Hadron mass spectra in extreme conditions -from Lattice Monte Carlo to Tensor Network

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CREST

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Outline

1. (4+0)-dim. Nf=2+1 QCD at physical point

   new configurations for HAL QCD collaboration

   HAL QCD collaboration: Phys.Rev.D 110 (2024) 9, 094502

2. (4+0)-dim. Nf=2 2color QCD at finite density

   Monte Carlo study

   K. Iida, EI, K. Murakami, D. Suenaga, arXiv2606.13974
   to appear in JHEP

3. (1+1)-dim. Nf=2 QED (Schwinger model) at nonzero topological theta regime

   Hamiltonian Lattice gauge theory using tensor network method

   E.I., Akira Matsumoto, Yuya Tanizaki, JHEP 11 (2023) 231
   E.I., Akira Matsumoto, Yuya Tanizaki, JHEP 09 (2024) 155

4. Summary

1. Nf=2+1 QCD at physical point

new configurations for HAL QCD collaboration

Supercomputer in Japan

K computer
(2012-2019)

京 (Kei)
= 10^{16}(flops)

Fugaku computer
(2021-)

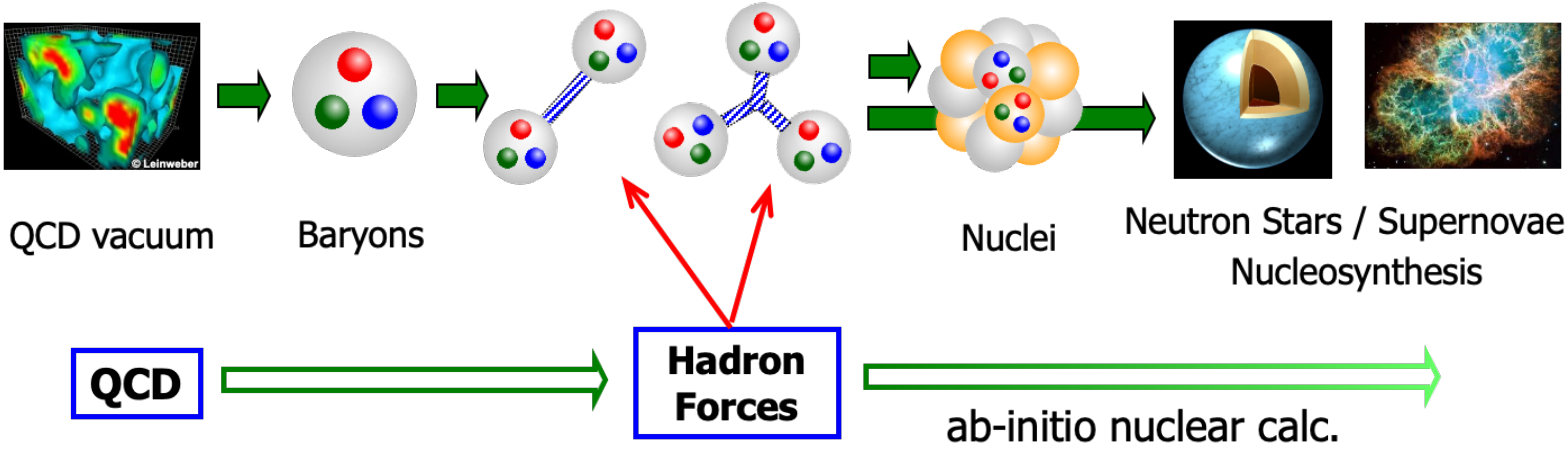
富岳 (Fugaku)
= Mt. Fuji

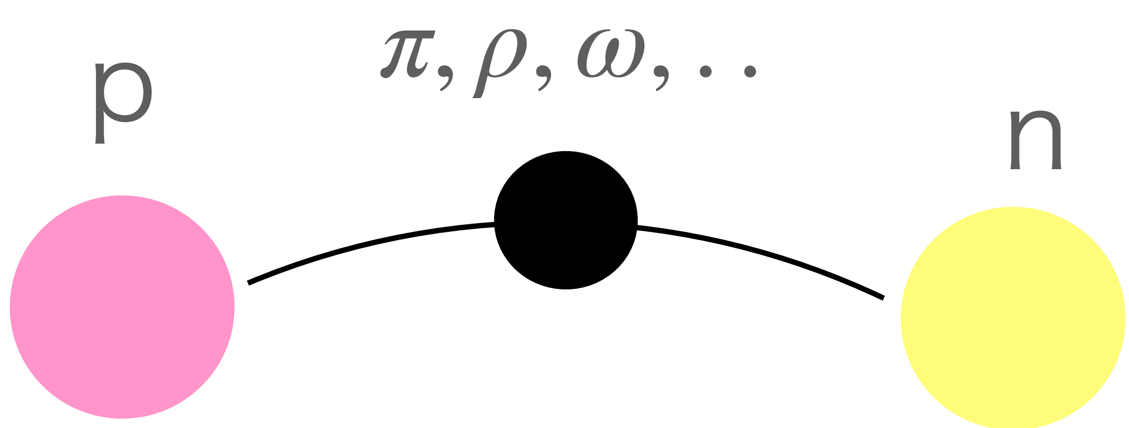
HAL QCD: $\mathrm{m}_\pi \approx 146\mathrm{MeV}$
$\mathrm{m}_K \approx 525\mathrm{MeV}$

At physical point!
$(m_\pi^{ave}, m_K^{ave}) = (138{,}496)$ MeV

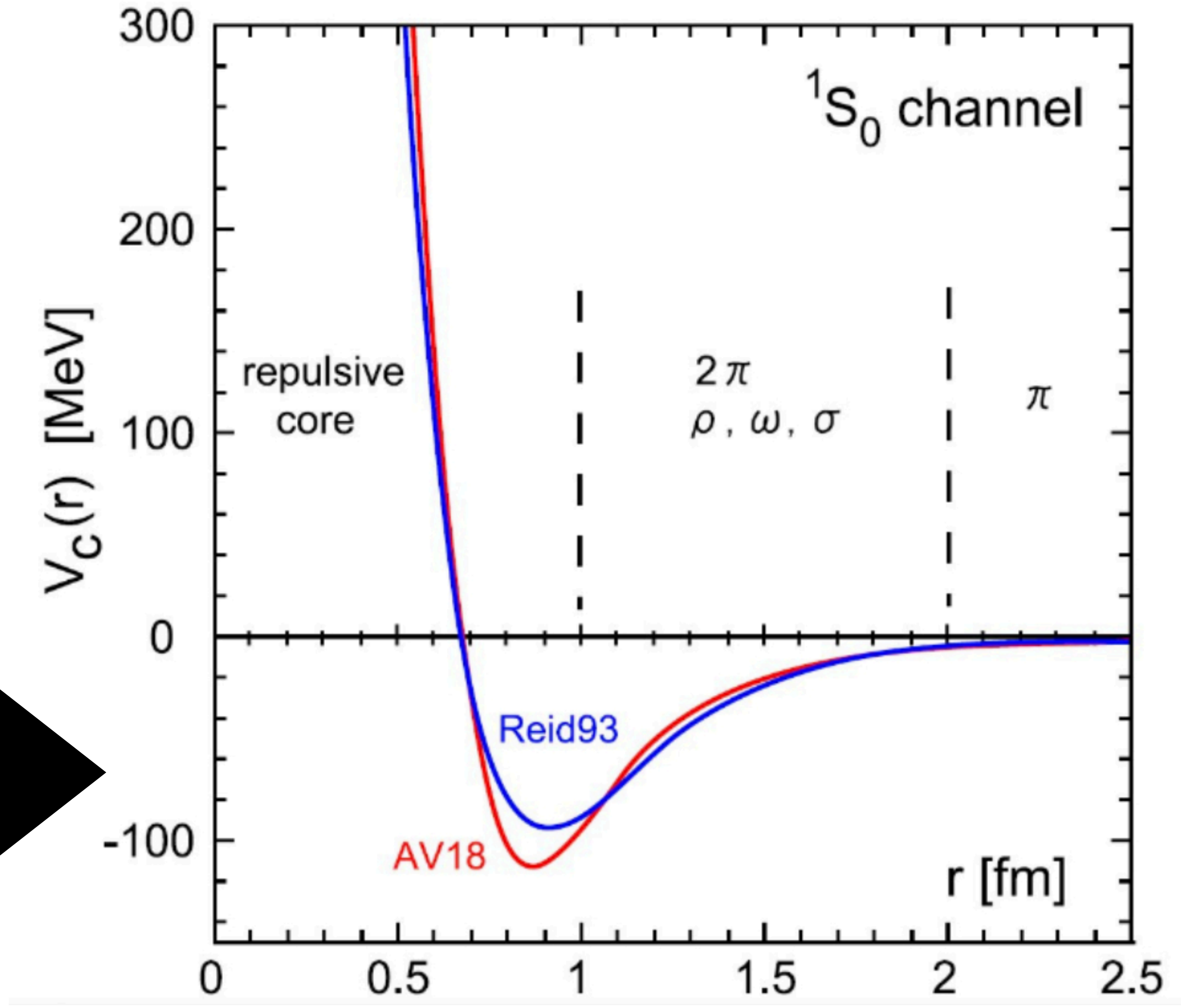
Cf.) "Fugaku-Next" computer
(2030 -)

