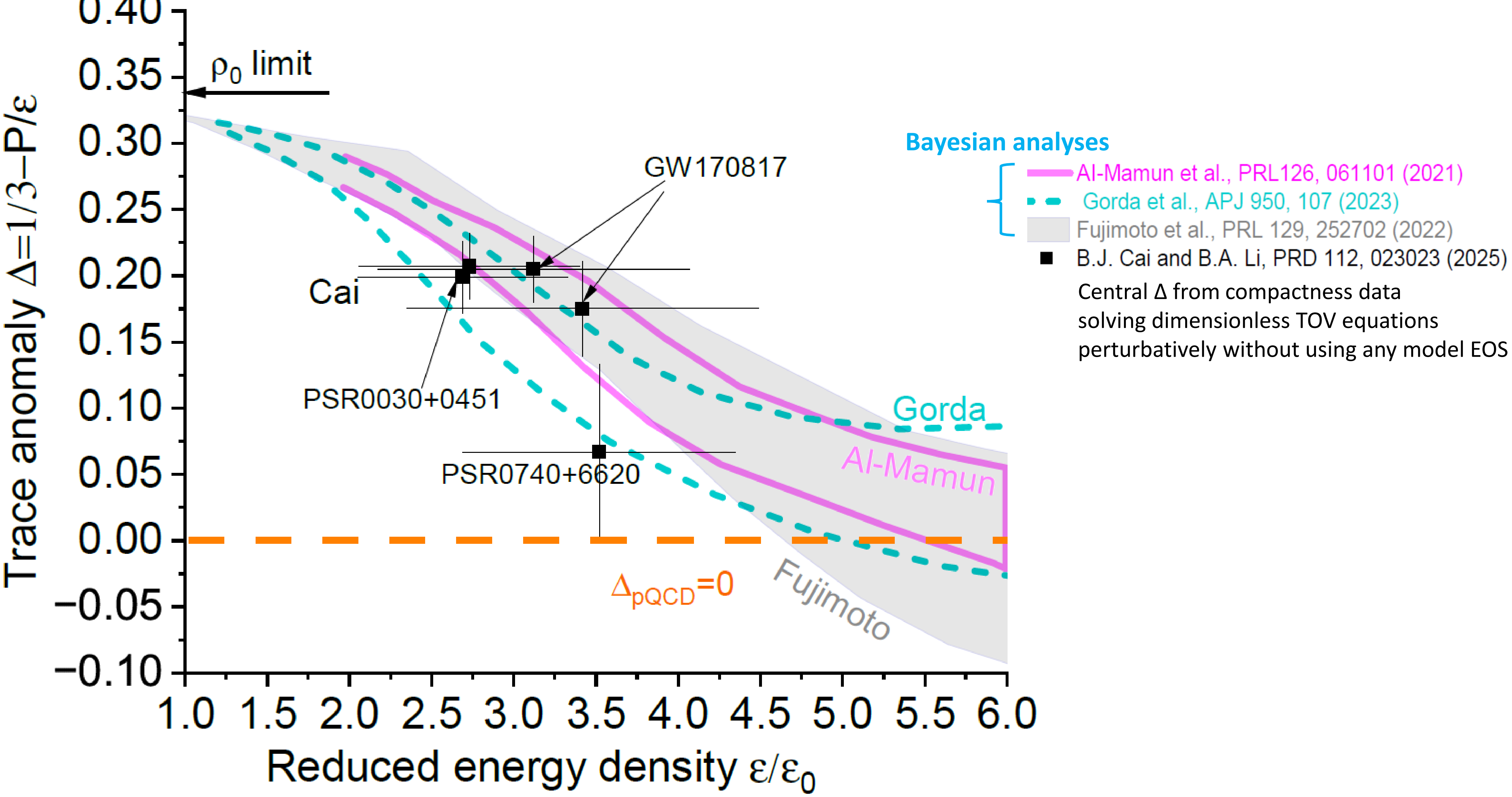
where the derivative and the nonderivative terms are

$$v^2_{s,\mathrm{deriv}} \equiv -\frac{d\Delta}{d\eta}, \qquad v^2_{s,\mathrm{nonderiv}} \equiv \frac{1}{3} - \Delta. \tag{6}$$





Universal behavior? Why?

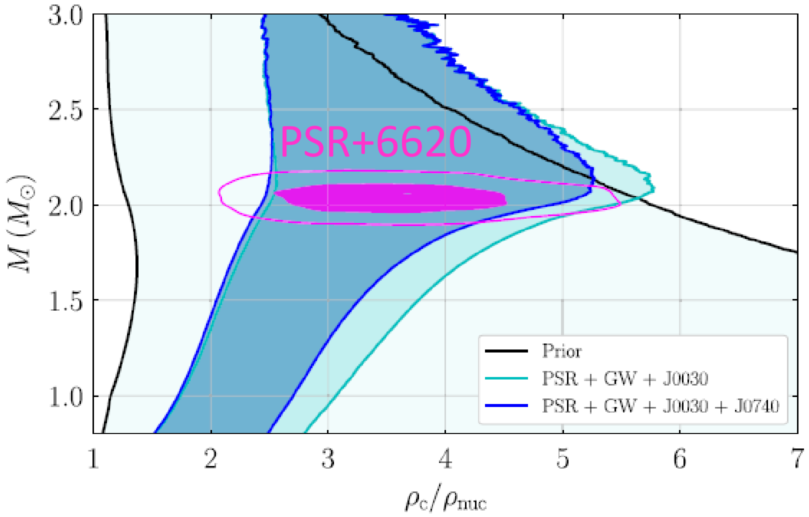


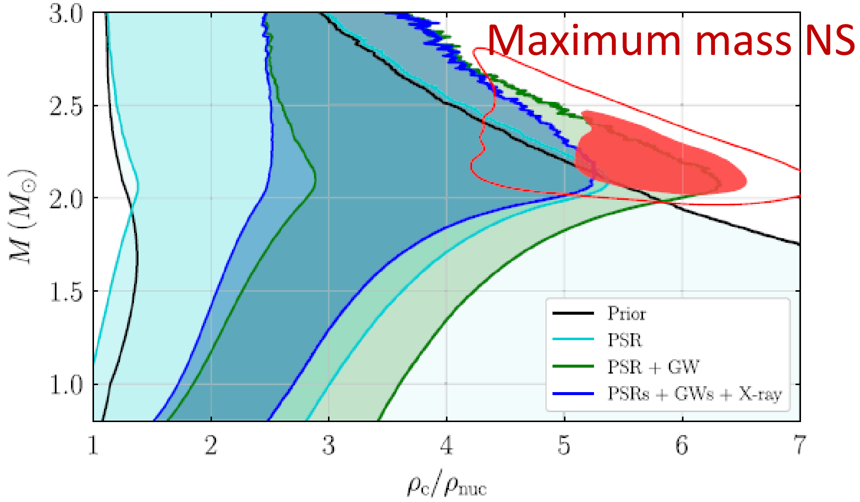
Impact of the PSR J0740 + 6620 radius constraint on the properties of high-density matter

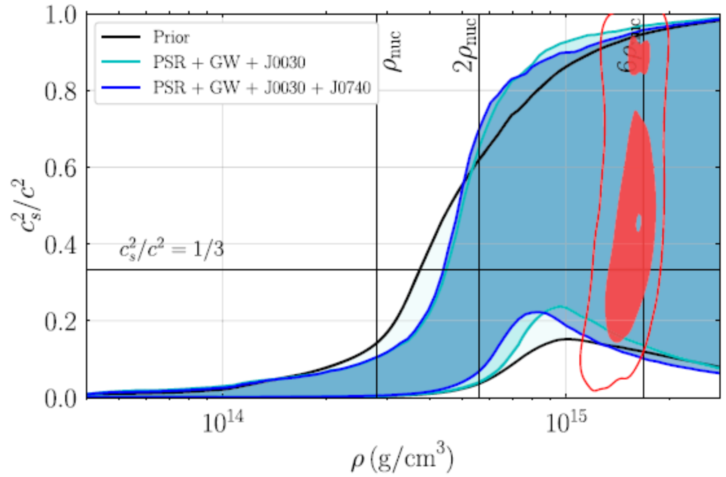
Isaac Legred[1,2,*] Katerina Chatziioannou[1,2,†] Reed Essick[3,‡] Sophia Han (韩君)[4,5,§] and Philippe Landry[6,∥]

PHYSICAL REVIEW D **104,** 063003 (2021)

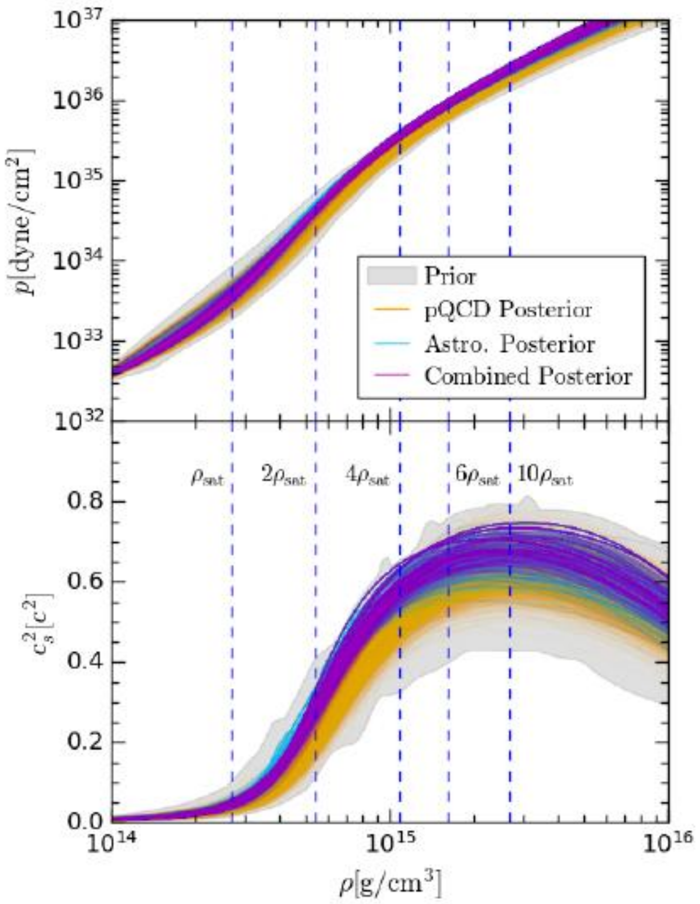
Maximum density reached at 68% CFL







At what density the sound speed may peak?



Possibly biased impression:

EOS at intermediate densities (2~10)ρ_{sat} is model dependent, leading to different locations/size of the peaks in speed of sound

Many attempts of LEAST-model dependent interpolations between CEFT and pQCD

Multiple mechanisms for the appearance of peaks in the density/radial profile of speed of sound

No consensus that the peak is actually reached in any NS observed so far

Iuliu Cuceu, Sandra Robles,

PRD 111 (2025) 12, 123029

Fundamental physics leading to a peaked speed of sound density profile

From hadrons to quarks in neutron stars: a review

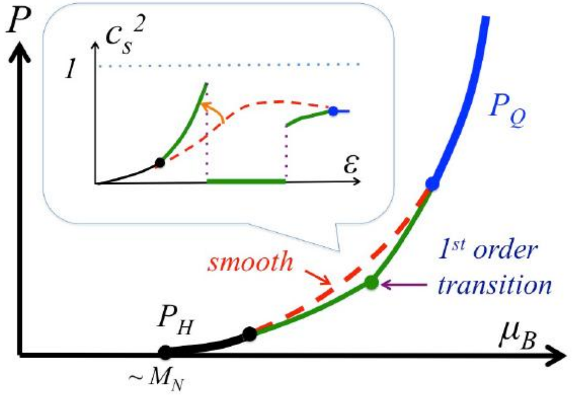
Reports on Progress in Physics, 81,. 056902 (2018)

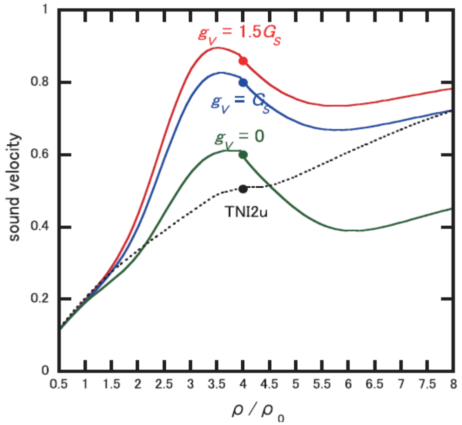
Gordon Baym,[1,2,3] Tetsuo Hatsuda,[2,4,5] Toru Kojo,[6,1] Philip D. Powell,[1,7] Yifan Song,[1] and Tatsuyuki Takatsuka[5,8]

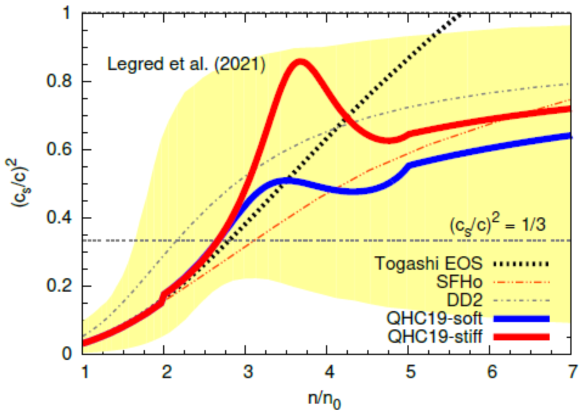
Hadron–quark crossover and massive hybrid stars

Kota Masuda[1,2,*], Tetsuo Hatsuda[2,3], and Tatsuyuki Takatsuka[4†]

Prog. Theor. Exp. Phys. **2013**, 073D01







Merger and post-merger of binary neutron stars with a quark-hadron crossover equation of state

Yong-Jia Huang[1,2,3], Luca Baiotti[4], Toru Kojo[5,6], Kentaro Takami[7,3], Hajime Sotani[8,3], Hajime Togashi[6], Tetsuo Hatsuda[3], Shigehiro Nagataki[3,8] and Yi-Zhong Fan[1,2]

PRL129, 181101 (2022)

Velocity of sound beyond the high-density relativistic limit from lattice simulation of dense two-color QCD

Prog. Theor. Exp. Phys. **2022** 111B01

Kei Iida[1] and Etsuko Itou[2,3,4,*]

Tripling Fluctuations and Peaked Sound Speed in Fermionic Matter

PHYSICAL REVIEW LETTERS **135**, 042701 (2025)

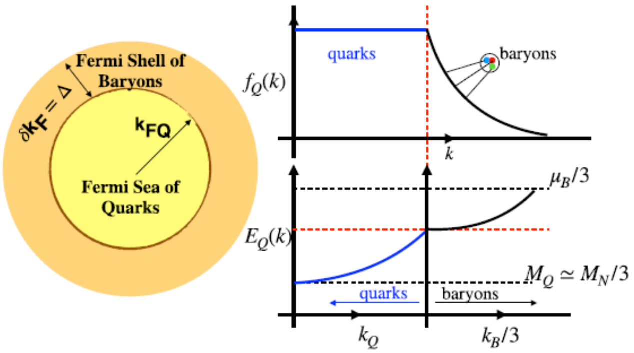
Hiroyuki Tajima,[1,2,3] Kei Iida,[4,2,5] Toru Kojo,[6] and Haozhao Liang[1,7,3]

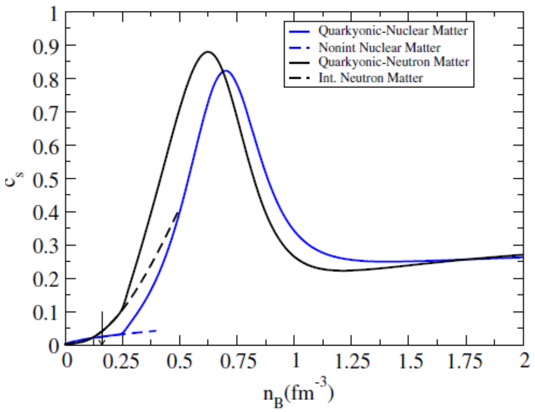
PHYSICAL REVIEW LETTERS **122**, 122701 (2019)

Quarkyonic Matter and Neutron Stars

Larry McLerran and Sanjay Reddy

Pure neutron matter





NS matter at beta-equilibrium

Quarkyonic Matter Equation of State in Beta-Equilibrium
Tianqi Zhao, James M. Lattimer, Phys. Rev. D 102, 023021 (2020)

Quarkyonic stars with isospin-flavor asymmetry

Jérôme Margueron,[1] Hubert Hansen,[1] Paul Proust,[1] and Guy Chanfray[1]

Phys. Rev. C104,055803(2021)

A lot more by Toru Kojo

Journal of Subatomic Particles and Cosmology

Volume 4, December 2025, 100088

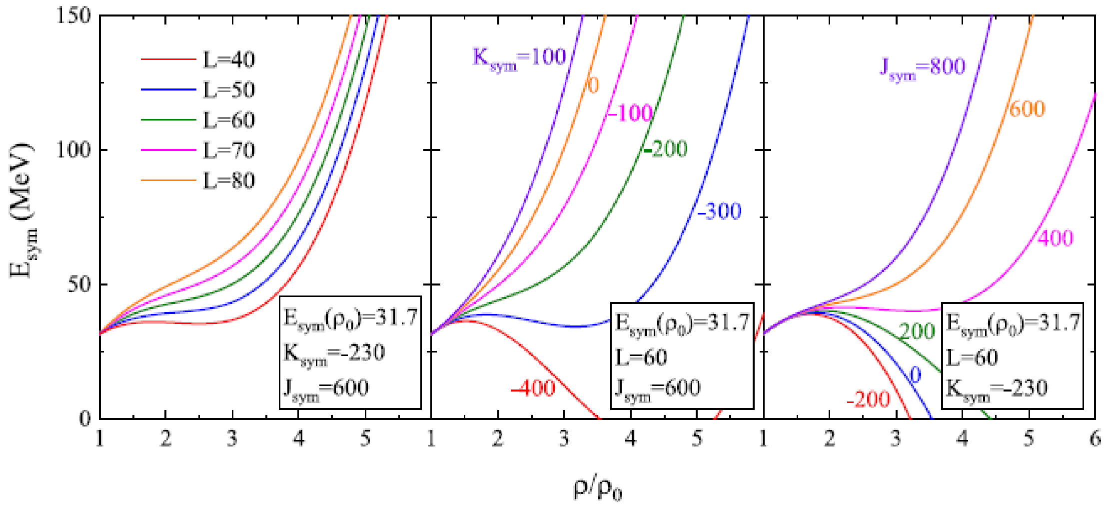
Stiffening of matter in quark–hadron continuity:
A mini-review

Dense matter = a single strongly interacting phase where
- •quarks dominate bulk properties
- •baryons survive as Fermi-surface excitations
- •gluons remain confining
- •the transition from hadrons to quarks is continuous

Impact of symmetry energy on sound speed and spinodal decomposition in dense neutron-rich matter

Nai-Bo Zhang[1,a], Bao-An Li[2,b]

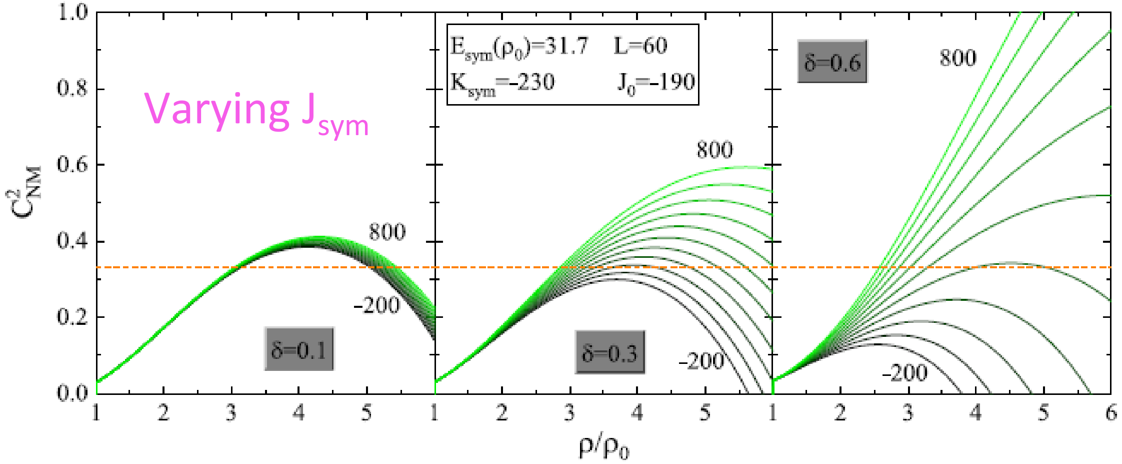
Eur. Phys. J. A (2023) 59:86



In nucleonic matter at fixed isospin asymmetry

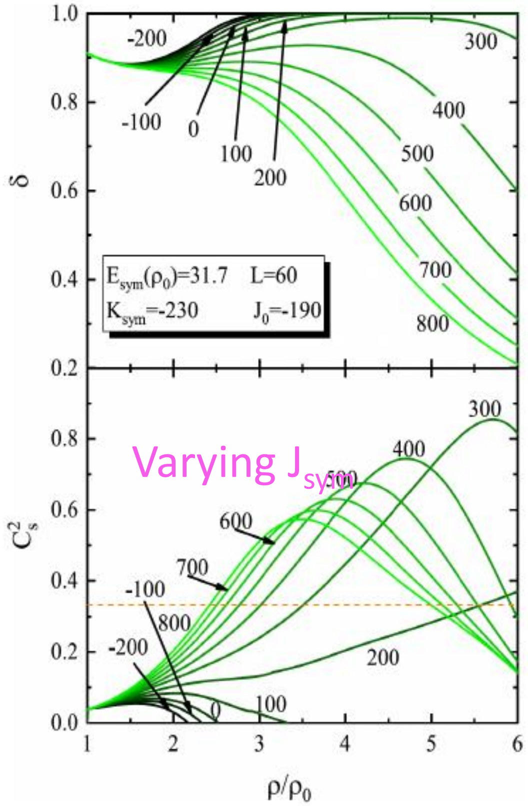
$$C_{NM}^2 \equiv \left(\frac{\partial P}{\partial \epsilon}\right)_\delta$$

$$C_{NM}^2 = \frac{K}{9(M_N + E(\rho,\delta) + P/\rho)}.$$

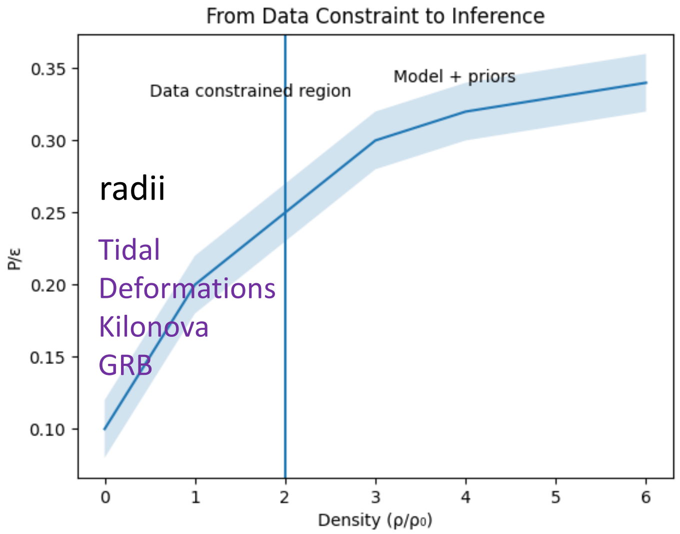


"Simple" mechanisms leading to peaked speed of sound in npeμ matter at beta-equilibrium

$$C_s^2 \equiv \frac{dP}{d\epsilon}.$$



How well do we know the EOS deep in the core?



- Radii probe ~1–2ρ_0; deep core not directly measured
- High-density constraints depend on models + priors
- Global constraints + TOV correlations propagate information between low and high density regions
- P/ε (thus the trace anomaly Δ) shows reduced model dependence
  The speed of sound involves derivatives of Δ, thus more uncertain

Key Insight: P/ε is robust across models

Why is the P/ϵ ratio less model-dependent than the EOS P(ϵ) itself?

(1) Dimensionless, insensitive to model details and absolute scales, as long as P and ϵ are from the same model

(2) Hugenholtz-Van Hove (HVH) Theorem (1st law of thermal dynamics at T=0) $P/\varepsilon = \rho E_{\rm F}/\varepsilon - 1 = E_{\rm F}/E - 1$

insensitive to how the energy density is partitioned into kinetic, 2-body, 3-body or momentum-dependent interactions

Independent extraction of P/ε from heavy-ion reactions is important

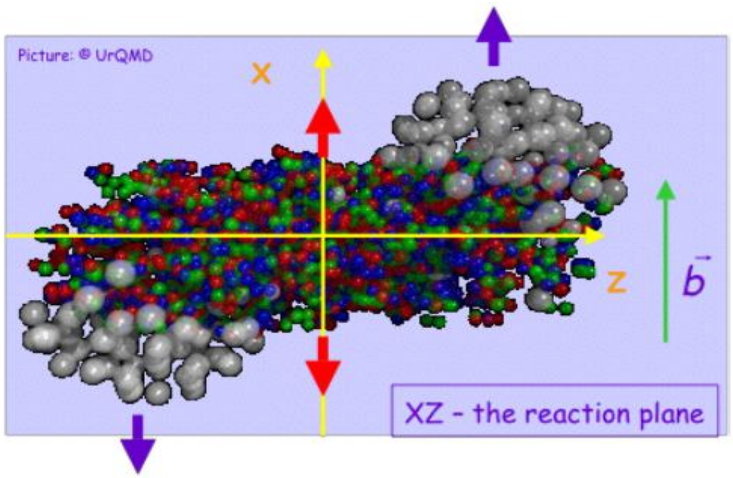
Flow observables are degenerate to details of P and ϵ (analogous to the mass and radius from the TOV equations)

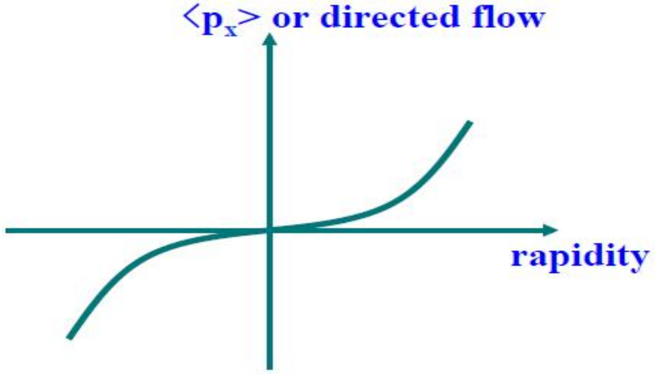
The Euler equation governing the collective flow velocity in a relativistic ideal fluid $(\varepsilon + P)\,\partial_t \vec{v} = -\nabla P,$

Flow is driven by pressure gradients

Directed flow (v_1)

Directed flow is quantified by the first harmonic (v_1)





The Euler equation is a guiding principle for an extreme case. Real analyses are done with viscous hydro and/or Boltzmann transport models. Viscosity reduces but does NOT remove the correlation between flow and P/ε.

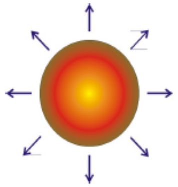
Why is flow correlated with P/ϵ at the maximum compression?

Azimuthal Anisotropy

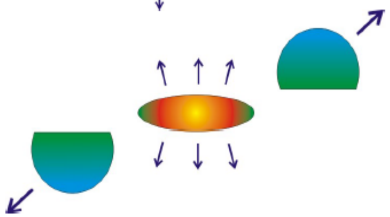
$$E\frac{dN^3}{d^3p} = \frac{1}{2\pi}\frac{d^2N}{p_t dp_t dy}\left(1 + 2v_1\cos(\phi - \psi_R) + 2v_2\cos 2(\phi - \psi_R) + \ldots\right)$$

isotropic directed elliptic

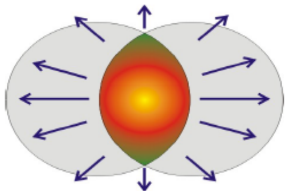
Radial flow
- Isotropic expansion of participant zone
- Measurable via slope parameter of spectra



Directed flow (v_1)
- Spectators deflected from dense reaction zone
- Sensitive to pressure
- Strong sensitivity to EoS



Elliptic flow (v_2)
- Asymmetry out- vs. in-plane emission
- Emission mostly during early phase
- Sensitive to EoS

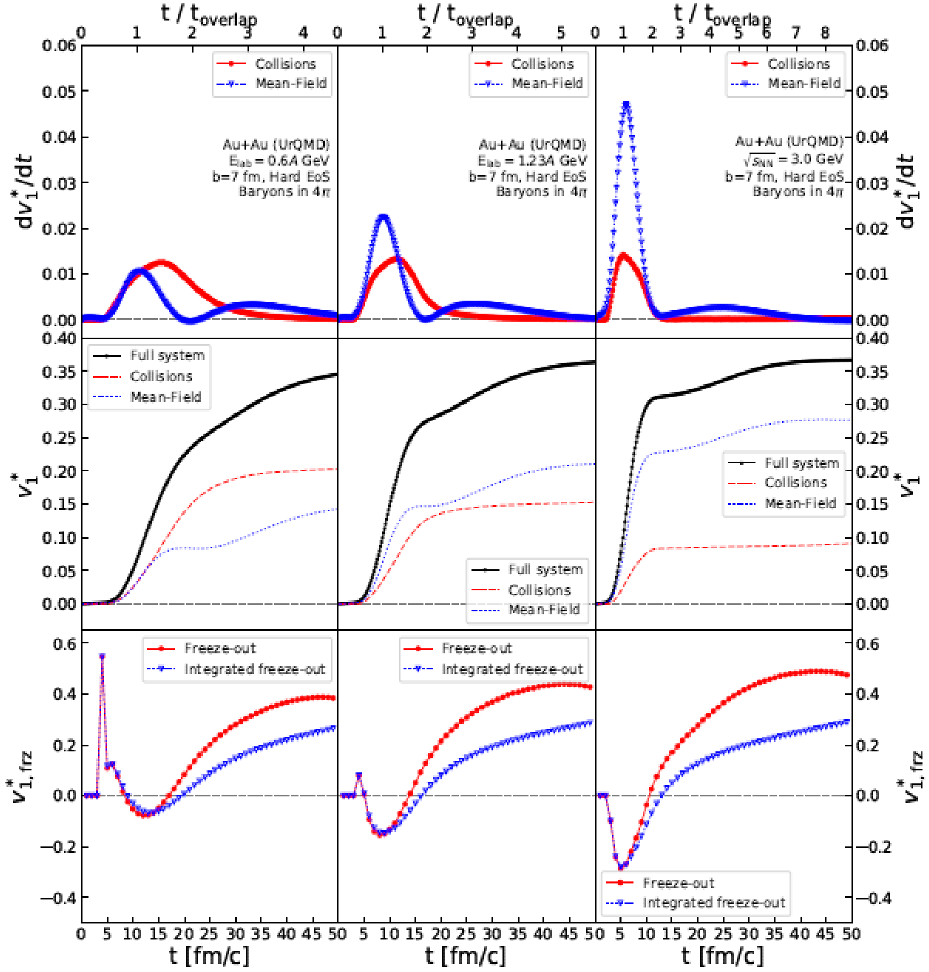


12

Space-time-averaged stiffness $c_s^2(\varepsilon) \equiv \frac{dP}{d\varepsilon}$

$$\langle c_s^2(\varepsilon)\rangle = \frac{1}{\varepsilon}\int_0^{\varepsilon} c_s^2(\varepsilon')\, d\varepsilon' = \frac{P(\varepsilon)}{\varepsilon} = w(\varepsilon).$$

Sources and evolution of directed flow



Tom Reichert[1,2,3] and Jörg Aichelin[4,2]

PRC 111 (2025) 5, 054916

COLD matter P/ϵ from HOT heavy-ion reactions

At intermediate energies before the formation QGP, flow depends on both the mean-field potential and elementary Xsections

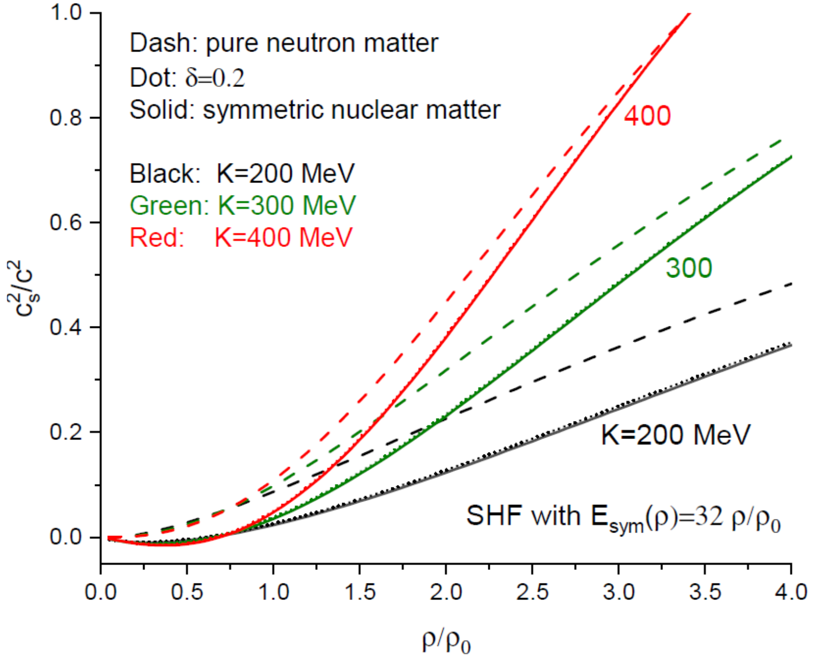
Boltzmann-Uehling-Uhlenbeck (BUU) equation   G.F. Bertsch and Subal DasGupta, Physics Reports 160, 189 (1988)

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \vec{v}\cdot\vec{\nabla}_r f - \vec{\nabla}_r V\cdot\vec{\nabla}_p f = I_c(f,\sigma_{NN})$$ collision integrals in hot matter

mean-field $V=\partial W/\partial\rho$
COLD matter EOS

In-medium modification factor
of NN scattering cross sections

$$X \equiv \sigma_{NN}^{med}/\sigma_{NN}^{free})$$



The simplest Skyrme potential still widely used

$$V_q(\rho,\delta) = a(\rho/\rho_0) + b(\rho/\rho_0)^\sigma + V_{asy}^q(\rho,\delta) + V_{Coulomb}^q.$$

Decoupling the cold matter EOS from thermal pressure

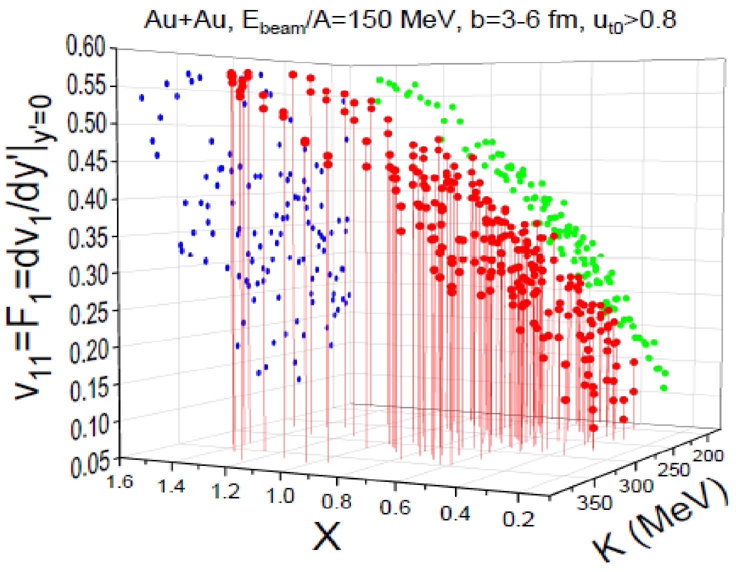
(1) If momentum-dependent interaction (poorly known) is used, cold matter EOS is coupled to thermal pressure

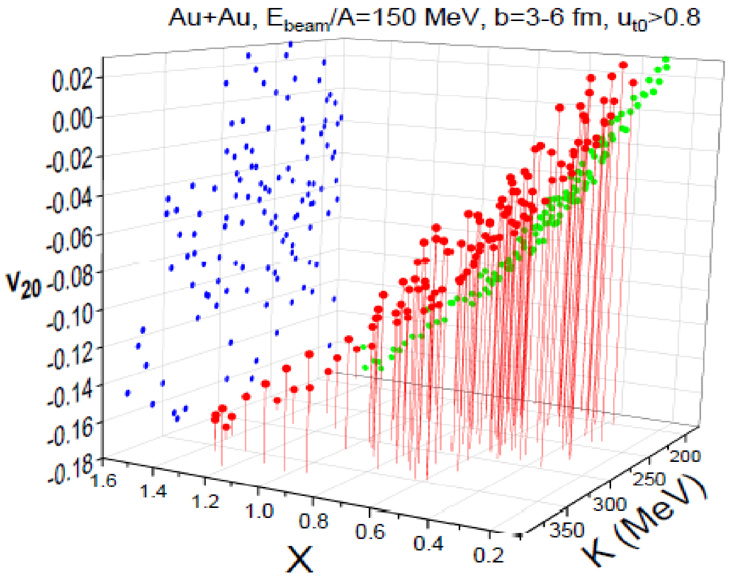
(2) In cold matter, P/ϵ is insensitive to how the EOS is constructed

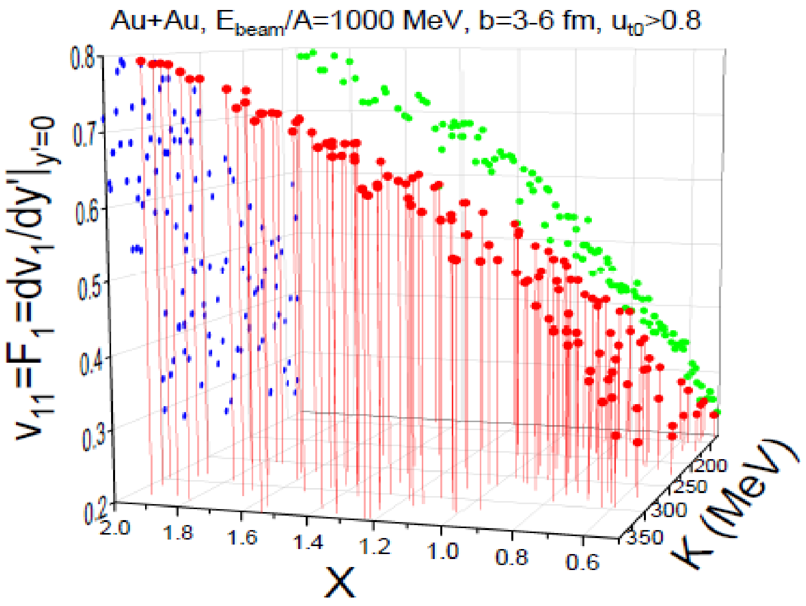
$$P/\varepsilon = \rho E_F/\varepsilon - 1 = E_F/E - 1$$

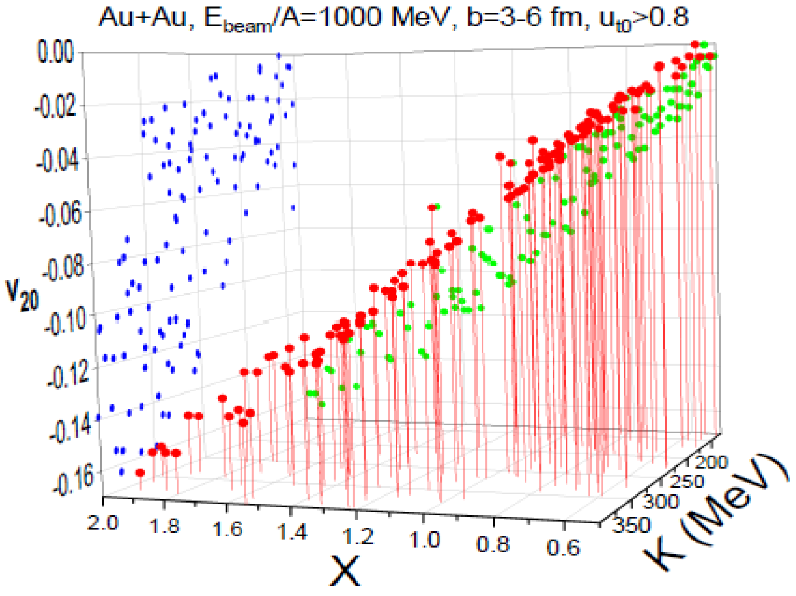
(3) Momentum-independent BUU code is 10 times faster

120 datasets at each beam energy for training and testing a Gaussian Process emulator