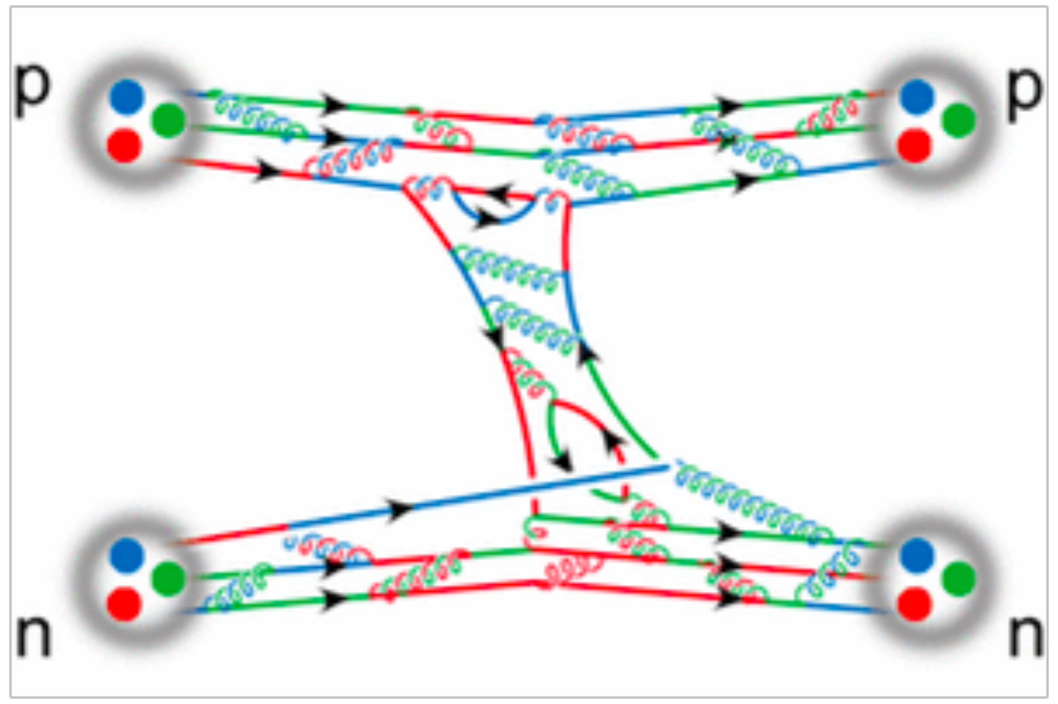
Ultimate goal of the HAL QCD collaboration