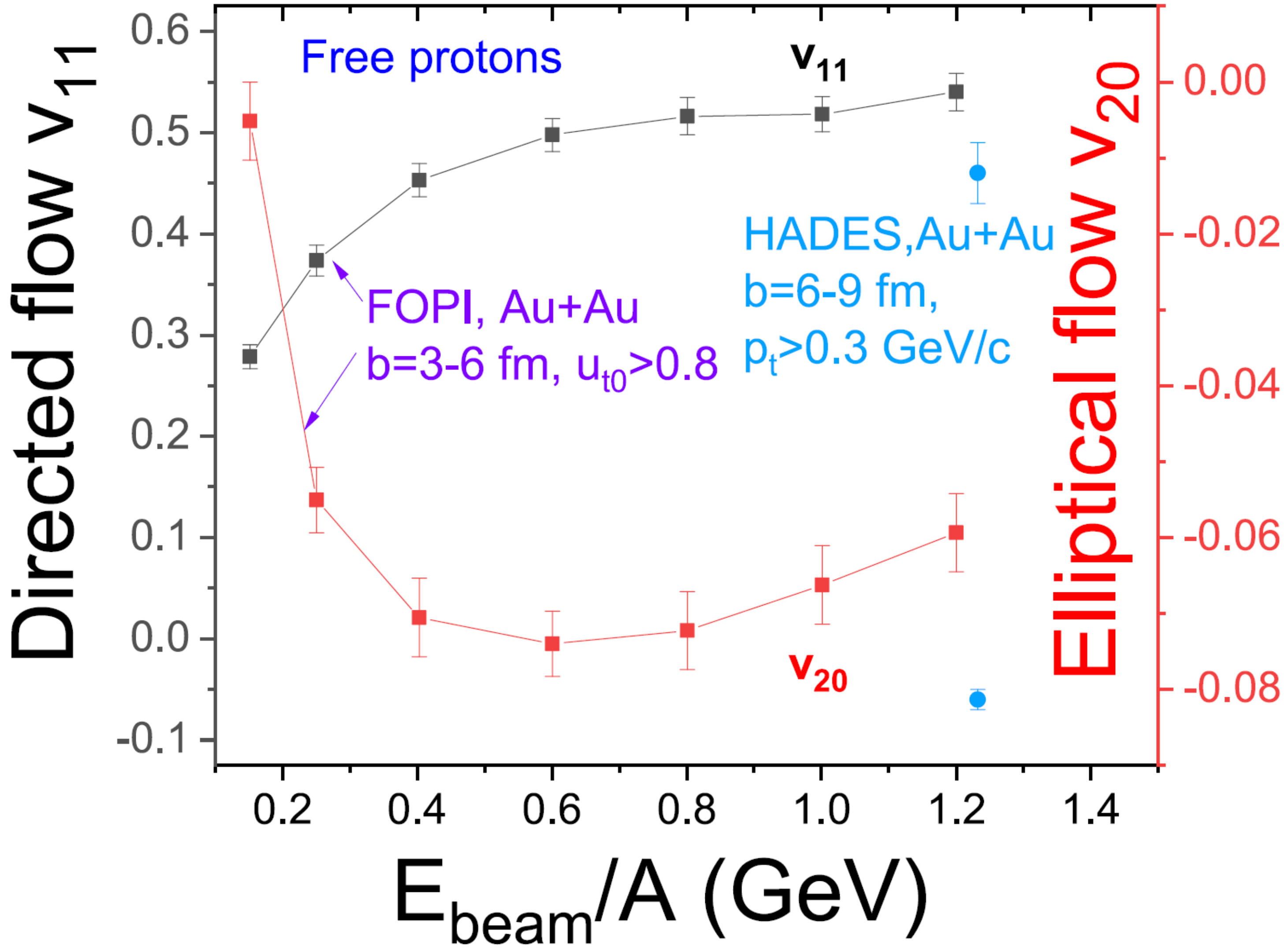




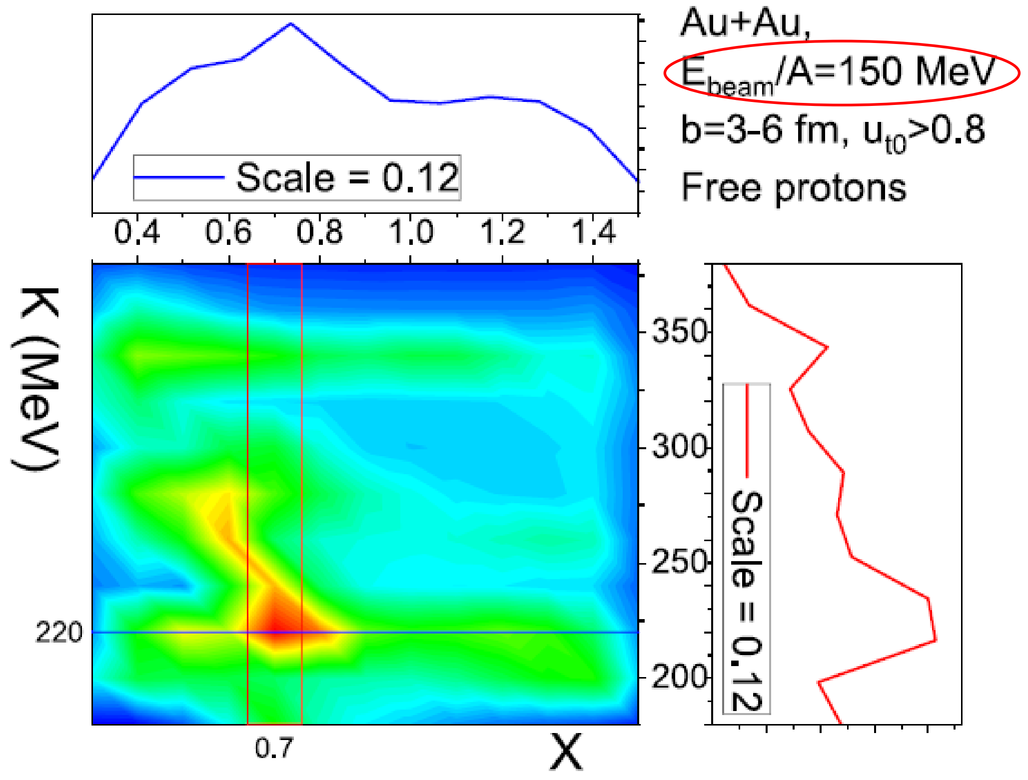


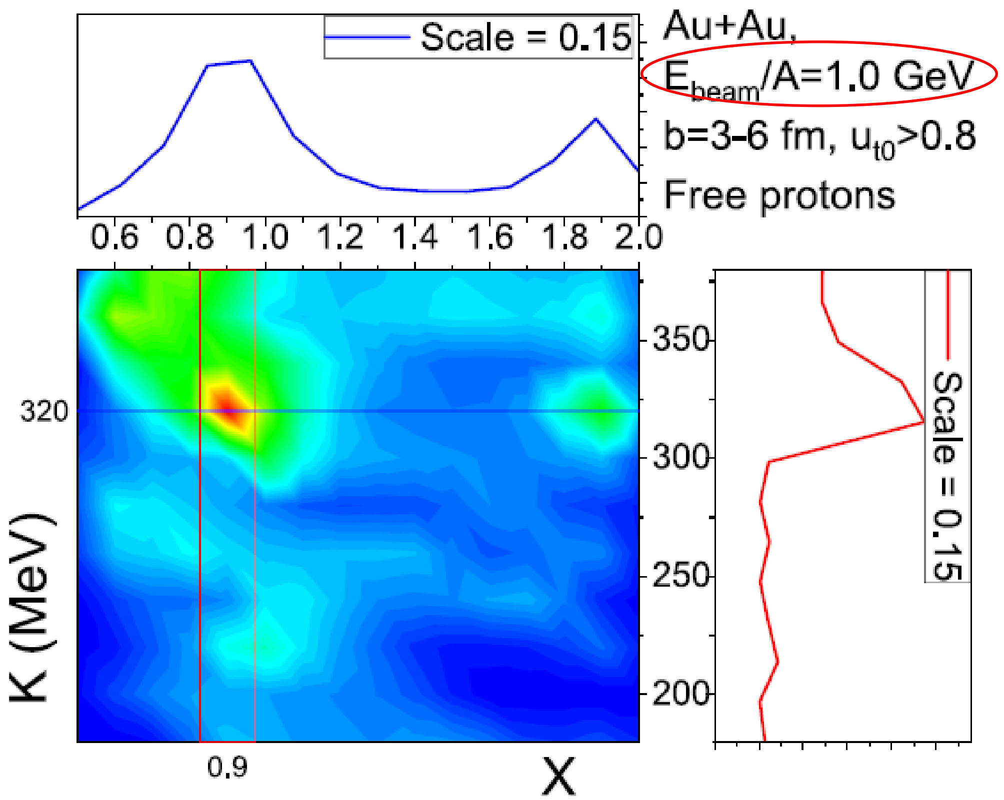


Excitation functions of directed and elliptical flow from GSI



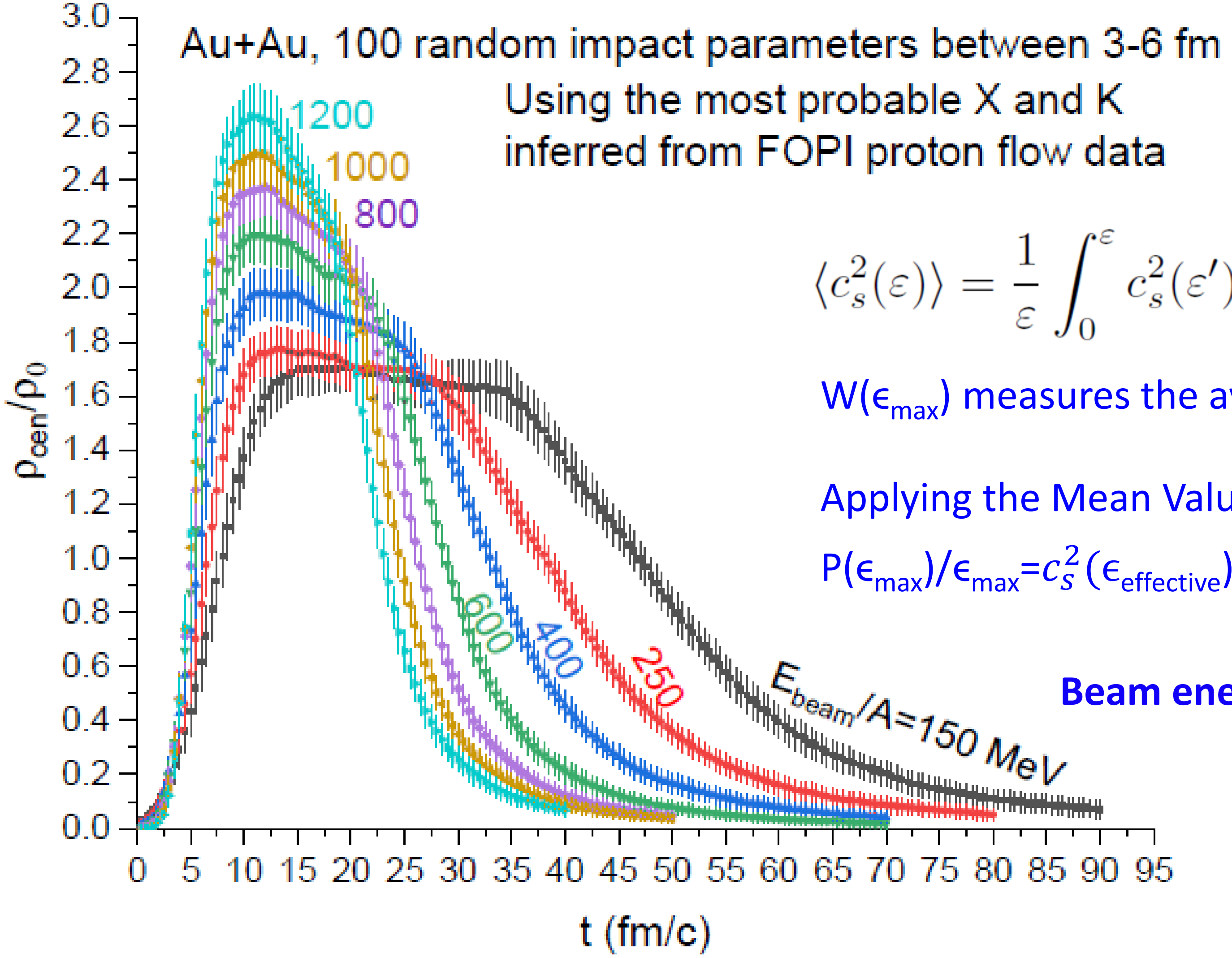
Typical results of Bayesian analyses of flow data from GSI





Bao-An Li, Wen-Jie Xie,
NPA 1039, 122726 (2023),
PRC 111, 054602 (2025)

Why is the P/ϵ robust?



$$\langle c_s^2(\varepsilon)\rangle = \frac{1}{\varepsilon}\int_0^{\varepsilon} c_s^2(\varepsilon')\,\mathrm{d}\varepsilon' = \frac{P(\varepsilon)}{\varepsilon} = w(\varepsilon)$$

$W(\epsilon_{max})$ measures the average stiffness up to ϵ_{max}

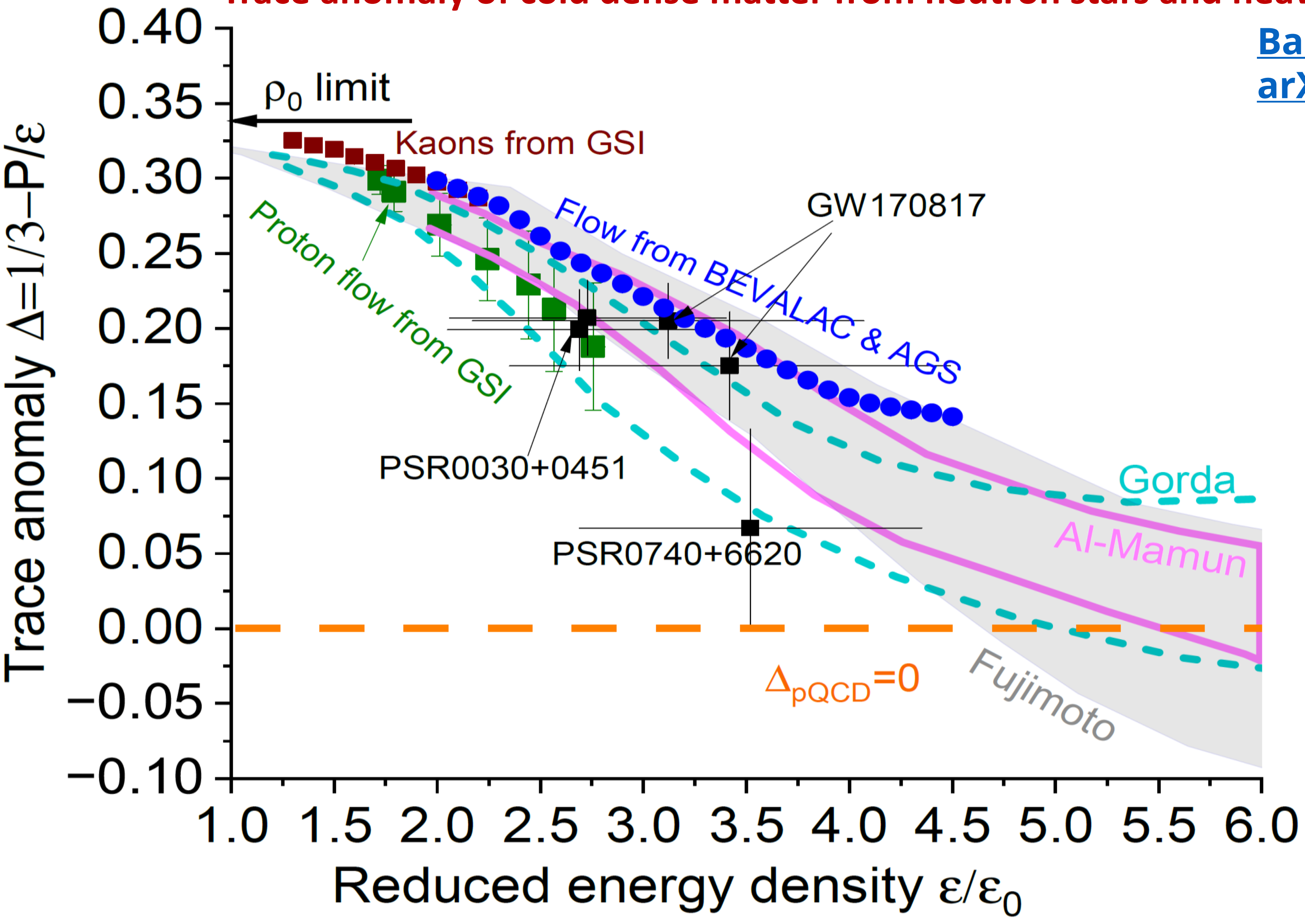
Applying the Mean Value Theorem to $P(\epsilon)$

$P(\epsilon_{max})/\epsilon_{max} = c_s^2(\epsilon_{effective})$ with $0<\epsilon_{effective}<\epsilon_{max}$

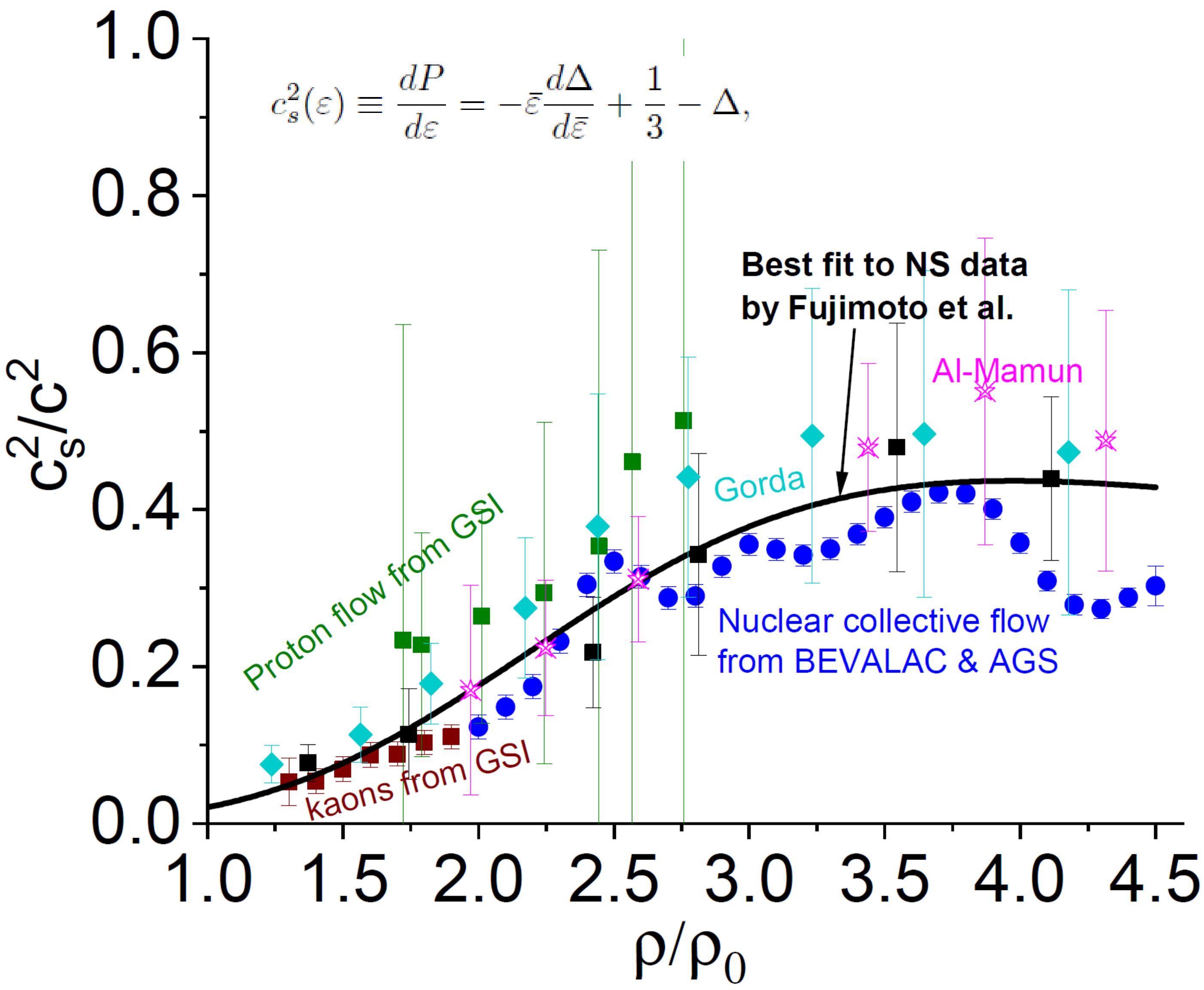
Beam energy dependence of flow → P/ϵ vs ϵ

Trace anomaly of cold dense matter from neutron stars and heavy-ion collisions

Bao-An Li, arXiv:2601.13374



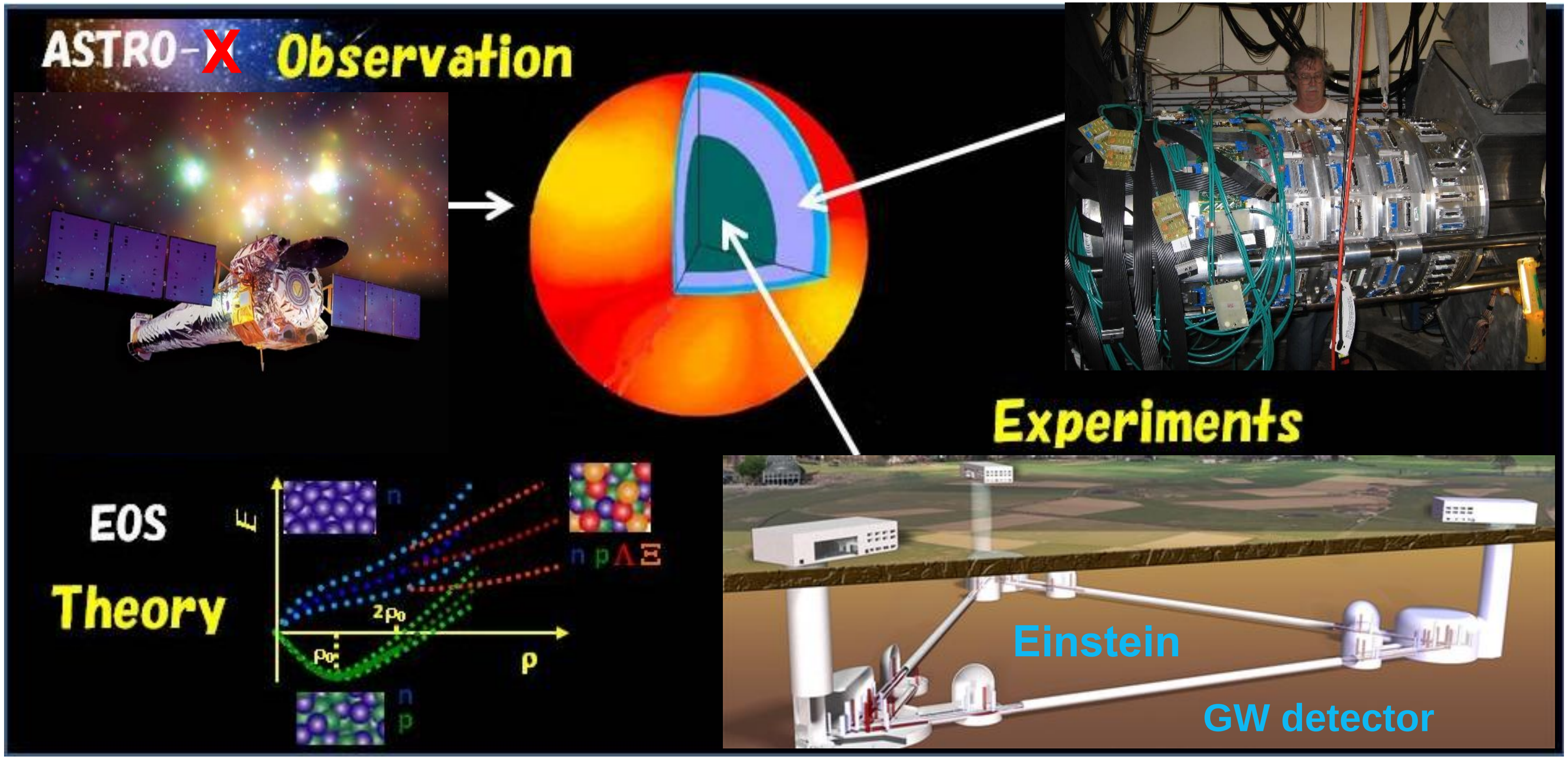
Unified density profile of the speed of sound squared



Consistent trace anomaly of cold dense matter

from heavy-ion collisions and neutron stars

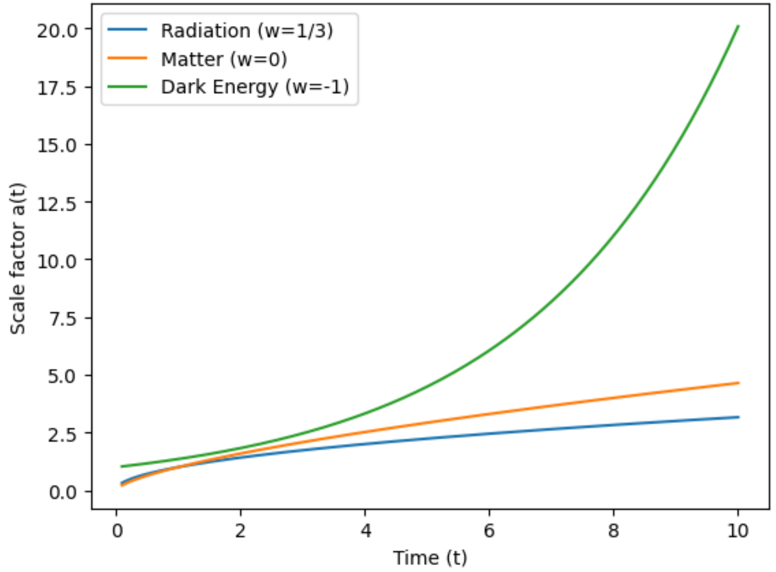
Inferred quantity P/ε is a robust global measure of EOS stiffness

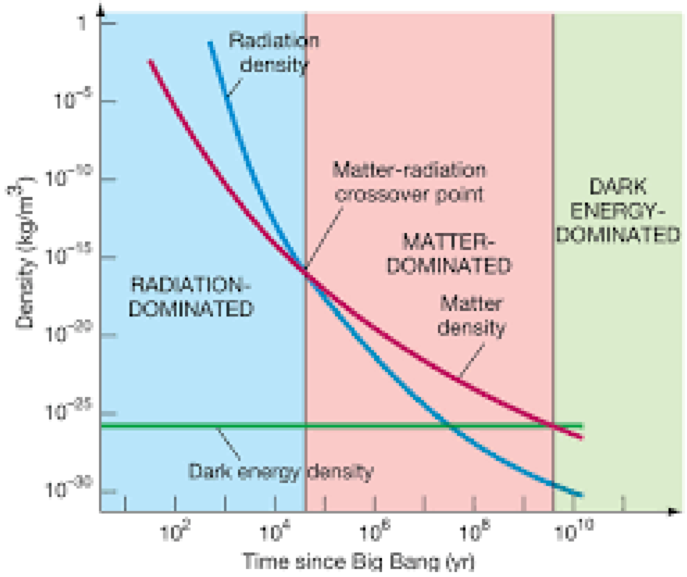


EOS P/ϵ and Cosmic Expansion

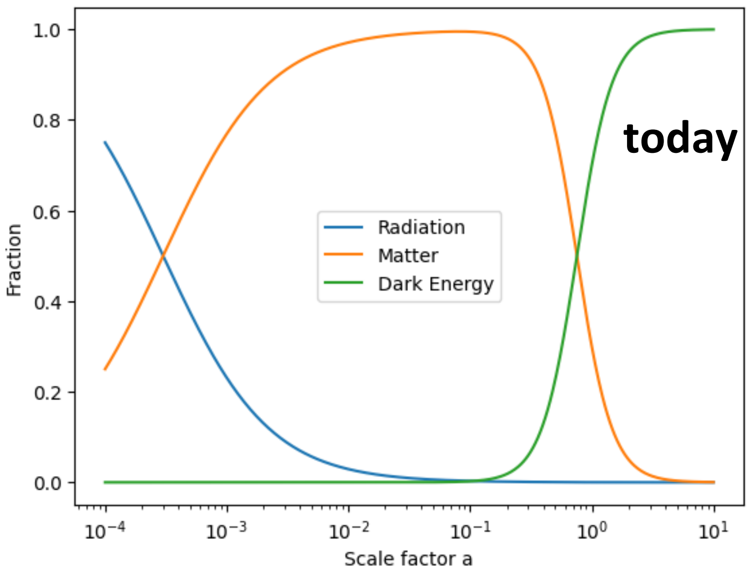
- Friedmann equation: **: $\ddot{a}/a = -(4\pi G/3)\,\rho\,(1 + 3w)$**
- Scale factor a(t): measures expansion (a=1 today)
- $w = P/\rho$ (**$\rho=\epsilon$**) determines how density evolves **$\rho(a) \propto a^{-3(1+w)}$**
- Acceleration if: $w < -1/3$
  - $w = 1/3$ (Radiation): strong deceleration
  - $w = 0$ (Matter): deceleration
  - $w = -1$ (Dark Energy in ΛCDM): acceleration

Evolution of the Universe





Energy sharing

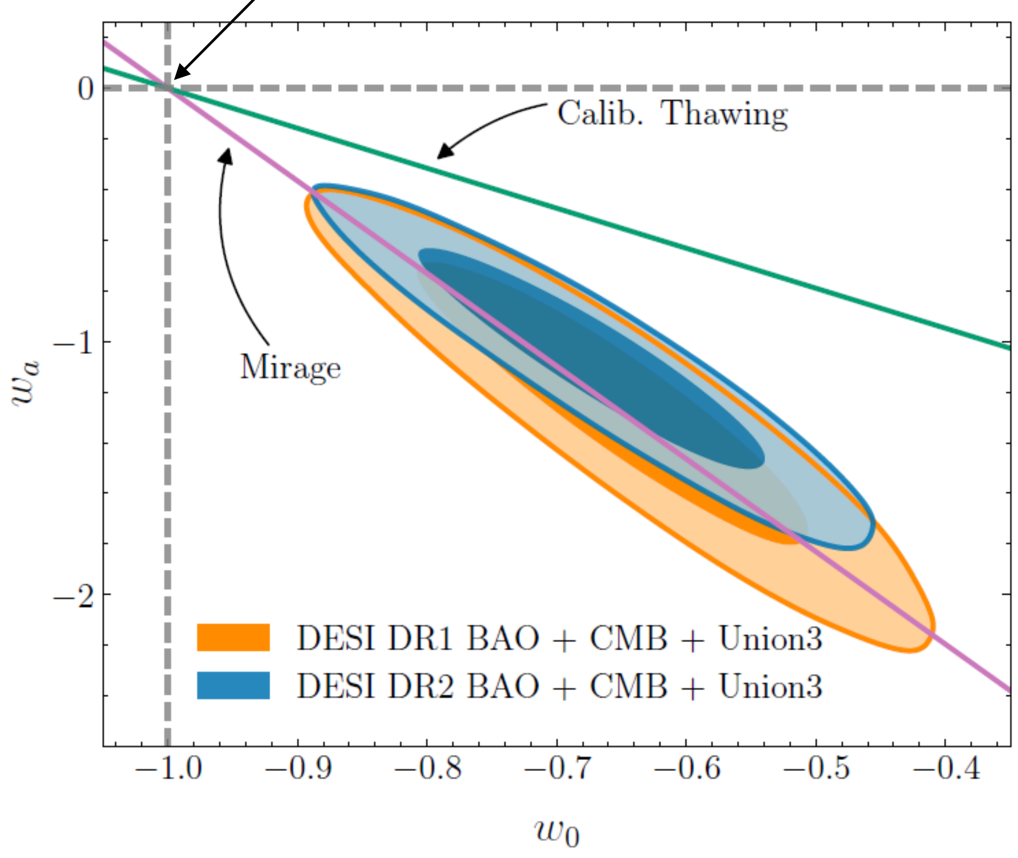


Today a=1

Indication for a time-dependent dark energy EOS: $w(a) = w_0 + w_a(1 - a)$

Dark Energy Spectroscopic Instruments (DESI) Collaboration, Phys. Rev. D 112 (2025) 8, 083511

ΛCDM: w_0=-1, w_a=0



From Heavy-Ion Collisions, Neutron Stars to the Expanding Universe

Pressure drives expansion across laboratory, astrophysics, and cosmology

EOS controls the expansion dynamics

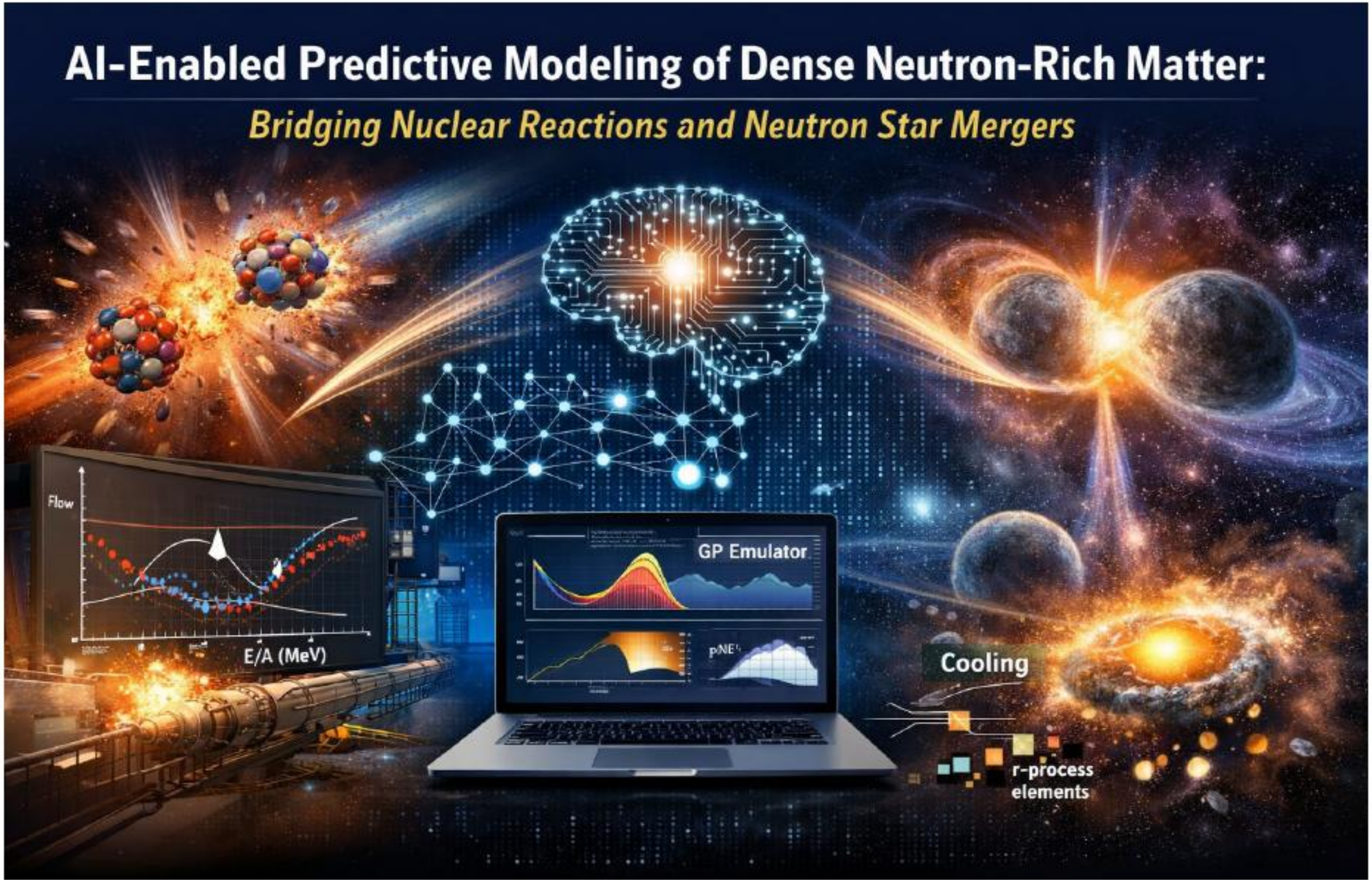
Take-Home Messages

- EOS uncertain at high density
- Degeneracy limits inference
- P/ε is robust
- Multi-messenger approach essential

Thanks!

Back-up slides

Looking ahead



Bayesian Inference of the Dense Matter EOS: From Heavy-Ion Collisions to Neutron Stars

Unified framework combining nuclear experiments and astrophysical observations to constrain the EOS with quantified uncertainties

Bayesian Framework: $P(M|D) \propto P(D|M)\,P(M)$
Prior (theory) + Likelihood (data) → Posterior (constraints)

Key Challenges: degeneracies and model dependencies

- EOS–gravity degeneracy and model uncertainties
- High-density symmetry energy is poorly known
- Collective flow probes high-density EOS, model-dependent, degenerate
- High-precision radius measurements are costly and nonlinear in impact

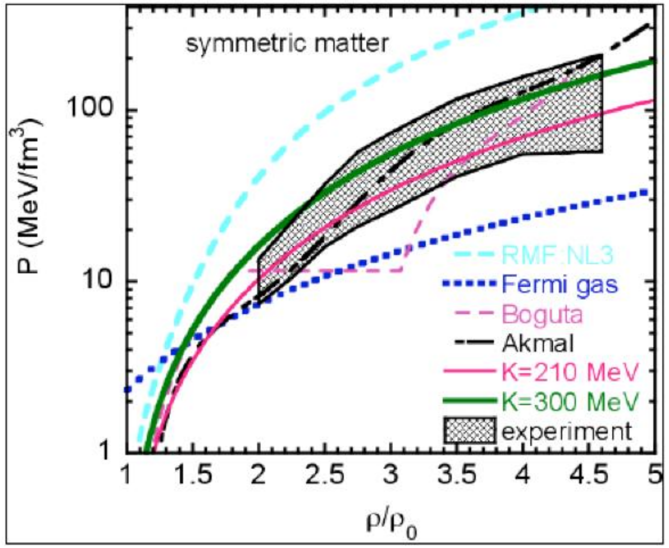
New Insights: degeneracy → universality or consistent patterns across fields

- Trace anomaly $\Delta = 1 - P/\varepsilon$ as EOS stiffness measure
- Combined HIC + NS Bayesian constraints improve EOS parameters

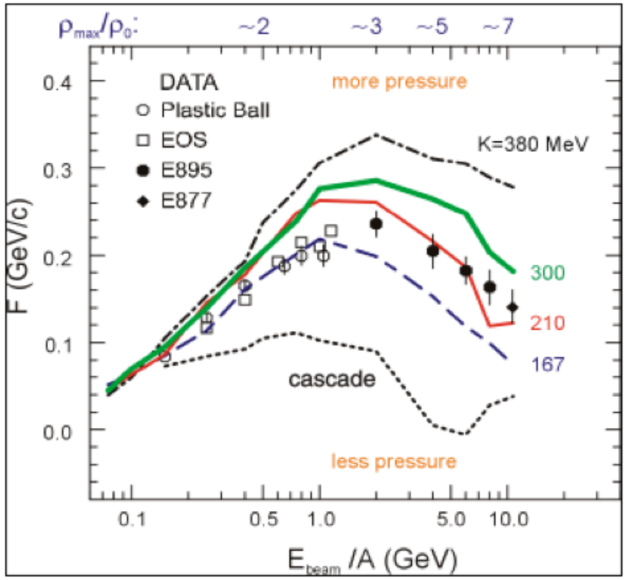
Take-Home Message: Combining lab experiments, astrophysics, and Bayesian inference is essential

Historical:

Danielewicz P. I., Lacey R and Lynch W G 2002 Science 298 1592



Flow favors K=210-300 MeV



Cautions necessary

(1) Labeling the P with a single K is sufficient but obscure the true physics ingredients

(2) The pressure band from 1.3 to 4.5ρ_0 does NOT constrain K but J_0 and Z_0

(3) NOTHING is WRONG to see the necessary K to increase with beam energy

(4) One should NOT try to fit the flow excitation function with a single K

Trace Anomaly in Dense Matter

Quantifies the breaking of scale invariance and provides a macroscopic measure of the EOS.

1. What is the Trace Anomaly at T=0?

$$\Delta = \frac{1}{3} - \frac{P}{\varepsilon}$$

- Measures deviation from conformal behavior
- $\Delta = 0$ for scale-invariant systems
- $\Delta \neq 0$ indicates masses and interactions

2. Energy–Momentum Tensor

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

$$T^\mu{}_\mu = -\varepsilon + 3P$$

- Trace encodes deviation from conformal symmetry
- $\varepsilon - 3P \neq 0$ signals interactions

3. Conformal Symmetry

- $T^\mu{}_\mu = 0 \Rightarrow$ scale invariance
- Photon gas: $P = \varepsilon/3 \Rightarrow \Delta = 0$
- Dense matter: $\Delta \neq 0$

4. Finite Temperature Form

$$\frac{\varepsilon - 3P}{T^4}$$

- Natural dimensionless quantity
- Measures interaction strength
- Used in lattice QCD and hot matter studies

5. Physical Picture

- Flow in heavy-ion collisions $\Rightarrow P/\varepsilon$
- Neutron stars $\Rightarrow P/\varepsilon$
- Both probe macroscopic EOS

6. Key Insight

Trace anomaly = bridge observable

- Connects laboratory experiments and astrophysics
- Encodes stiffness of dense matter
- Independent of microscopic details

Why $T^{\mu\mu} = 0 \Rightarrow$ Scale (Conformal) Invariance

- $T^{\mu\mu}$ measures breaking of scale invariance
- Scale transform: $x \rightarrow \lambda x$ (no preferred length scale)
- If $T^{\mu\mu} = 0 \rightarrow$ action invariant under scaling
- Implies no intrinsic mass/energy scale (massless theory)
- Example: ultra-relativistic gas $\rightarrow P = \varepsilon/3 \Rightarrow \varepsilon - 3P = 0$
- Quantum trace anomaly can break this symmetry

Why $w = p/\varepsilon = -1, 0, 1/3$ in cosmology

- $w \equiv p/\varepsilon$ controls cosmic expansion
- Radiation: relativistic → $p = \varepsilon/3 \Rightarrow w = 1/3$
- Matter: negligible pressure → $w = 0$
- Dark energy (Λ): constant density → $p = -\varepsilon \Rightarrow w = -1$
- Pressure determines acceleration vs deceleration
- DESI data: hints $w(z)$ may evolve (not exactly -1)

Energy–Momentum Tensor and Its Trace

Ideal fluid energy–momentum tensor:

$$T^{\mu\nu} = (\varepsilon + P)\, u^\mu u^\nu + P\, g^{\mu\nu}$$

Local rest frame (LRF):

$$u^\mu = (1,0,0,0), \quad g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

Components:

$$T^{00} = \varepsilon, \quad T^{ii} = P, \quad T^{0i} = 0$$

$$\Rightarrow \quad T^{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Trace:

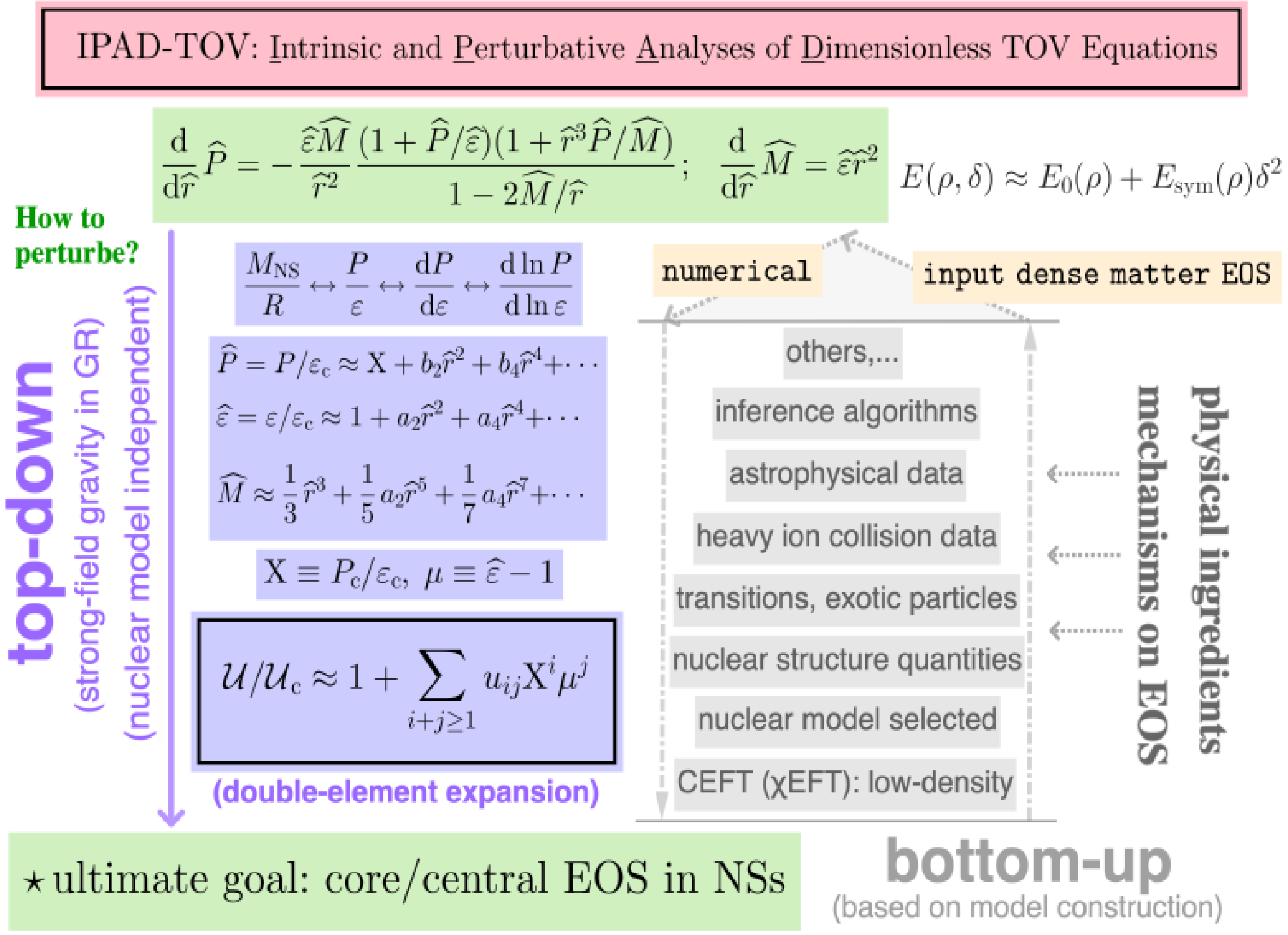
$$T^\mu_{\ \mu} = g_{\mu\nu} T^{\mu\nu} = -\varepsilon + 3P$$

X17, Symmetry, and Dense Matter EOS

- X17 anomaly (~16.7 MeV) from nuclear transitions (^{8}Be, ^{4}He)
- Possible new boson → hints of beyond Standard Model / new force
- Represents new symmetry or hidden sector coupling to nuclear matter
- Symmetry breaking & new degrees of freedom affect EOS at high density
- Analogy: emergence of new particles/phases in neutron star cores
- Key question: Do new light mediators modify $P(\varepsilon)$ or P/ε at high density?
- Bridge: nuclear experiments ↔ QCD matter ↔ neutron stars ↔ cosmology

Reduced variables (all smaller than 1): $\hat{P} = P/\varepsilon_c,\ \hat{\varepsilon} = \varepsilon/\varepsilon_c,\ \hat{r} = r/Q$ and $\hat{M} = M/W.$

Mass and radius scaling factors to maintain the TOV equations $W = \frac{1}{G}\frac{1}{\sqrt{4\pi G\varepsilon_c}} = \frac{1}{\sqrt{4\pi\varepsilon_c}},\ Q = \frac{1}{\sqrt{4\pi G\varepsilon_c}} = \frac{1}{\sqrt{4\pi\varepsilon_c}},$



Bao-Jun Cai, Bao-An Li and Zhen Zhang, ApJ 952,147(2023); PRD 108,103041(2023)

Bao-Jun Cai and Bao-An Li, PRD 109,083015(2024); PRD 112,023023(2025)

M, R and compactness scalings with $X \equiv P_c/\varepsilon_c,$

(1) Two TOV equations have three unknows: P(r), m(r) and ε(r), need the input EOS P(ε)

(2) Without using any input EOS P(ε), only scalings of M and R as functions of P_c and ε_c are obtained