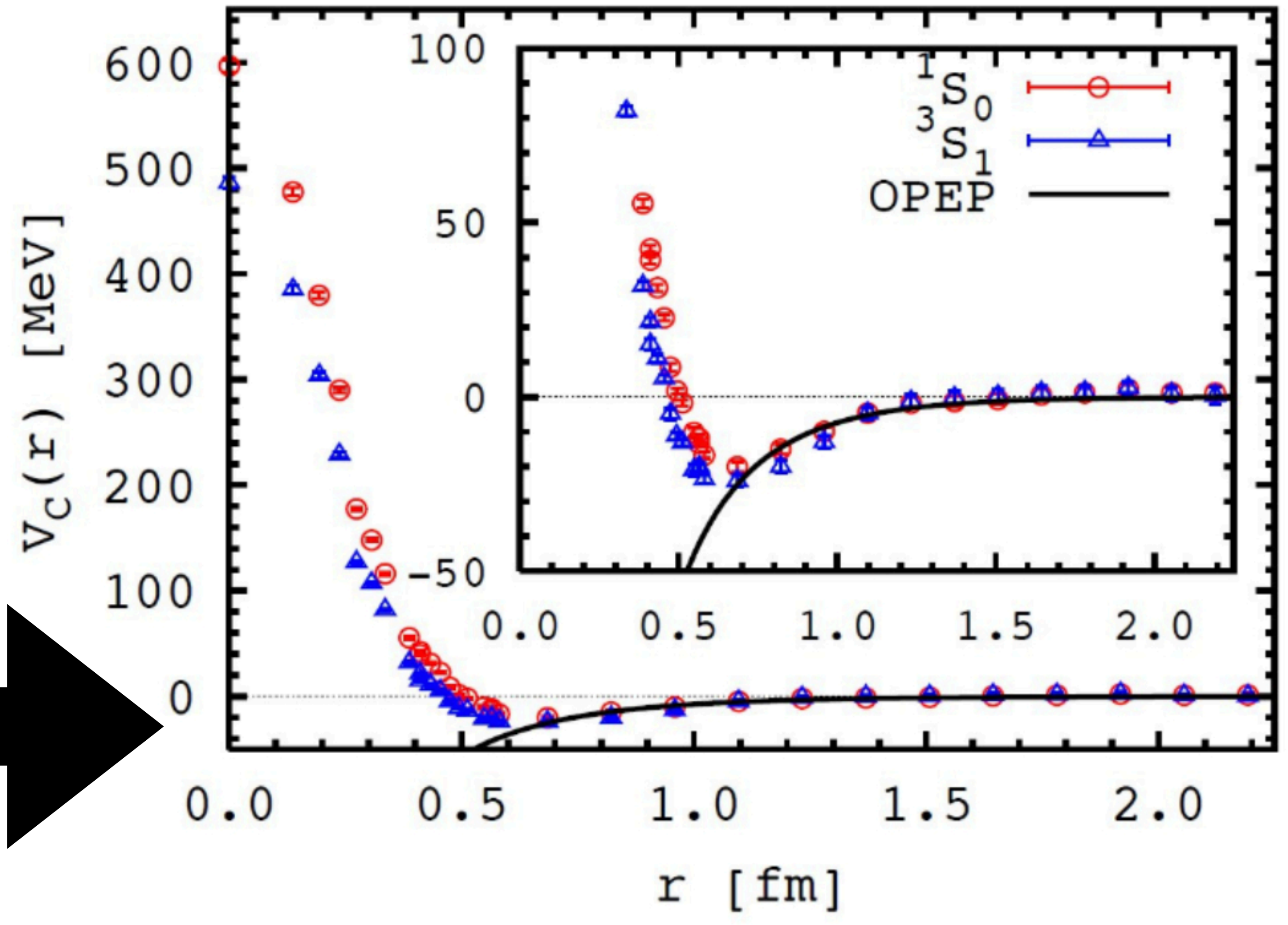
Phenomenological nuclear force