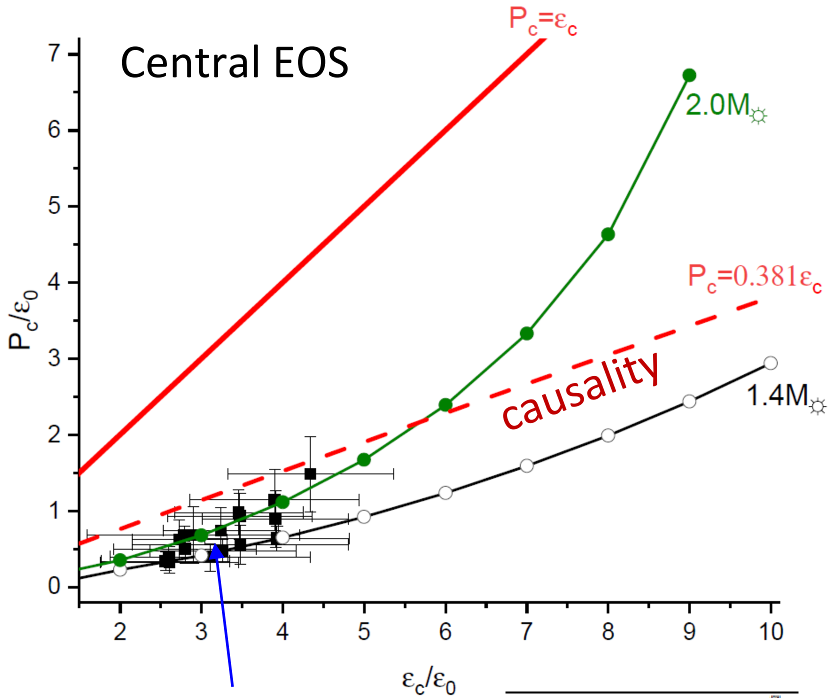
$$M_{NS} = \frac{1}{3}\hat{R}^3 W = \left(\frac{6}{\pi G^3}\right)^{1/2}\Gamma_c, \quad \text{with} \quad \Gamma_c \equiv \frac{1}{\sqrt{\varepsilon_c}}\left(\frac{X}{1+3X^2+4X}\right)^{3/2}.$$

$$R = \hat{R}Q = \left(\frac{3}{2\pi G}\right)^{1/2}\nu_c, \quad \text{with} \quad \nu_c \equiv \frac{1}{\sqrt{\varepsilon_c}}\left(\frac{X}{1+3X^2+4X}\right)^{1/2}.$$

Compactness

$$\xi \equiv \frac{M_{NS}}{R} = \frac{2}{G}\frac{X}{1+3X^2+4X} = \frac{2\Pi_c}{G}, \quad \text{with} \quad \Pi_c \equiv \frac{X}{1+3X^2+4X}.$$

Central trace anomaly $\Delta_c = 1/3 - X$



$\varepsilon_0 \approx 150\ \text{MeV/fm}^3$

Scaling coefficients from 10^5 meta + hundreds of microscopic and phenomenological EOSs existing

Verified within (2-6)% accuracy for $M_{1.4}$ by Lattimer using all EOSs he collected
B.Y. Sun and J. Lattimer, APJ 984, 30 (2025)

(3) If M or R is known from observations, the central EOS $P_c(\varepsilon_c)$ is then obtained. If both M and R are know, both P_c and ε_c are obtained

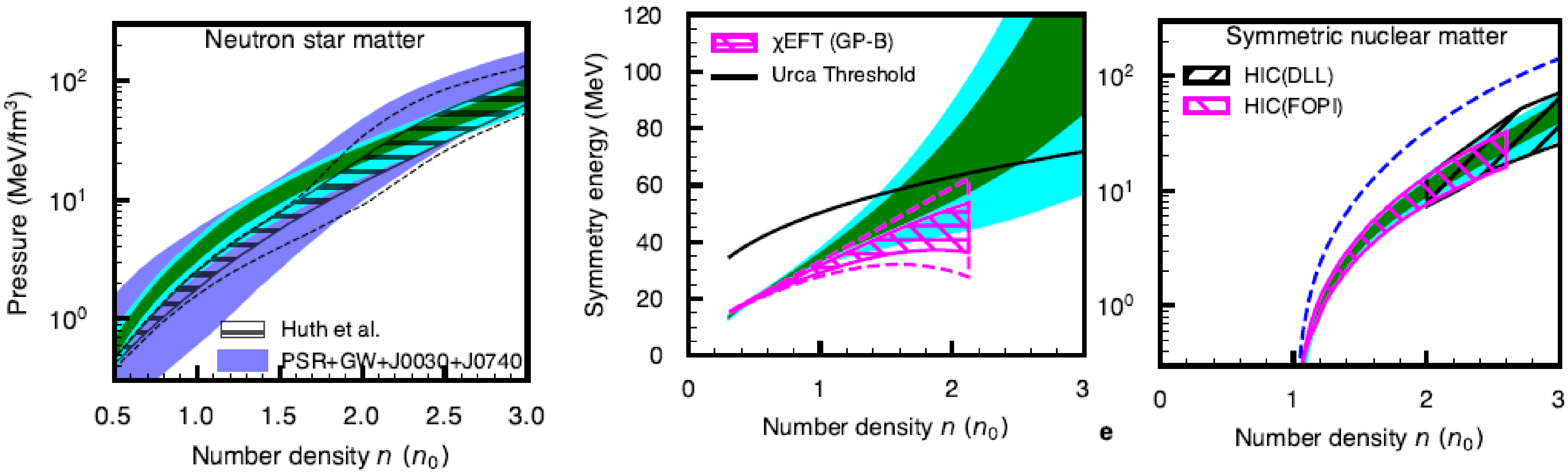
Bao-Jun Cai and Bao-An Li,
EPJA,61,55(2025)

Nature Astronomy **volume 8**, pages328–336 (2024)

Determination of the equation of state from nuclear experiments and neutron star observations

Chun Yuen Tsang, ManYee Betty Tsang, William G. Lynch, Rohit Kumar & Charles J. Horowitz

EOS=Meta-model EOS for charge neutral n+p+e+μ matter at beta-equilibrium



Typical hadronic meta models for the EOS of NSs:

Margueron, J., Hoffmann Casali, R. & Gulminelli, F. Phys. Rev. C 97,025805 (2018).

Nai-Bo Zhang, Bao-An Li, Jun Xu, *Astrophys. J.* 859, 90 (2018).

The simplest Skyrme potential still widely used

Much more complicated, isospin-momentum dependent potentials are available

$$V_q(\rho,\delta) = a(\rho/\rho_0) + b(\rho/\rho_0)^\sigma \; + V^q_{\rm asy}(\rho,\delta) + V^q_{\rm Coulomb}.$$

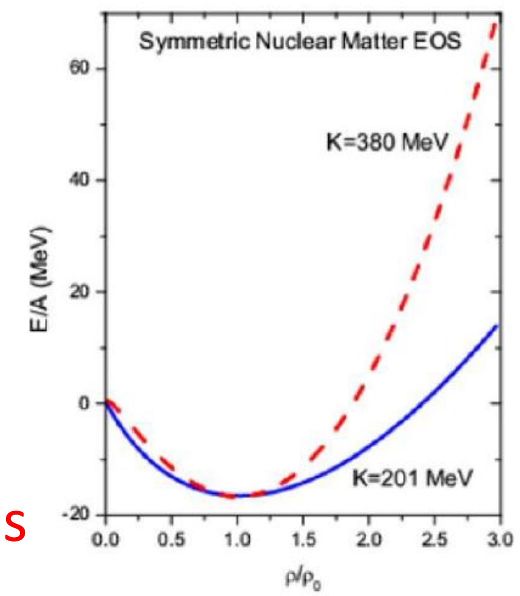
a,b and σ are fixed by (1) $E_{bin}(\rho_0)$=-16 MeV, (2) $P(\rho_0)$=0 and (3) a specified incompressibility K

$$a = -29.81 - 46.90\frac{K+44.73}{K-166.32}\ ({\rm MeV}),$$

$$b = 23.45\frac{K+255.78}{K-166.32}\ ({\rm MeV}),$$

$$\sigma = \frac{K+44.73}{211.05}.$$

EOS of symmetric matter



K, J_0 and Z_0 are degenerate: once K is fixed, J_0 and Z_0 are fixed, efficient & sufficient but leads to confusions

$$P(\rho) = \rho^2\frac{dE_0(\rho)}{d\rho} = \frac{\rho^2}{\rho-\rho_0}[K(\frac{\rho-\rho_0}{3\rho_0})^2 + \frac{J_0}{2}(\frac{\rho-\rho_0}{3\rho_0})^3 + \frac{Z_0}{6}(\frac{\rho-\rho_0}{3\rho_0})^4].$$

Bayesian calibration of SNM EOS parameters using an empirical pressure band

	K_0	J_0	Z_0
Av	235	−200	−146
σ	30	200	1728
Min	145	−800	−5330
Max (3σ)	325	400	5038

Prior ranges of SNM EOS parameters based on theories and data available

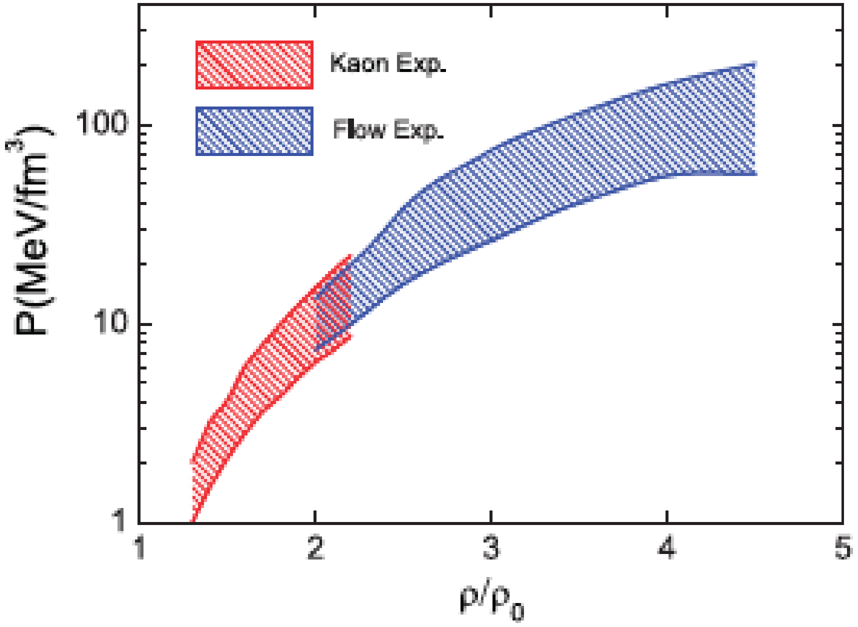
Margueron J, Hoffmann C R and Gulminelli F 2018 PRC97, 025805 and 025806

Antic S, Chatterjee D, Carreau T and Gulminelli F 2019 J. Phys. G: Nucl. Part Phys. 46 065109

The pressure in symmetric nuclear matter

$$P(\rho) = \rho^2 \frac{dE_0(\rho)}{d\rho} = \frac{\rho^2}{\rho - \rho_0}\left[K_0\left(\frac{\rho-\rho_0}{3\rho_0}\right)^2 + \frac{J_0}{2}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^3 + \frac{Z_0}{6}\left(\frac{\rho-\rho_0}{3\rho_0}\right)^4\right].$$

K_0=incompressibility
J_0=skewness
Z_0=kurtosis



Constraints on the EOS of symmetric nuclear matter from heavy-ion collisions

Danielewicz P. I., Lacey R and Lynch W G 2002 *Science* 298 1592

Fuchs C 2006 *Prog. Part. Nucl. Phys.* 56 1
Lynch W G *et al* 2009 *Prog. Part. Nucl. Phys.* 62 427

The bottom line: the extracted pressure band from 1.3 to 4.5ρ_0 from kaon yields & flow does **NOT** constrain K_0 but J_0 and Z_0

Bayesian calibration of the SNM EOS parameters

Wen-Jie Xie and Bao-An Li
JPG 48, 025110 (2021)

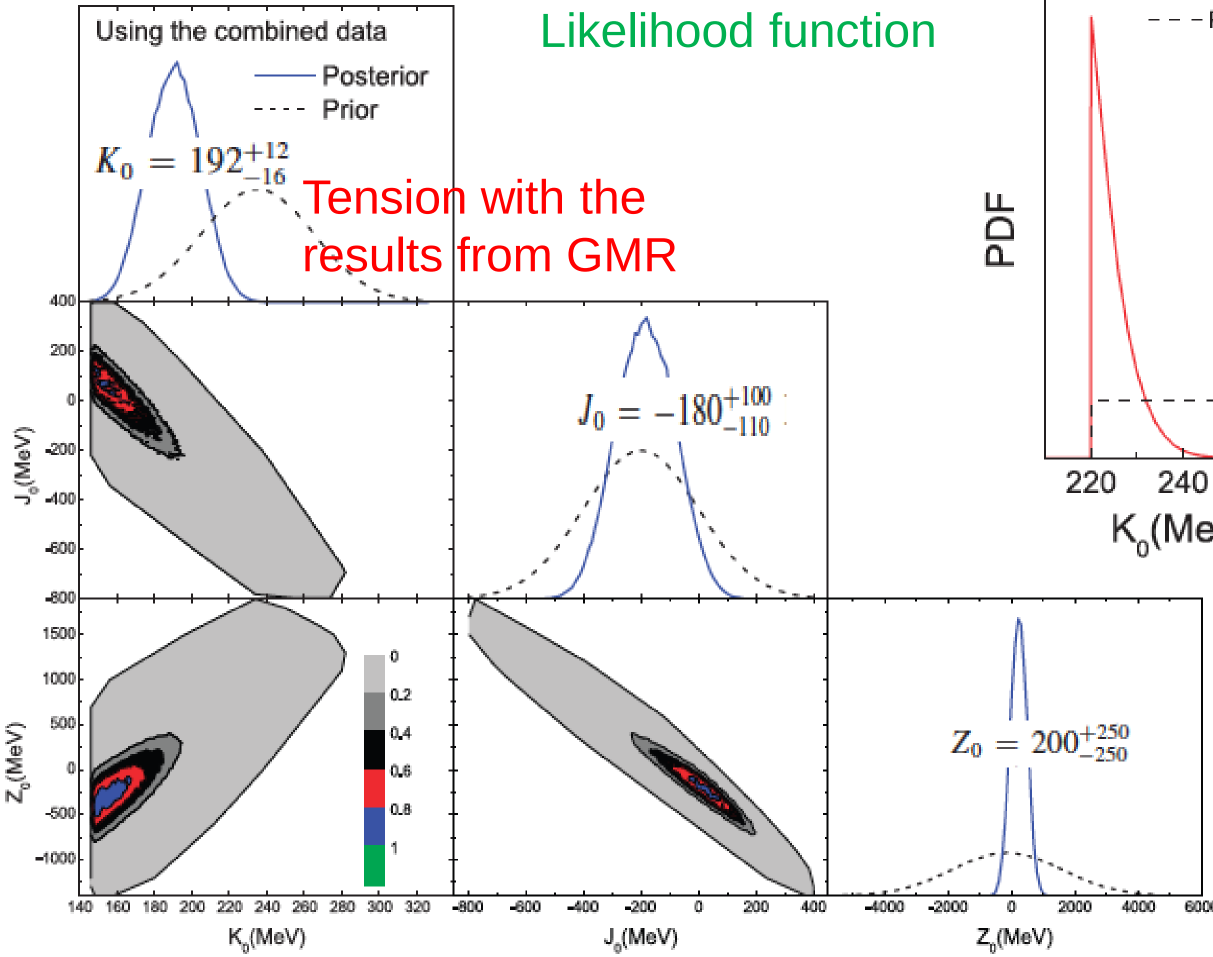
Using the empirical pressure band as quasi-data

Posterior PDF: $P(\mathcal{M}(K_0, J_0, Z_0)|D) = CP(D|\mathcal{M}(K_0, J_0, Z_0))P(\mathcal{M}(K_0, J_0, Z_0)),$

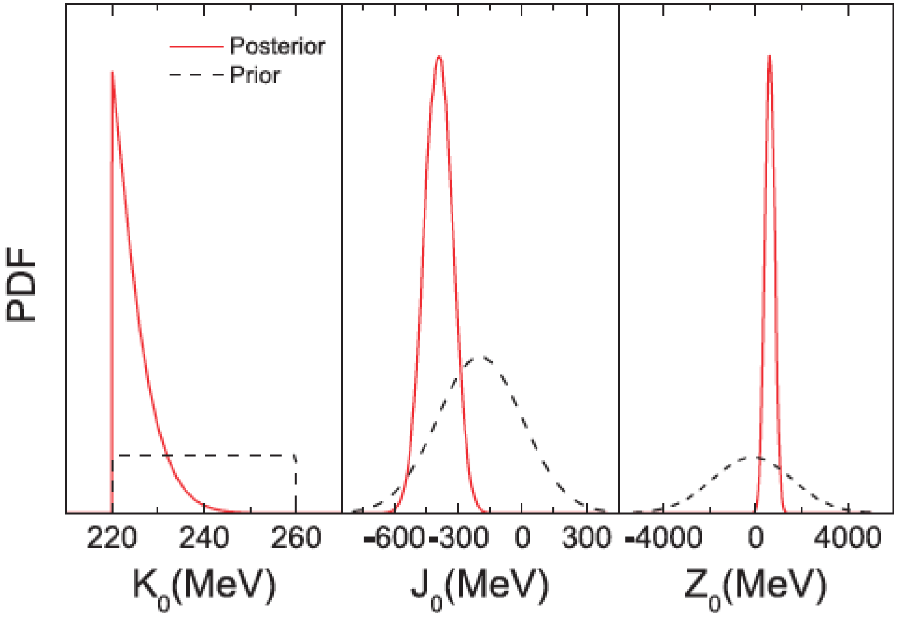
$$P[D|\mathcal{M}(K_0, J_0, Z_0)] = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_{D,j}} \exp\left[-\frac{(P_{\mathrm{th},j} - P_{D,j})^2}{2\sigma_{D,j}^2}\right]$$

Likelihood function

Prior PDF (Probability Dist. Func.)



Tension with the results from GMR

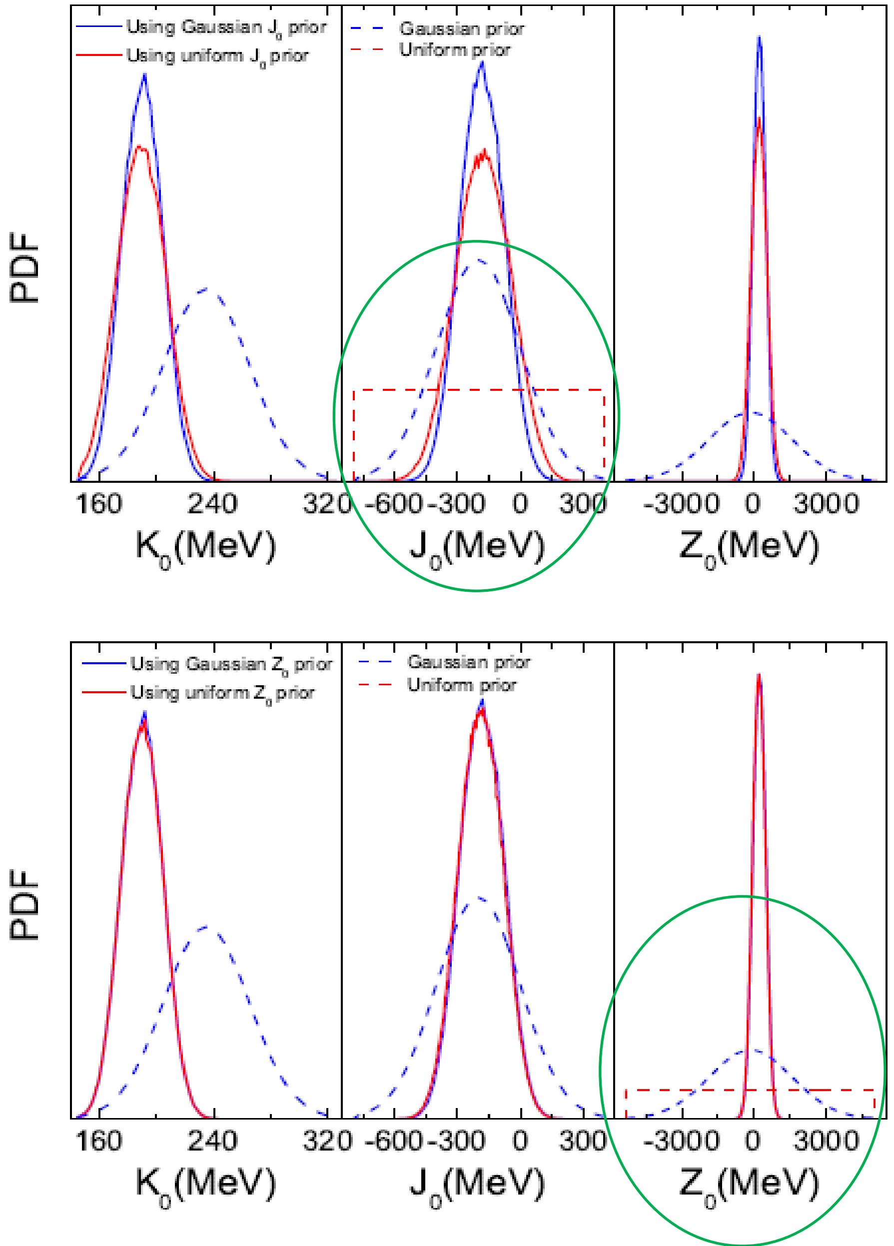


Kaon production and elliptical flow data from GSI favors a K_0 smaller than the fiducial value from ISGMR data: 235±30 MeV

A. Le Fe`vre et al., NPA945, 112 (2016).

Y.J. Wang et al., PLB778, 207 (2018)

Except for K, the posterior PDFs are Independent of the priors because of the strong constraining power of the pressure band from flow and kaon production

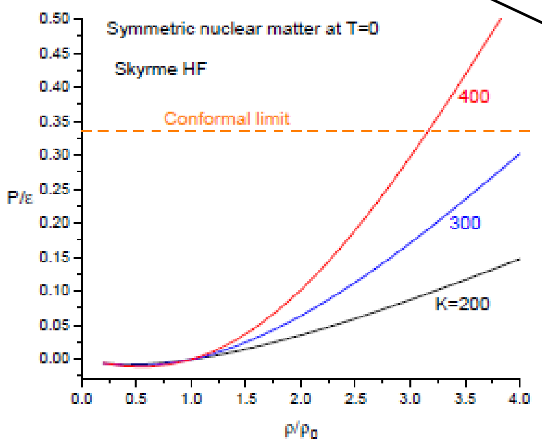


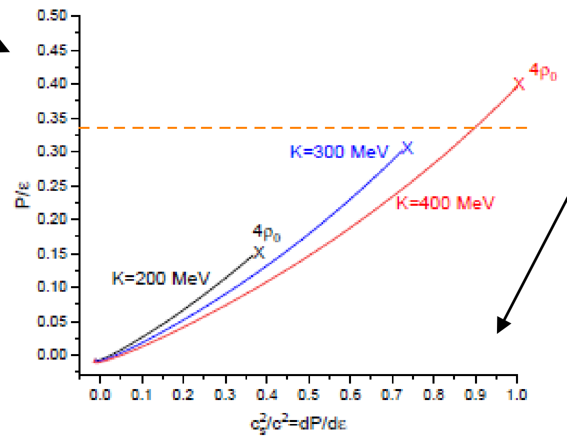
Φ=P/ϵ is expected to be LESS model dependent, compared with P(ρ) and ϵ(ρ)

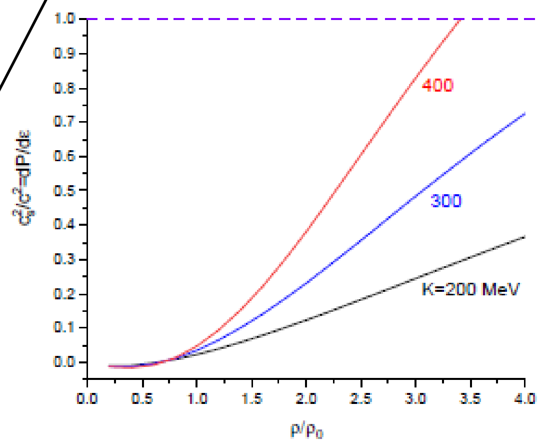
SHF SNM EOS used in the IBUU transport model

EOS parameter P/ϵ

Speed of sound squared

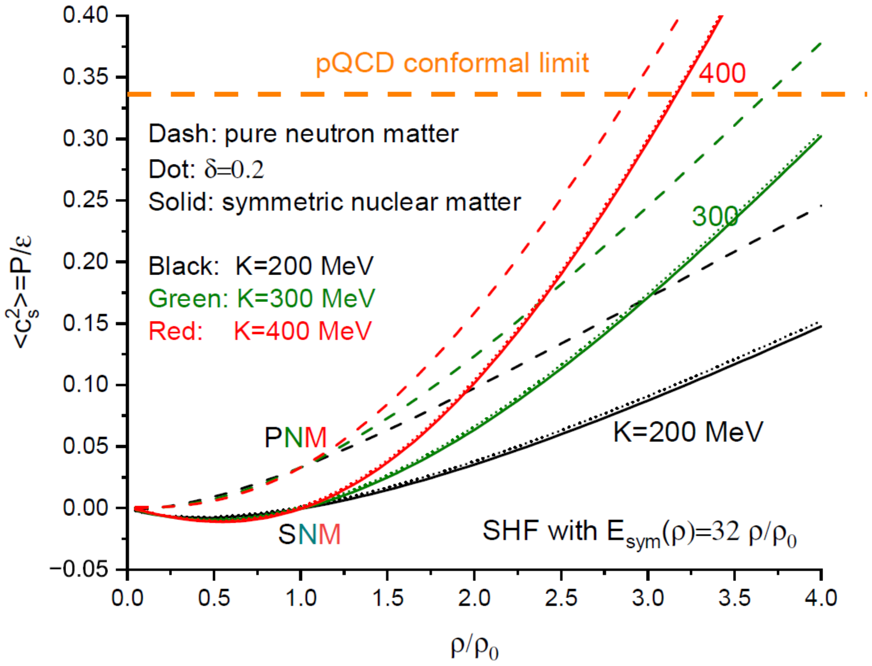






E_{sym} contribution is neglectable

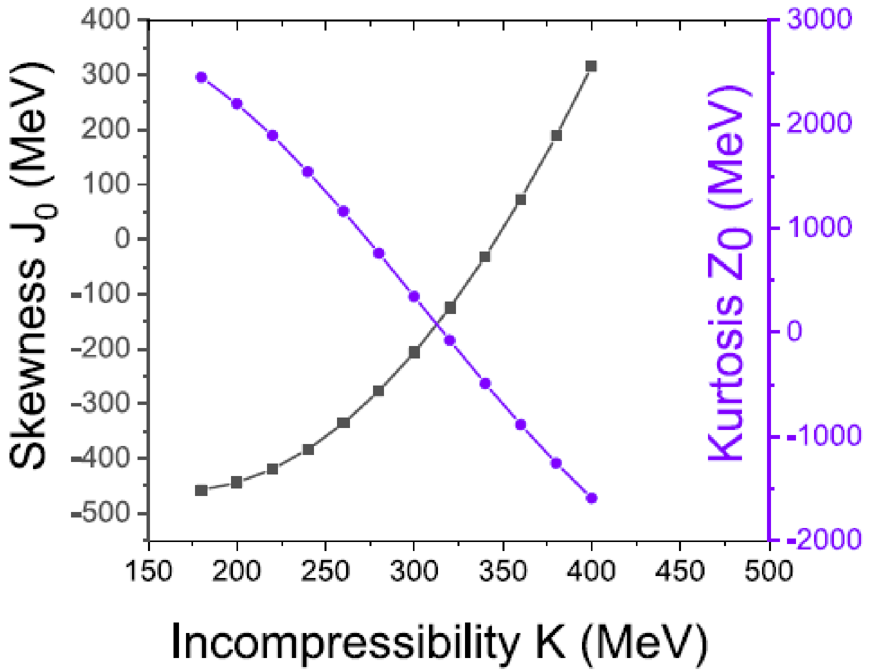
Evolution of the δ(ρ) is not recorded in the IBUU+GP+Bayesian analyses



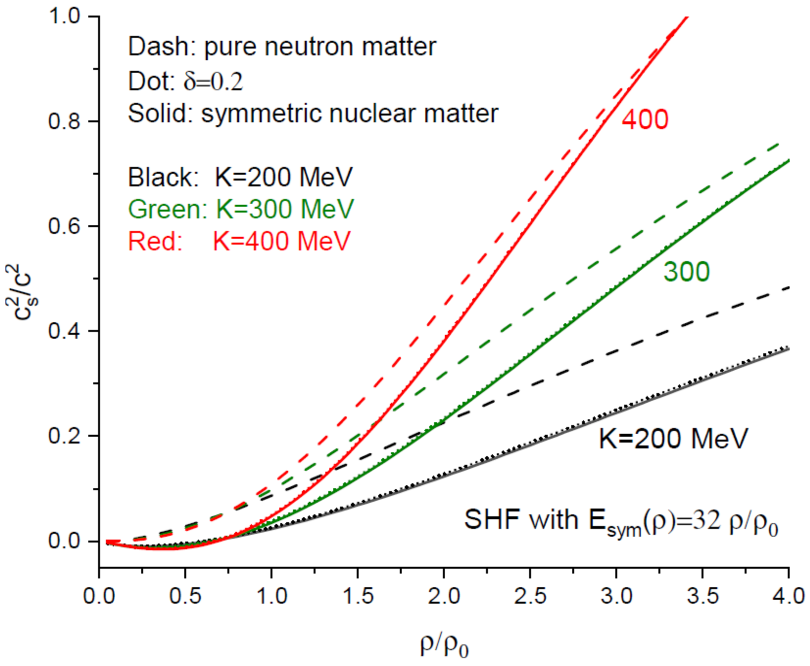
The simplest Skyrme potential still widely used

$$V_q(\rho,\delta) = a(\rho/\rho_0) + b(\rho/\rho_0)^{\sigma} + V^q_{\rm asy}(\rho,\delta) + V^q_{\rm Coulomb}.$$

Correlations and anti-correlations among incompressibility, skewness and kurtosis



The resulting stiffness of high-density matter scan a broad range necessary for Bayesian analyses



Why it is important to study in-medium baryon-baryon scattering cross sections

- In-medium effects on baryon-baryon scattering cross sections : Pauli blocking (modeled independently) , shifts in thresholds, interaction matrix elements, shifts in kinematics and phase space due to effective mass changes with momentum-dependent potentials, modifications of angular distributions due to tensor force and/or quantum entanglement.
- Essential for understanding transport properties and validating nuclear many-body theories.
- Critical for extracting the nuclear EOS from heavy-ion collisions.
- Isospin-dependent in-medium cross sections impact E_{sym} studies via isospin transport.
- Direct connect to neutron star structure, mergers, and dense matter in astrophysics.

Current status

- Quantitative knowledge is limited, and most results are model-dependent.
- Key source of uncertainty in simulations of nuclear matter, heavy-ion collisions, and neutron star dynamics.

What we do

the simplest: modify the total cross section by a factor $X \equiv \sigma_{NN}^{med}/\sigma_{NN}^{free})$

What we want to know

How X and K_0 evolve with beam energy in a 2-parameter Bayesian inference from FOPI flow excitation function

Some technical details of the Bayesian + GP + IBUU:

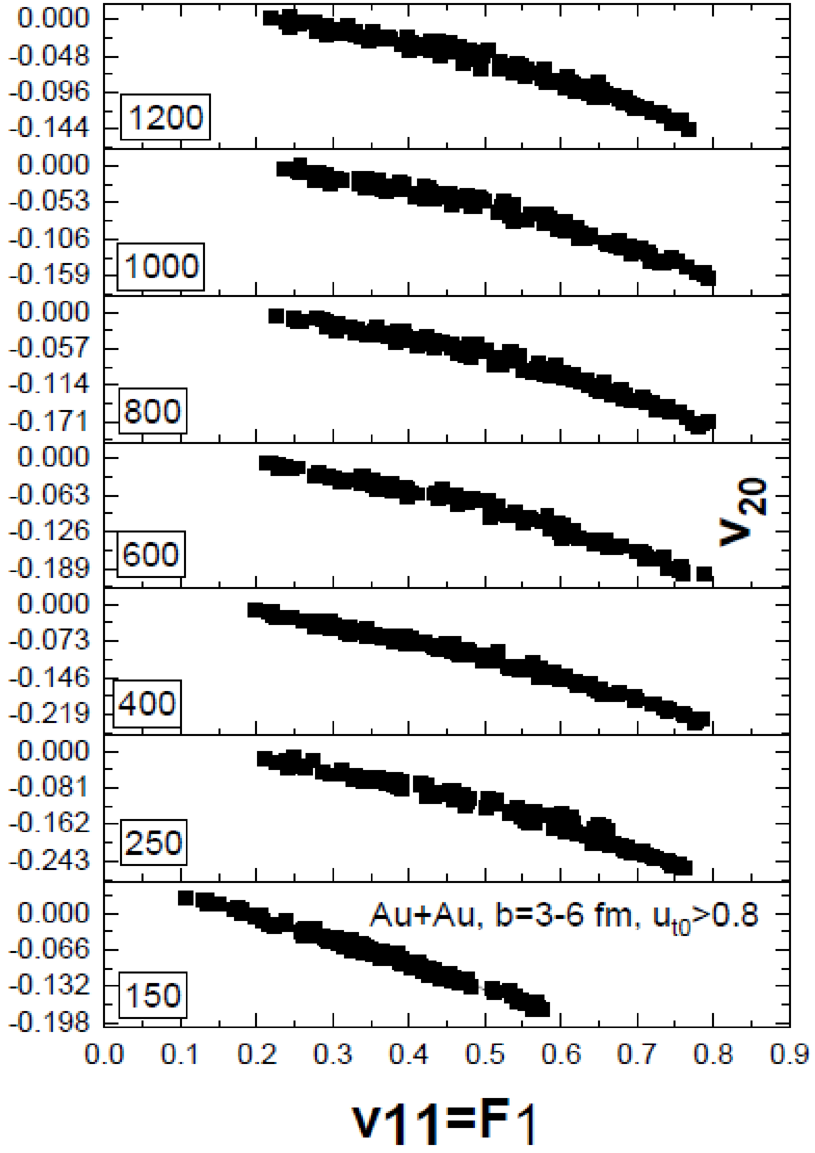
1. **Two parameters: X and K, Skyrme force with no-momentum dependence**
2. **Uniform priors, X between 0.3 and 2, K between 180 and 400 large enough to account effect of missing momentum-dependence**
3. **120 sets of (X,K) at each beam energy on Latin hyper lattice for training, each training uses 20,000 events with 100 impact parameter randomly generated between 3-6 fm**
4. **NO clusters, free nucleons at freeze-out identified by a sharp density cut at $\rho_0/8$**
5. **Gaussian Process emulator used**
6. **Only the v_{11} and v_{20} data from FOPI & HADES used (+ stopping observables in progress)**

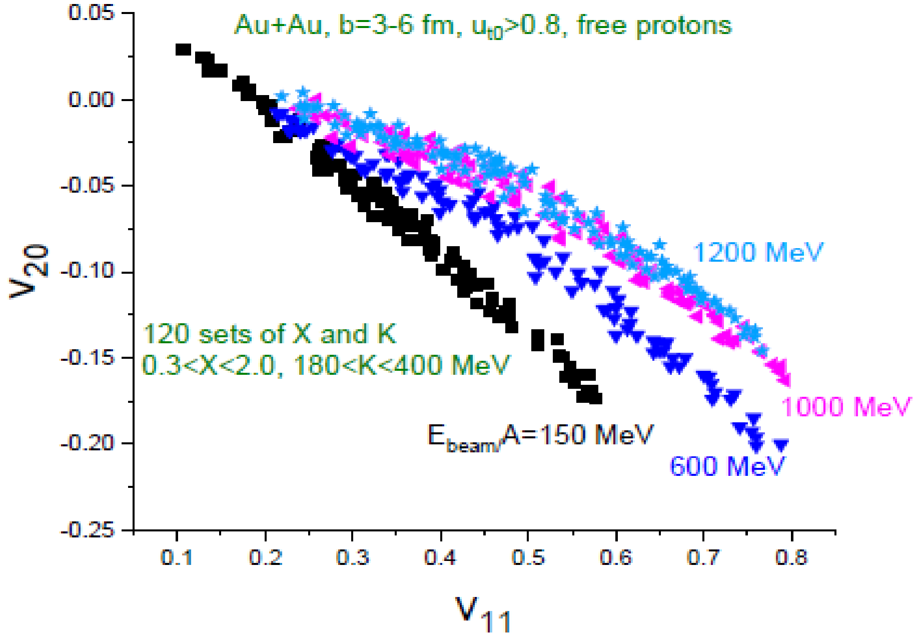
We did what we could in a poor place

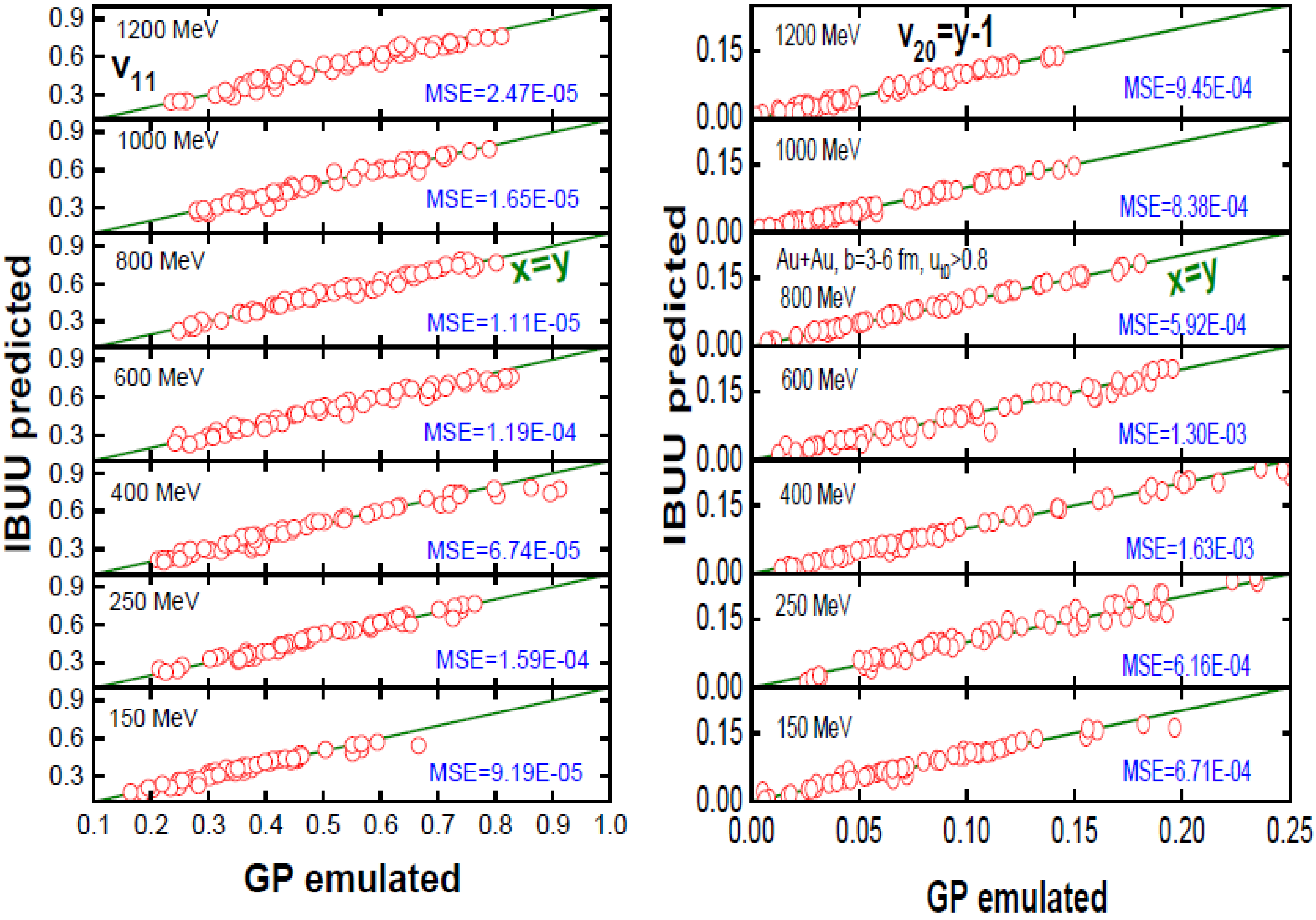
Similar 2-parameters (controlling n-p effective mass splitting and density-dependence of E_{sym}) within MDI was developed/trained

12-parameter MDI with short-range correlations were developed, BUT we can NOT afford to train it at this time

Anti-correlation between directed and elliptical flow



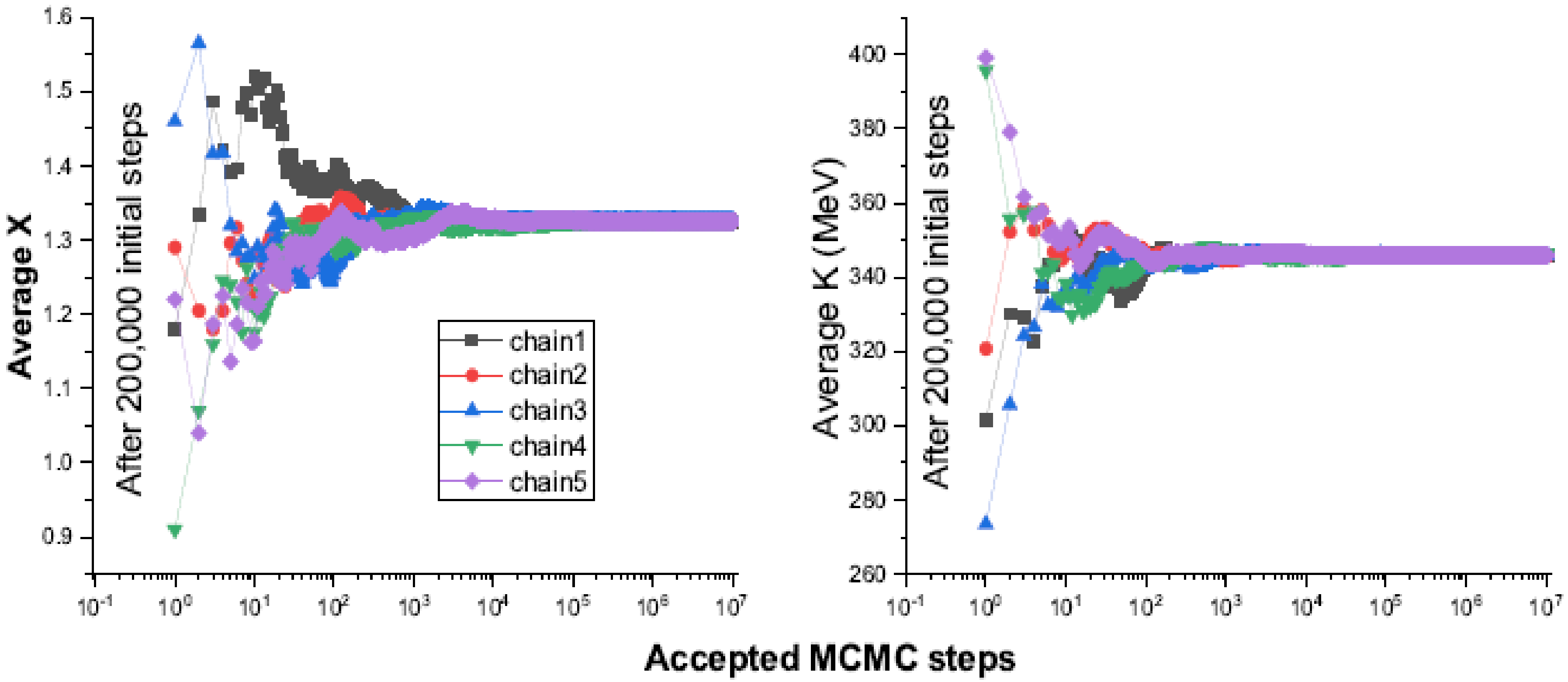


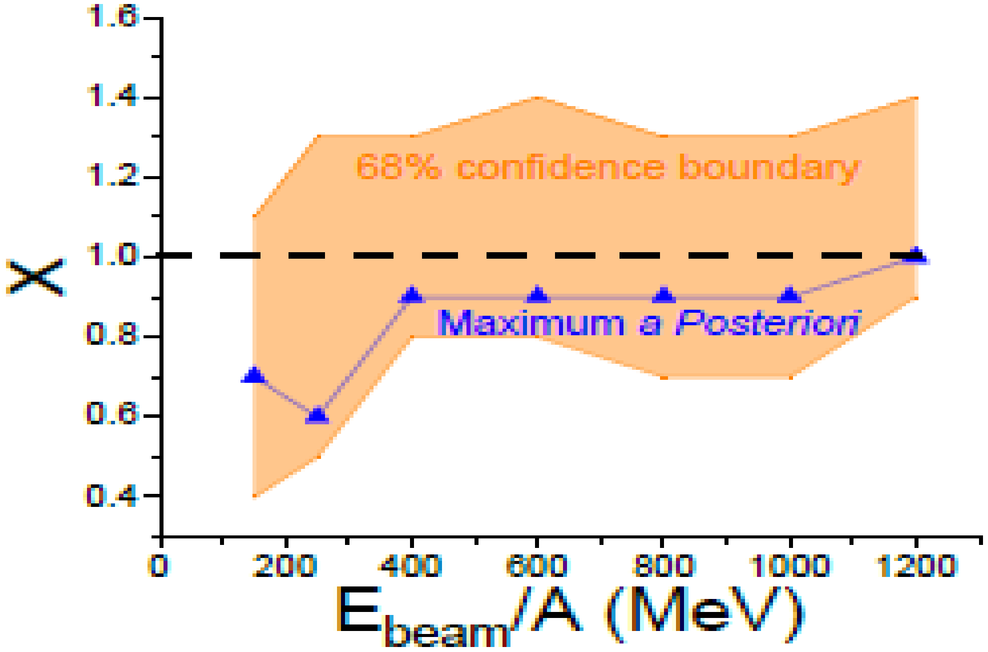


IBUU predicted
GP emulated
v11
x=y
1200 MeV
MSE=2.47E-05
1000 MeV
MSE=1.65E-05
800 MeV
MSE=1.11E-05
600 MeV
MSE=1.19E-04
400 MeV
MSE=6.74E-05
250 MeV
MSE=1.59E-04
150 MeV
MSE=9.19E-05
v20=y-1
Au+Au, b=3-6 fm, ut0>0.8
1200 MeV
MSE=9.45E-04
1000 MeV
MSE=8.38E-04
800 MeV
MSE=5.92E-04
600 MeV
MSE=1.30E-03
400 MeV
MSE=1.63E-03
250 MeV
MSE=6.16E-04
150 MeV
MSE=6.71E-04