HAL QCD

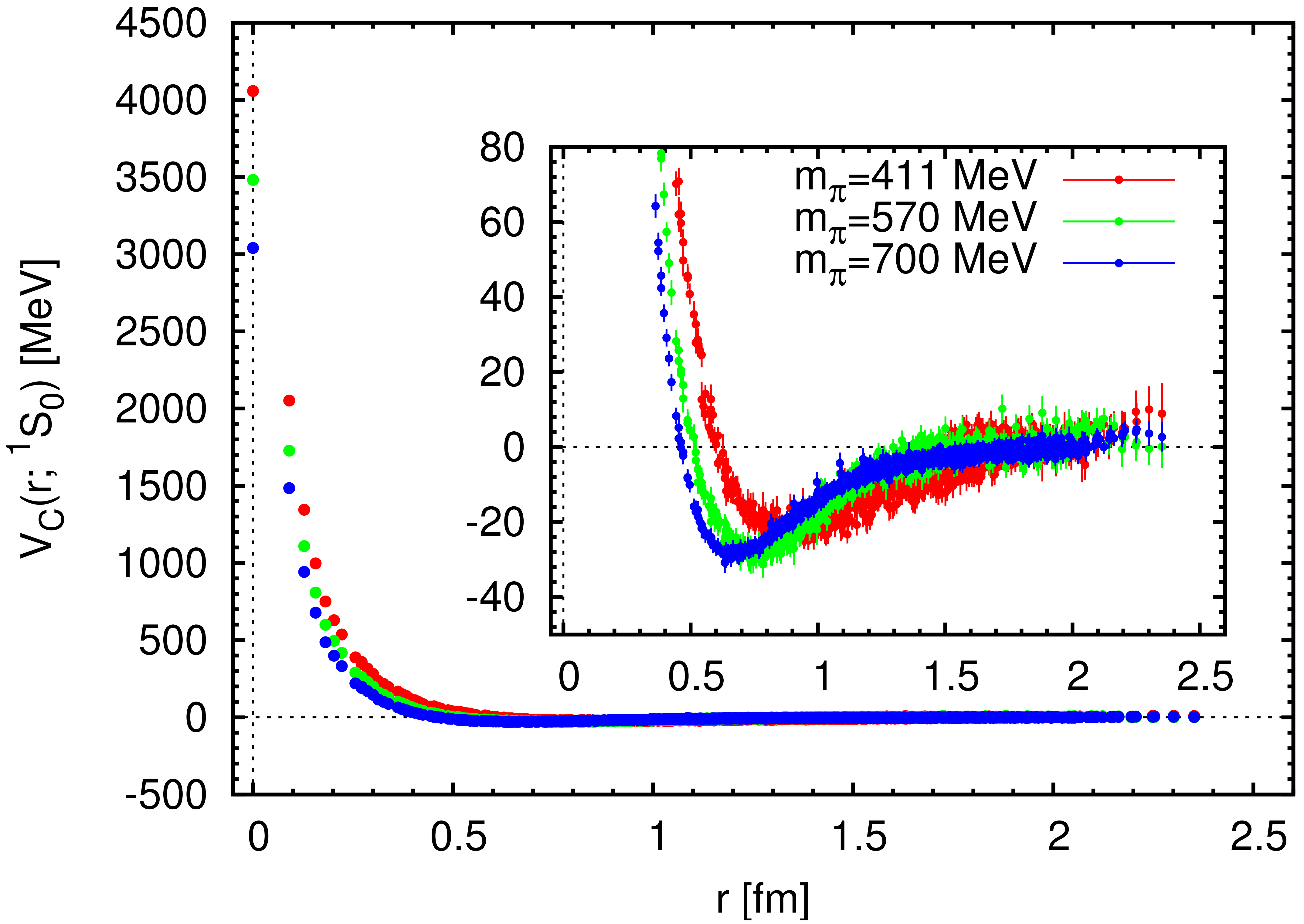
potential from QCD!