Quick convergence of MCMC chains

Au+Au, E_{beam}=1.23 GeV/A, HADES





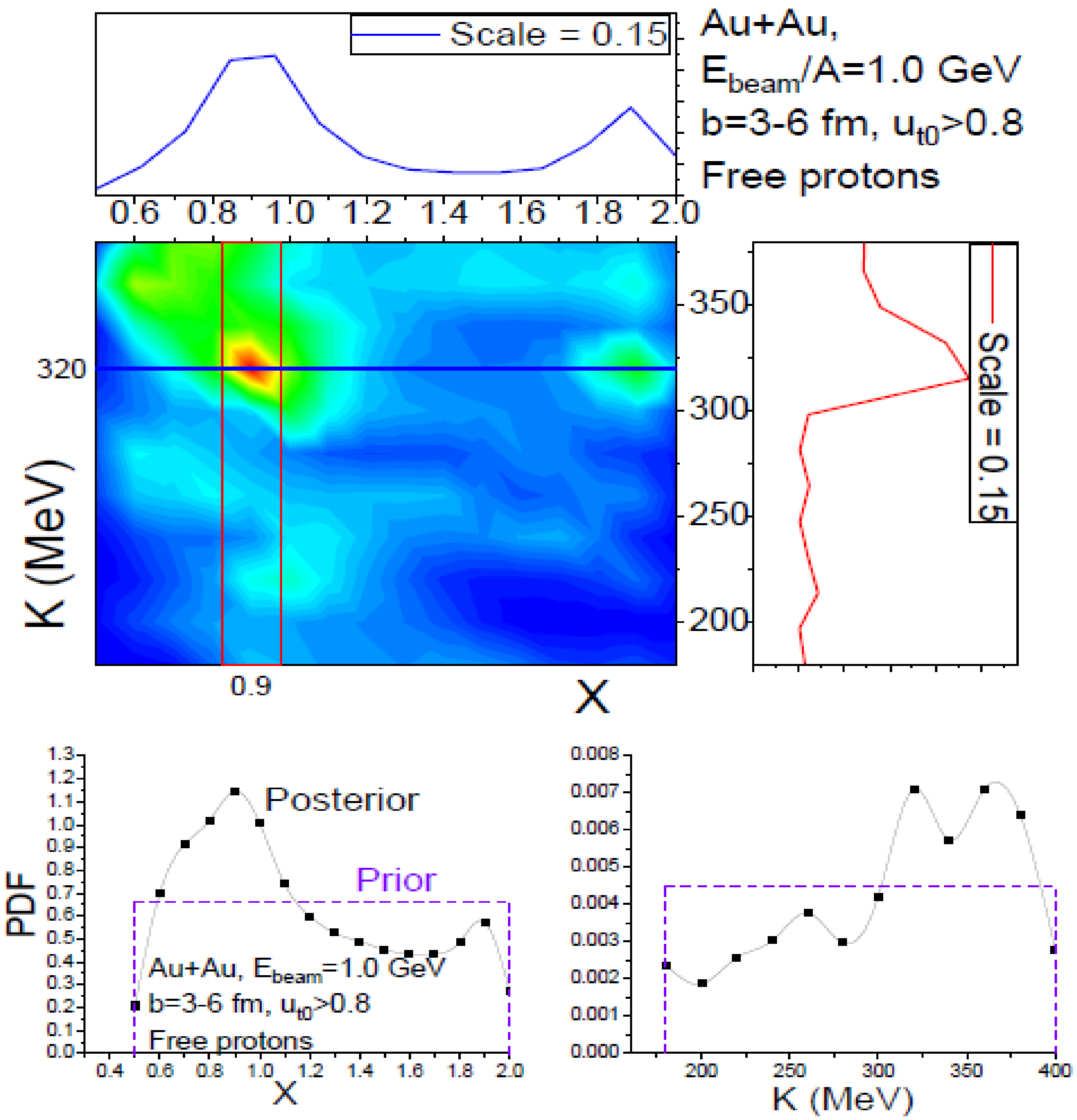
matter becomes harder
(measured by speed of sound)

E_{beam} (MeV)	$\langle X \rangle$	$\langle K \rangle$ (MeV)	X (MaP)
150	0.84 ± 0.34	267.6 ± 55.0	$0.7^{+1.1}_{-0.4}$
250	1.09 ± 0.43	286.9 ± 61.8	$0.6^{+1.3}_{-0.5}$
400	1.12 ± 0.41	306.1 ± 63.4	$0.9^{+1.3}_{-0.8}$
600	1.19 ± 0.41	312.1 ± 60.2	$0.9^{+1.4}_{-0.8}$
800	1.15 ± 0.41	309.0 ± 62.7	$0.9^{+1.3}_{-0.7}$
1000	1.15 ± 0.42	310.4 ± 61.5	$0.9^{+1.3}_{-0.7}$
1200	1.23 ± 0.40	323.0 ± 52.8	$1.0^{+1.4}_{-0.9}$

Measuring mostly K_0 & J_0

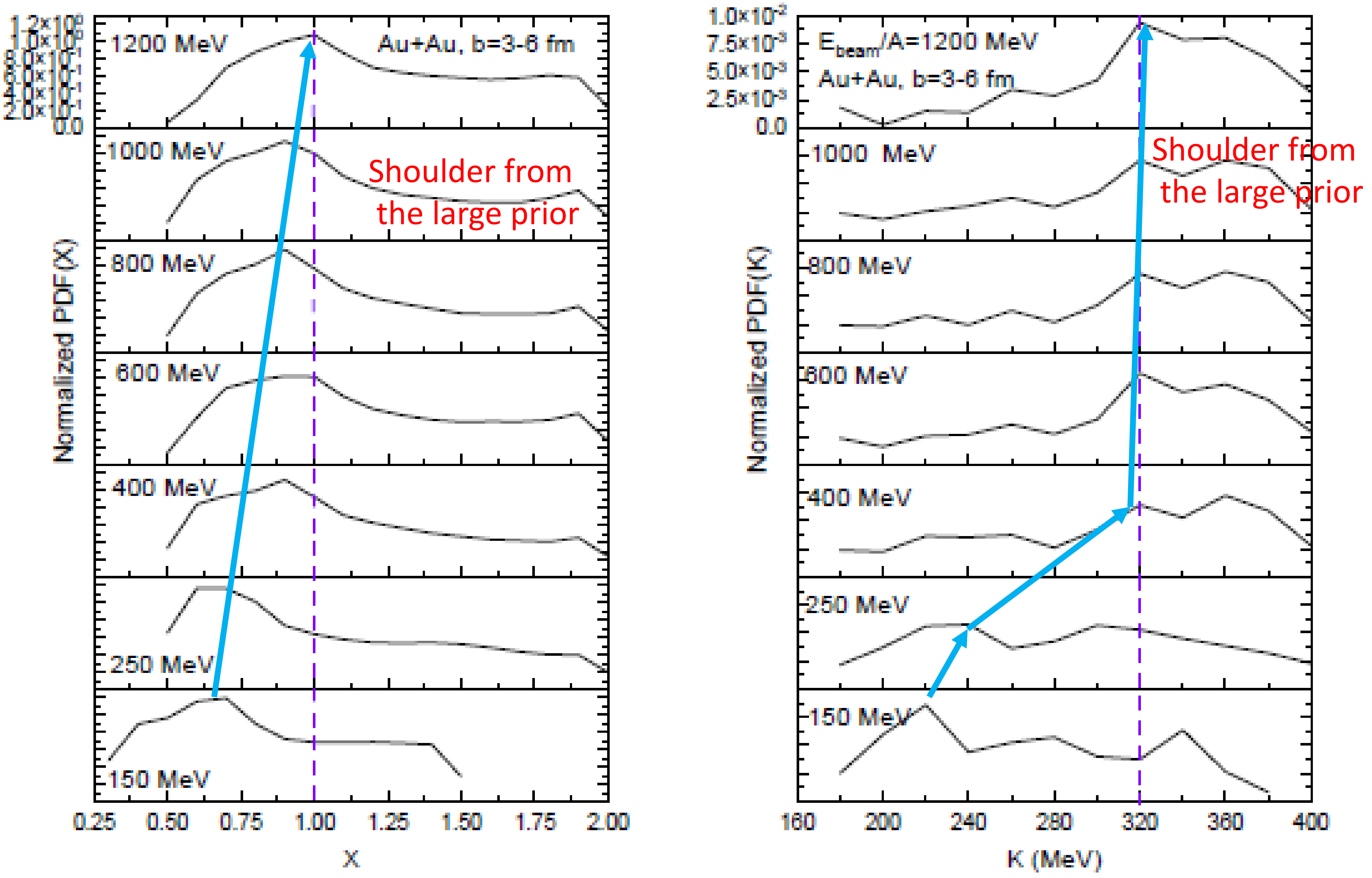
Measuring mostly J_0 & Z_0 corresponding to these K_0

E/A=1000 MeV



Scale = 0.15
Au+Au,
E_{beam}/A=1.0 GeV
b=3-6 fm, u_{t0}>0.8
Free protons
0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
320
K (MeV)
350
300
250
200
Scale = 0.15
0.9
X
PDF
Posterior
Prior
Au+Au, E_{beam}=1.0 GeV
b=3-6 fm, u_{t0}>0.8
Free protons
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
X
0.000 0.001 0.002 0.003 0.004 0.005 0.006 0.007 0.008
200 250 300 350 400
K (MeV)

Evolution of X and K with beam energy



Bayesian constraints on neutron star EOS with a smooth hadron-quark crossover

Xavier Grundler and Bao-An Li, PRD (2026) in press.

$$
\begin{aligned}
P(\varepsilon) &= P_{\rm HM}(\varepsilon) f_{-}(\varepsilon) + P_{\rm QM}(\varepsilon) f_{+}(\varepsilon), \\
f_{\pm}(\varepsilon) &= \frac{1}{2}\left[1 \pm \tanh\left(\frac{\varepsilon - \overline{\varepsilon}}{\Gamma}\right)\right],
\end{aligned}
$$

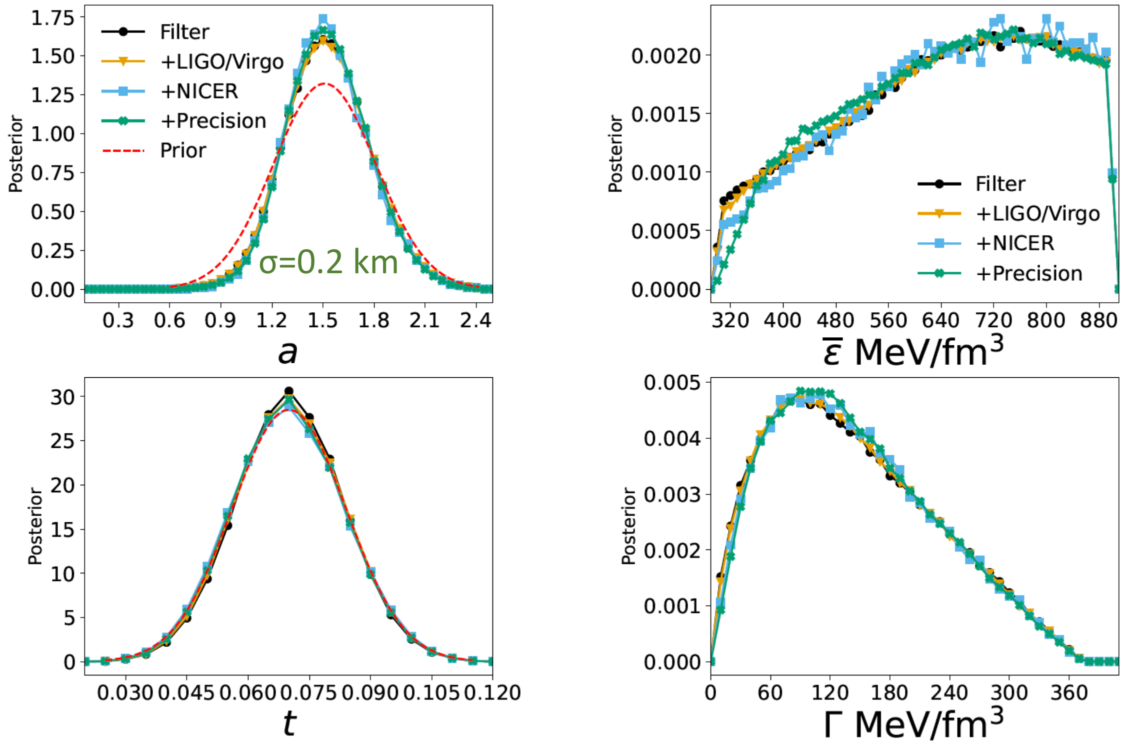
QM EOS: trace anomaly-based parameterization

$$\Delta = \frac{1}{3}\left(1 - f\, t\, \varepsilon_*^a\right) e^{-t\varepsilon_*^a}$$

$$\Delta \equiv \frac{1}{3} - \frac{P}{\varepsilon}.$$

Quark matter EOS parameters

Hadron-quark transition parameters



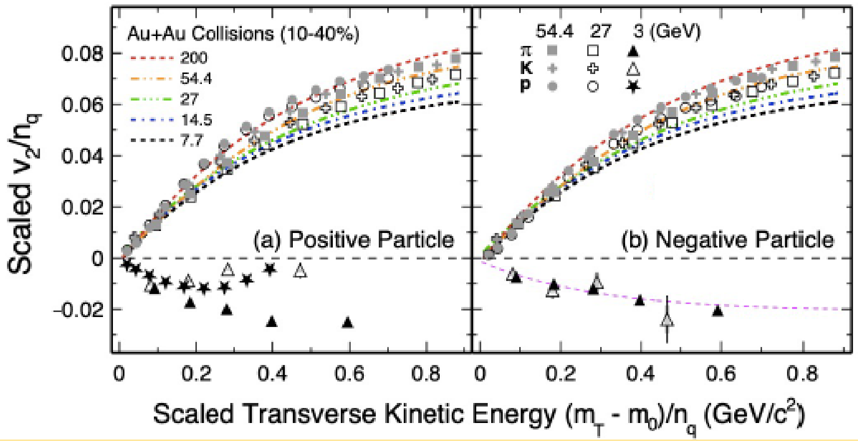
Given the limitations → What can we robustly infer?

Nu Xu,

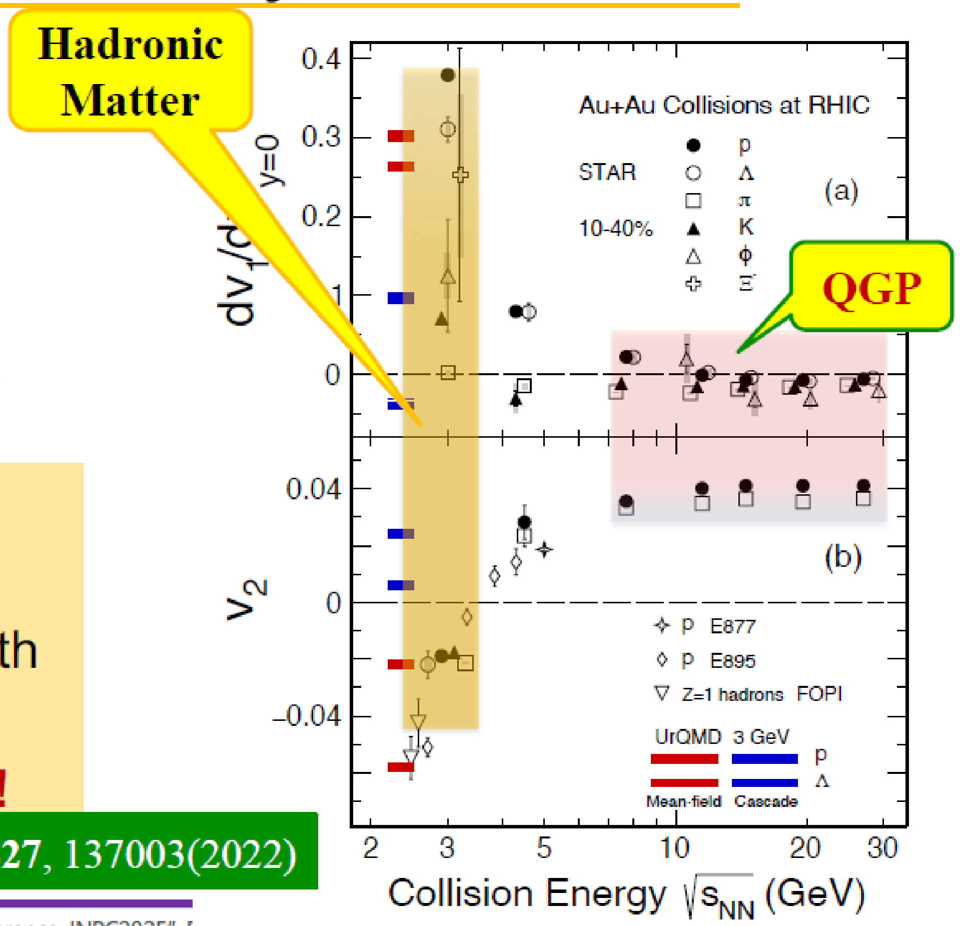
indications on hadron-quark transition density in hot dense matter from BES/RHIC

Partonic Collectivity or Not

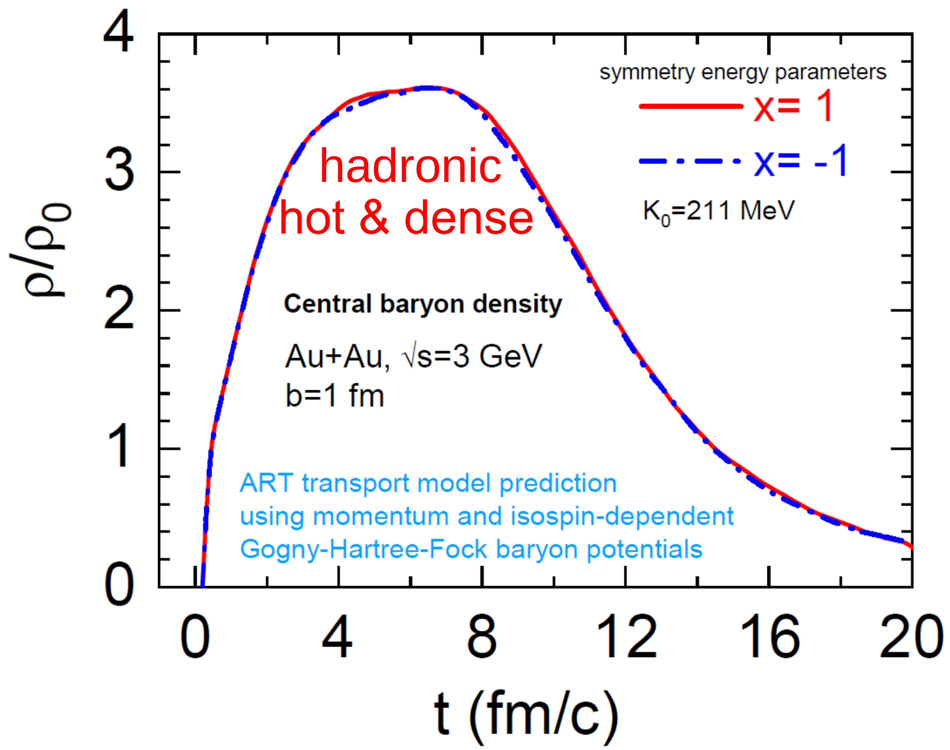
INPC 2025



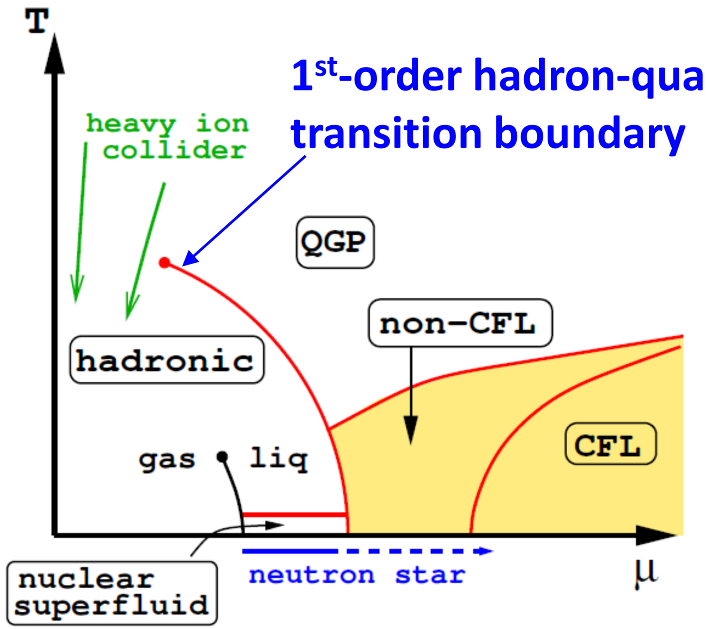
- At **3 GeV**, NCQ scaling is absent;
- Transport model calculations, with baryonic mean field, reproduce both v_1 and v_2 results;
- **Hadronic interactions dominant!**



STAR: PLB**827**, 137003(2022)



G.C. Yong, B.A. Li, Z.G. Xiao and Z.W. Lin
Phys. Rev. C **106**, 024902 (2022)



Mark G. Alford, Andreas Schmitt,
Krishna Rajagopal and Thomas Schafer,
Review of Modern Physics, 80, 1455 (2008)

Indication (BES/RHIC data and transport model simulation):
The 1st order hadron-quark phase transition can NOT happen below about 3.6ρ_0 in cold (T=0) neutron stars

➡ **Modify the prior range of hadron-quark transition density in Bayesian analyses**

pQCD probably does NOT constrain the NS EOS

algorithmic inference

PHYSICAL REVIEW C **107**, L052801 (2023)

Letter

Perturbative QCD and the neutron star equation of state

R. Somasundaram,[1,*] I. Tews,[2] and J. Margueron[1]
[1]*Université de Lyon, Université de Claude Bernard Lyon 1, CNRS/IN2P3, IP2I Lyon, UMR 5822, F-69622 Villeurbanne, France*
[2]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

(Received 14 May 2022; accepted 4 May 2023; published 26 May 2023)

We construct a physics-agnostic approach to the neutron star (NS) equation of state (EoS) based on a sound speed model, which connects both low-density information from nuclear theory and high-density constraints from perturbative quantum chromodynamics (pQCD). Using this approach, we study the impact of pQCD calculations on NS EoS that have been constrained by astrophysical observations. We find that pQCD affects the EoS mainly beyond the densities realized in NS. Furthermore, we observe an interesting interplay between pQCD and astrophysical constraints, with pQCD preferring softer EoS for the heaviest NS while recent NICER observations suggest the EoS to be stiffer. We explore the sensitivity of our findings to pQCD uncertainties and study the constraining power of pQCD if future observations of heavy NS were to suggest radii larger than 13 km.

DOI: 10.1103/PhysRevC.107.L052801

"We find that pQCD affects the EOS mainly beyond the densities realized in NSs"

pQCD theories

PHYSICAL REVIEW C **111**, 015810 (2025)

Reexamining constraints on neutron star properties from perturbative QCD

Dake Zhou*
Department of Physics, University of Washington, Seattle, Washington 98195, USA;
Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195, USA;
Department of Physics, University of California Berkeley, Berkeley, California 94720, USA;
and Department of Physics Astronomy, Northwestern University, Evanston, Illinois 60208, USA

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The implications of perturbative QCD (pQCD) calculations on neutron stars are carefully examined. While pQCD calculations above baryon chemical potentials $\mu_B \approx 2.4$ GeV demonstrate the potential of ruling out a wide range of neutron star equations of state (EOSs), such constraints only affect the most massive neutron stars in the vicinity of the Tolman-Oppenheimer-Volkoff limit, resulting in constraints that are orthogonal to current or expected astrophysical bounds. In the most constraining scenario, pQCD considerations favor low values of the squared speed sound C_s at high μ_B relevant for the most massive neutron stars, but leave predictions of the radii and tidal deformabilities almost unchanged. Such considerations become irrelevant if the maximum speed of sound squared inside neutron stars does not exceed about $C_{s,\max} \approx 0.5$, or if pQCD breaks down below $\mu_B \approx 2.9$ GeV. Furthermore, the large pQCD uncertainties preclude any meaningful bounds on the neutron star EOS at the moment. Interestingly, if pQCD predictions for the pressure at around $\mu_B \approx 2.5$ GeV are refined and found to be low ($\lesssim 1.5\ \mathrm{GeV/fm^3}$), evidence for a soft neutron star inner core EOS in combination with the existence of two-solar-mass pulsars would indicate the presence of color superconductivity beyond neutron star densities. I point out that two-solar-mass pulsars place robust upper bounds on this nonperturbative effect and require the pairing gap to be less than $\Delta_{\mathrm{CFL}} \leqslant 500$ MeV at $\mu_B \approx 2.5$ GeV.

DOI: 10.1103/PhysRevC.111.015810

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estimate for chemical potential in NS cores:

$$P \approx 250\,\mathrm{MeV/fm^3},\ \varepsilon \approx 800\,\mathrm{MeV/fm^3},\ \rho \approx 6\rho_0 \approx 0.96\,\mathrm{fm^{-3}}$$

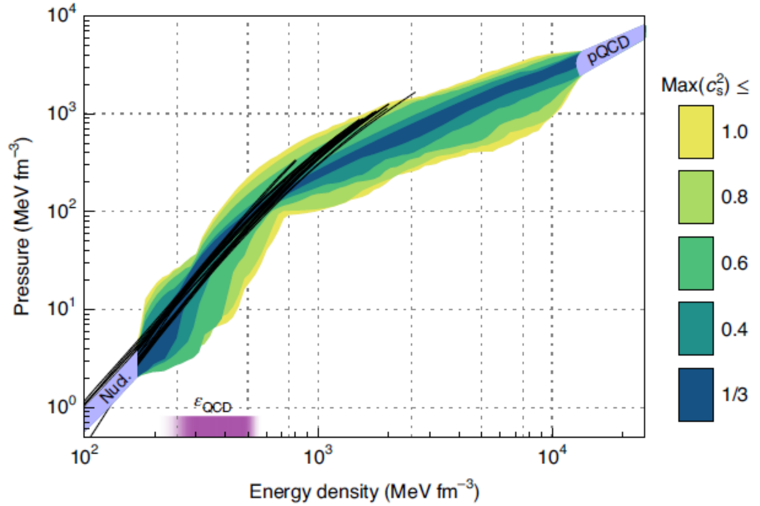
$$P + \varepsilon = \mu_B \rho \rightarrow \mu_B = \frac{P+\varepsilon}{\rho} \approx 1.1\,\mathrm{GeV} < \mu_B^{\mathrm{pQCD}} \approx 2.5\,\mathrm{GeV}$$

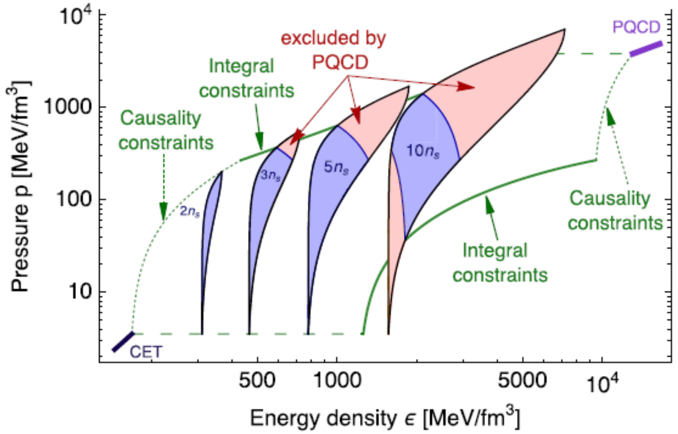
EOS above the maximum density reached in isolated NSs

possibly relevant for modeling post-merger physics at finite temperature

nature physics — LETTERS
https://doi.org/10.1038/s41567-020-0914-9

Eemeli Annala[1], Tyler Gorda[2] ✉, Aleksi Kurkela[3,4] ✉, Joonas Nättilä[5,6,7] and Aleksi Vuorinen[1] ✉





How Perturbative QCD Constrains the Equation of State at Neutron-Star Densities

Oleg Komoltsev and Aleksi Kurkela PRL 128, 202701 (2022)

$$\mathcal{L} = \mathcal{L}_{\rm GW} \times \mathcal{L}_{\rm NICER} \times \mathcal{L}_{M_{\rm max}} \times \mathcal{L}_{\rm pQCD}. \qquad \mathcal{L}_{\rm pQCD} = P(n_{\rm L}, \varepsilon(n_{\rm L}, {\rm EOS}), p(n_{\rm L}, {\rm EOS}))$$

Equation of state at neutron-star densities and beyond from perturbative QCD

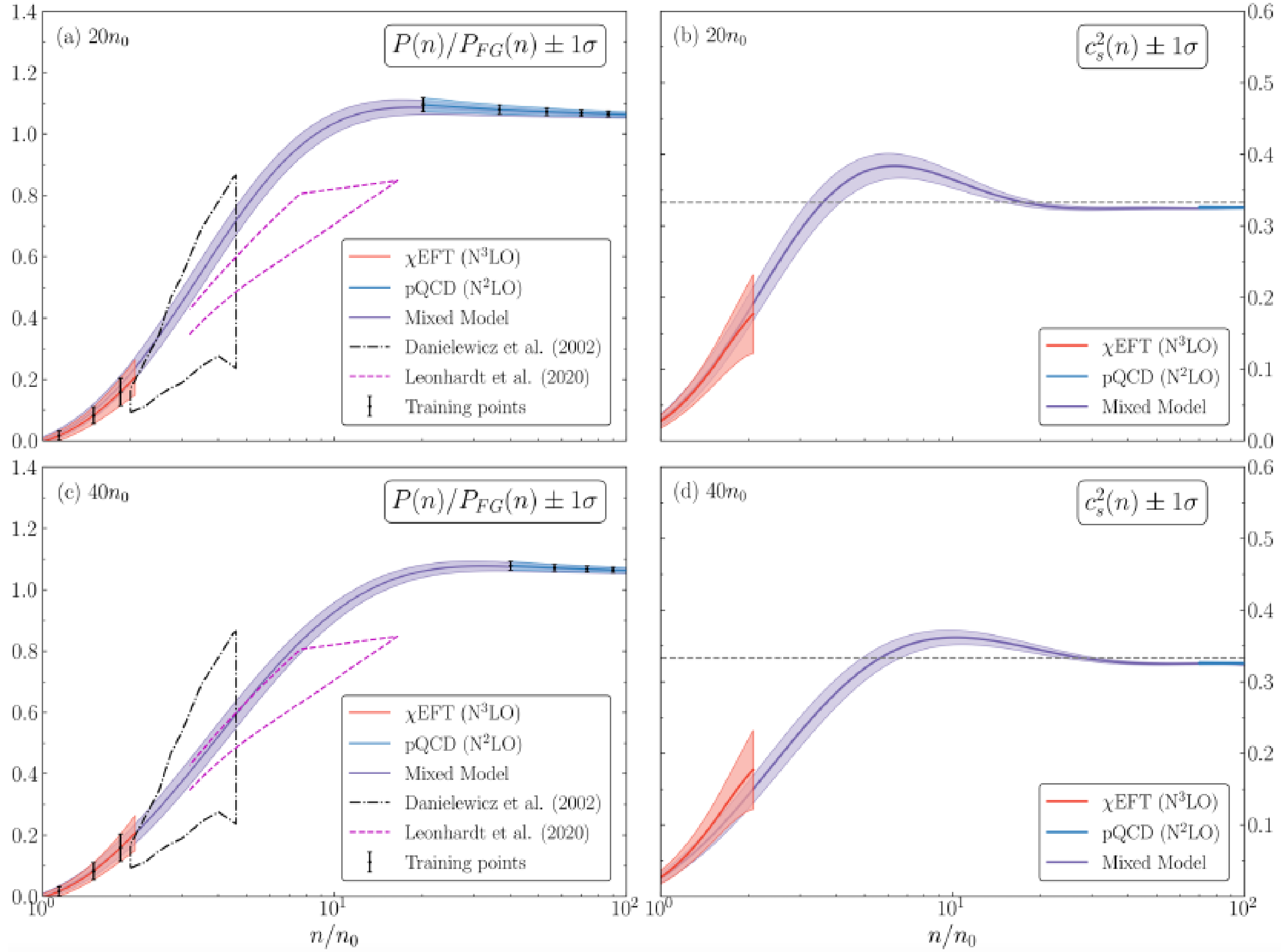
Oleg Komoltsev, Rahul Somasundaram, Tyler Gorda, Aleksi Kurkela, Jérôme Margueron, Ingo Tews

arXiv:2312.14127

From chiral effective field theory to perturbative QCD: A Bayesian model mixing approach to symmetric nuclear matter

A. C. Semposki,[1,*] C. Drischler,[1,2,†] R. J. Furnstahl,[3,‡] J. A. Melendez,[3,§] and D. R. Phillips[1,4,‖]

PHYSICAL REVIEW C **111**, 035804 (2025)



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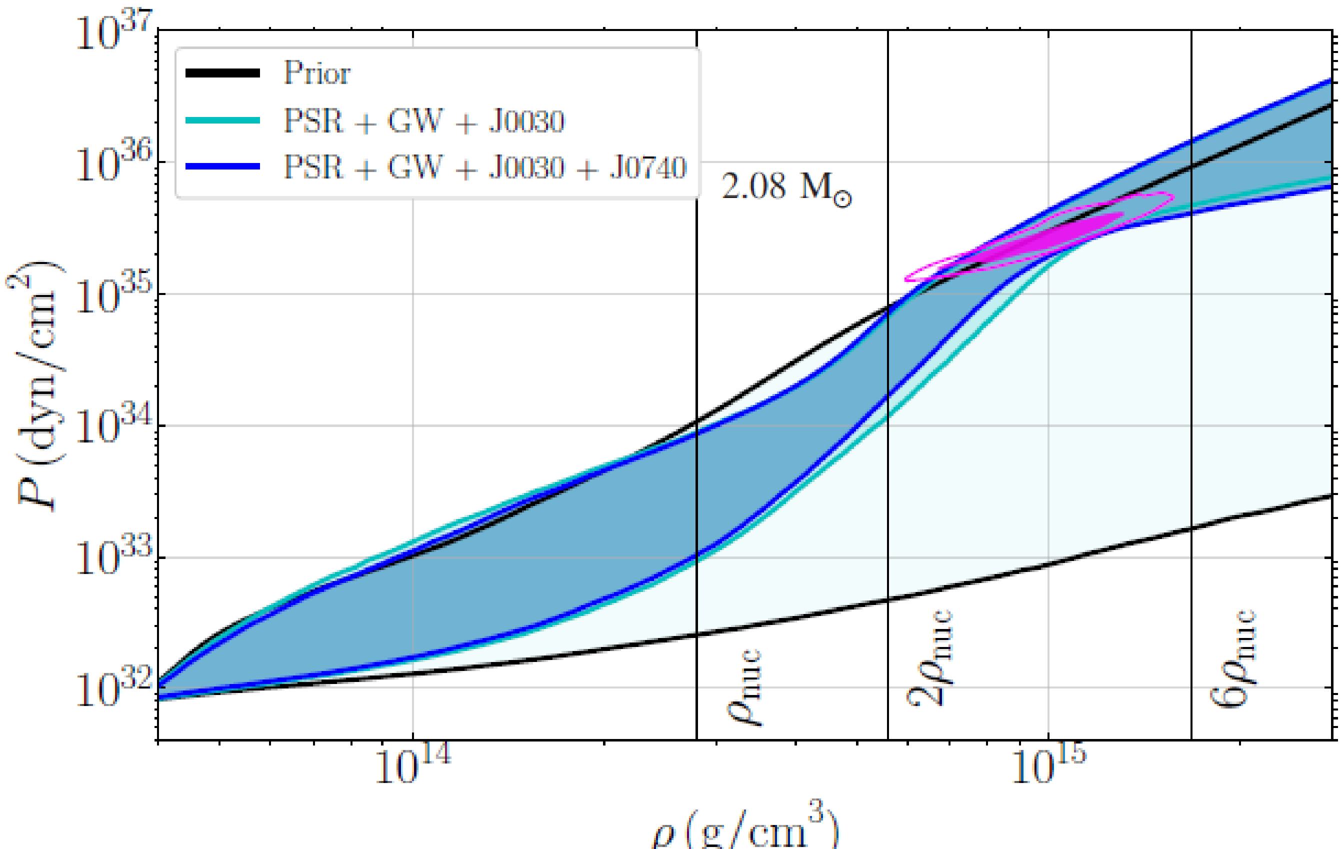
Impact of the PSR J0740 + 6620 radius constraint on the properties of high-density matter

Isaac Legred,[1,2,*] Katerina Chatziioannou,[1,2,†] Reed Essick,[3,‡] Sophia Han (韩君),[4,5,§] and Philippe Landry[6,‖]

PHYSICAL REVIEW D **104**, 063003 (2021)

EOS=Non-parametric, P(e) functions are generated by Gaussian Process



Single-nucleon potential in isospin-asymmetric nuclear matter

A.M. Lane, PRL 8, 171 (1962)

+ for neutrons

$$U_{n/p}(k,\rho,\delta) = U_0(k,\rho) \pm U_{sym1}(k,\rho)\cdot\delta + U_{sym2}(k,\rho)\cdot\delta^2 + o(\delta^3)$$

- for protons

Isovector

According to the Hugenholtz-Van Hove (HVH) theorem:

$$E_F = \frac{d\xi}{d\rho} = \frac{d(\rho E)}{d\rho} = E + \rho\frac{dE}{d\rho} = E + P/\rho$$

J. Dabrowski and P. Haensel, PLB 42, (1972) 163.

S. Fritsch, N. Kaiser and W. Weise, NPA. A750, 259 (2005).

C. Xu, B.A. Li, L.W. Chen, Phys. Rev. C 82 (2010) 054607.

$$E_{sym}(\rho) = \frac{1}{3}\frac{\hbar^2 k^2}{2m_0^*}|_{k_F} + \frac{1}{2}U_{sym,1}(\rho,k_F),$$

Potential

$$L(\rho) = \frac{2}{3}\frac{\hbar^2 k^2}{2m_0^*}|_{k_F} - \frac{1}{6}\left(\frac{\hbar^2 k^3}{m_0^{*2}}\frac{\partial m_0^*}{\partial k}\right)|_{k_F} + \frac{3}{2}U_{sym,1}(\rho,k_F) + \frac{\partial U_{sym,1}}{\partial k}|_{k_F}\cdot k_F + 3U_{sym,2}(\rho,k_F),$$

Kinetic

Rong Chen, Bao-Jun Cai, Lie-Wen Chen, Bao-An Li, Xiao-Hua Li, Chang Xu, PRC 85, 024305 (2012)

Nucleon effective mass in isospin symmetric matter

$$m_0^*(\rho,k) = \frac{m}{1+\frac{m}{\hbar^2 k}\frac{\partial U_0(\rho,k)}{\partial k}},$$

Neutron-proton effective mass splitting in neutron-rich matter

$$m^*_{n-p} \approx 2\delta\frac{m}{\hbar^2 k_F}\left[-\frac{dU_{sym,1}}{dk} - \frac{k_F}{3}\frac{d^2U_0}{dk^2} + \frac{1}{3}\frac{dU_0}{dk}\right]_{k_F}\left(\frac{m_0^*}{m}\right)^2$$

$$\approx 2\delta\left(\frac{M_s^*}{M}\right)^2\left[\frac{M}{M_v^*} - \frac{M}{M_s^*}\right]$$

B.A. Li, B.J. Cai, L.W. Chen and J. Xu,
Progress of Particle and Nuclear Physics, 99 (2018) 29.

What are the fundamental physics behind the symmetry energy?

$$U_{n/p}(k,\rho,\delta) = U_0(k,\rho) \pm U_{sym1}(k,\rho)\bullet\delta + U_{sym2}(k,\rho)\bullet\delta^2 + o(\delta^3)$$

- **Isospin dependence of strong interactions and correlations**

$V_{T0} = V'_{np}$ (n-p pair in the T=0 state)

$V_{T1} = V_{nn} = V_{pp} = V_{np}$ (charge independence in the T=1 state)

Tensor force due to pion and **ρ** meson exchange MAINLY in the T=0 channel

$V_{np}(T0) \neq V_{np}(T1)$

In a simple interacting Fermi gas model:

The direct term

Isospin-dependent correlation function

$$U_{sym}(k_F,\rho) = \frac{1}{4}\rho \int [V_{T1}(r_{ij}) f^{T1}(r_{ij}) - V_{T0}(r_{ij}) f^{T0}(r_{ij})] d^3 r_{ij}$$

Isospin-dependent effective 2-body interaction

M.A. Preston and R.K. Bhaduri, Structure of the Nucleus, 1975

Major issues relevant to high-density E_{sym}, heavy-ion reactions and neutron stars

- **Momentum dependence of the symmetry potential due to the finite-range of isovector int.**
- **Short-range correlations due to the tensor force in the isosinglet n-p channel**
- **Spin-isospin dependence of the 3-body force**
- **Isovector interactions of Δ(1232) resonances and their spectroscopy (mass and width)**
- **Possible sign inversion of the symmetry potential at high momenta/density**

Posterior thoughts

(1) EOS $P(\varepsilon)$ is model-dependent, but P/ε is a robust and unified EOS parameter from heavy-ion collisions, neutron stars to the expansion of the Universe

(2) Trace anomaly $\Delta = 1/3 - P/\varepsilon$ measures breaking of scale invariance (conformal symmetry $\Delta = 0$) (A leading expert Toru Kojo is here). Its energy dependence for cold dense matter, Bayesian inferred from neutron stars and heavy-ion collisions, is consistent

Uncertainties to be quantified by you

COLD matter P/ϵ from HOT heavy-ion reactions by decoupling the mean-field from thermal pressure

At intermediate energies before the formation QGP, flow depends on both the mean-field potential and elementary Xsections

Boltzmann-Uehling-Uhlenbeck (BUU) equation

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \vec{v}\cdot\vec{\nabla}_r f - \vec{\nabla}_r V \cdot \vec{\nabla}_p f = I_c(f, \sigma_{NN})$$ Collision integrals

mean-field V=∂W/∂ρ

In-medium modification factor
of NN scattering cross sections

$$X \equiv \sigma_{NN}^{med}/\sigma_{NN}^{free})$$

Hugenholtz-Van Hove (HVH) Theorem:

$$P/\varepsilon = \rho E_F/\varepsilon - 1 = E_F/E - 1$$

Potential energy density at T=0

$$W(\rho) = +\frac{a}{2}\frac{\rho^2}{\rho_0} + \frac{b}{1+\sigma}\left[\frac{\rho}{\rho_0}\right]^{\sigma}\rho + \frac{c\rho^2}{\rho_0}\left\langle\frac{1}{1+\left[\frac{\mathbf{p}'}{\Lambda}\right]^2}\right\rangle$$

momentum-dependent interaction

Pressure at T=0

$$P = \Pi_{zz} = \frac{1}{5}\frac{p_F^2}{m}\rho + \frac{a}{2}\frac{\rho^2}{\rho_0} + \frac{\sigma}{\sigma+1}b\frac{\rho^{\sigma+1}}{\rho_0^{\sigma}} + \frac{c\rho^2/\rho_0}{1+p_F^2/\Lambda^2}$$

G.F. Bertsch and Subal DasGupta, Physics Reports 160, 189 (1988)