Ishii–Aoki–Hatsuda, PRL 99 (2007) 022001

Necessity for precision scale-setting in nuclear-force study

NN interaction potential
depends on the pion mass
(spin-singlet S-wave channel with J=0)



S.Aoki, T.Hatsuda and N.Ishii
Prog.Theor.Phys. 123 (2010) 89-128
Noriyoshi Ishii, PoS(LAT2009)019

- Hadron interactions sometimes are very sensitive to pion (quark) masses
- In particular:
  Dibaryons, Hypernucleus, Tcc...
  (Target of HAL QCD coll. and also LHCb)
- Technically, we measure 2pt corr. fn. and 4pt corr. fn.
  High statistics is necessary

Lattice setup of recent HAL QCD configurations

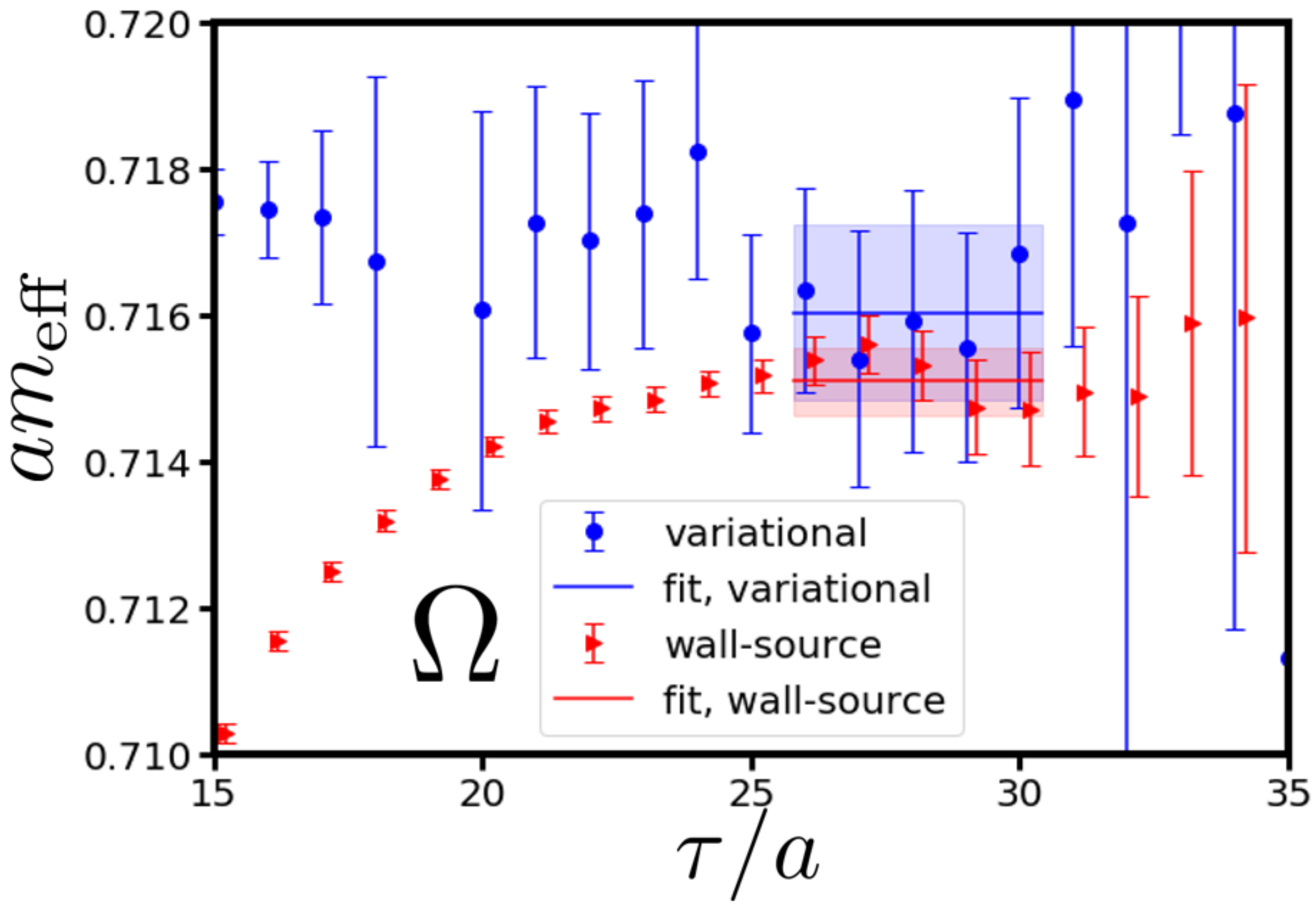
- Iwasaki gauge action and Nf=2+1 Wilson-Clover fermions

- $\beta = 1.82,\ \ N_x^4 = 96^4 \approx (8.1\mathrm{fm})^4$

- Physical point mass (QCD simulation without QED)
  target: isospin-averaged values of PS meson masses
  $(m_\pi^{\mathrm{ave}}, m_K^{\mathrm{ave}}) = (138{,}496)$MeV

- 8,000 Monte Carlo trj. = 1,600 conf.
  # of measurement of baryon correlation fn.(pt-sink and wall-source)
  rotational sym. (4) x temporal source locations (96) x fw-bw average (2),
  1600conf x 4 x 96 x 2 = 1,228,800 measurements

Physical scale setting using Omega



- Wall-source
  Variational w/ 2x2 smeared op.

- Effective mass: $m_{\rm eff} = \log \dfrac{C(\tau)}{C(\tau+1)}$

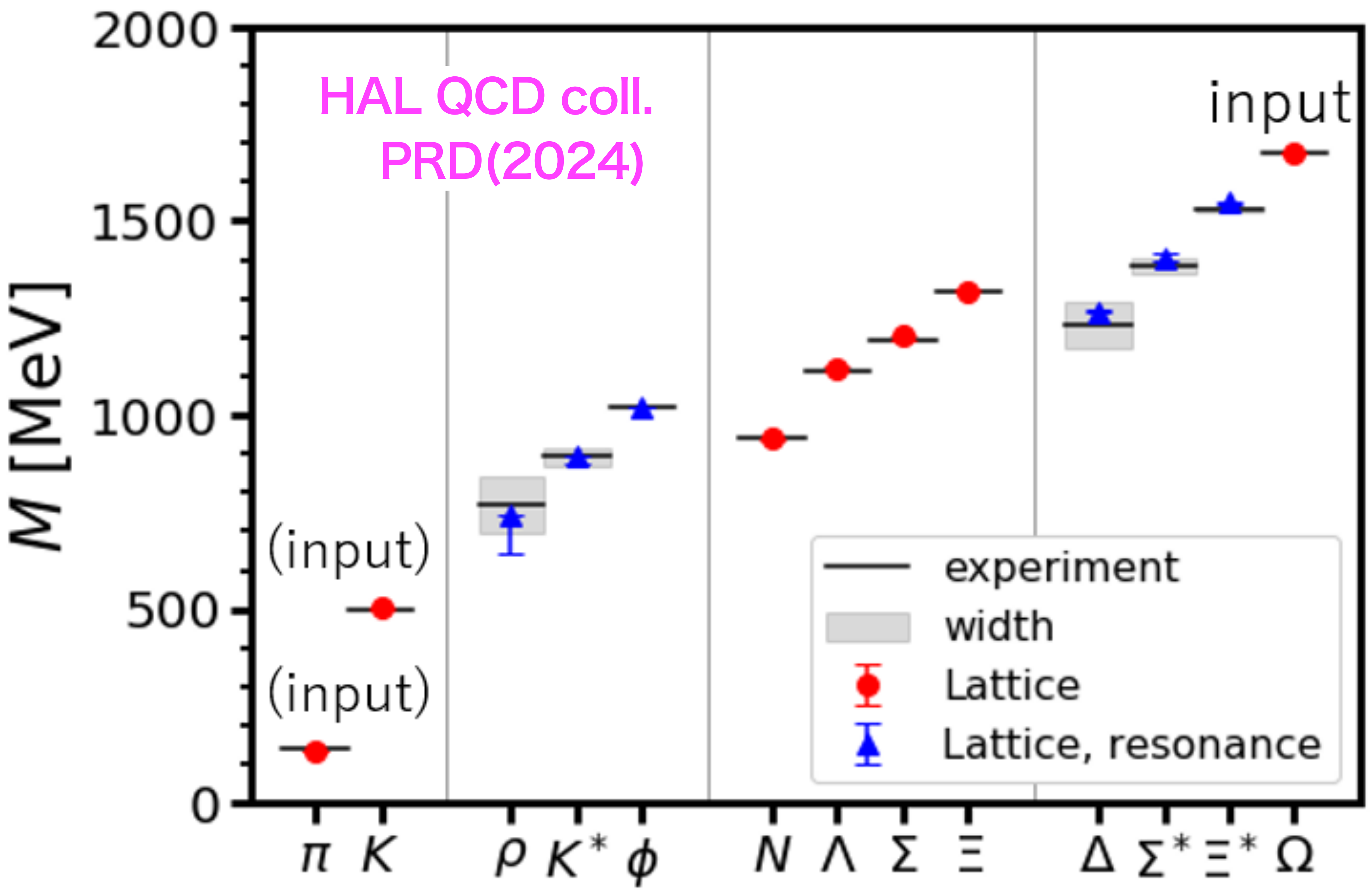
- Central value + stat. err.
  from wall-source result
  Sys. err. (1): fit range deps in wall-source result
  Sys. err. (2): difference between
  wall and variational

- Input: $m_\Omega = 2316.2(4.4)$ MeV

  our data: $am_\Omega = 0.71510(46)\left({}^{+93}_{-5}\right)$

  Result of a: $a^{-1} = 2338.8(1.5)\left({}^{+0.2}_{-3.0}\right)$MeV

Results for the hadron spectrum



- 3 input parameters

  g_0 ($\to a$), $\kappa_{u,d}$ (m^0_{ud}) , κ_s (m^0_s)

- Input data:

  am_Ω -> fix the scale $a \approx 2.3$GeV^{-1}

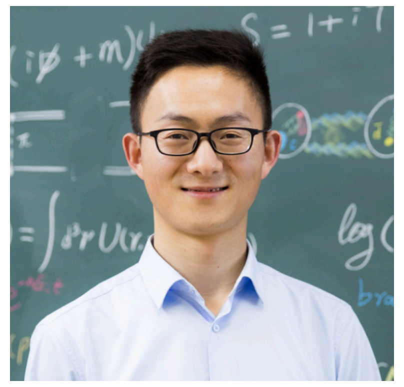
  am_π -> fix $\kappa_{u,d}$, ($\kappa_{u,d}$=0.126117)

  am_K -> fix κ_s, (κ_s=0.124902)

We also determined the topological susceptibility, decay constants, quark (PCAC) masses, and gradient-flow scales.

HAL QCD collaboration: Phys.Rev.D 110 (2024) 9, 094502

HAL QCD projects



- Lattice Projects:

$D^*D(T_{cc}^+)$, NJ/ψ, $J\eta_c$: Yan Lyu — Phys.Rev.Lett. 131 (2023) 16, 161901; Phys.Lett.B 860 (2025) 139178; Phys.Rev.D 114 (2026) 1, L011503

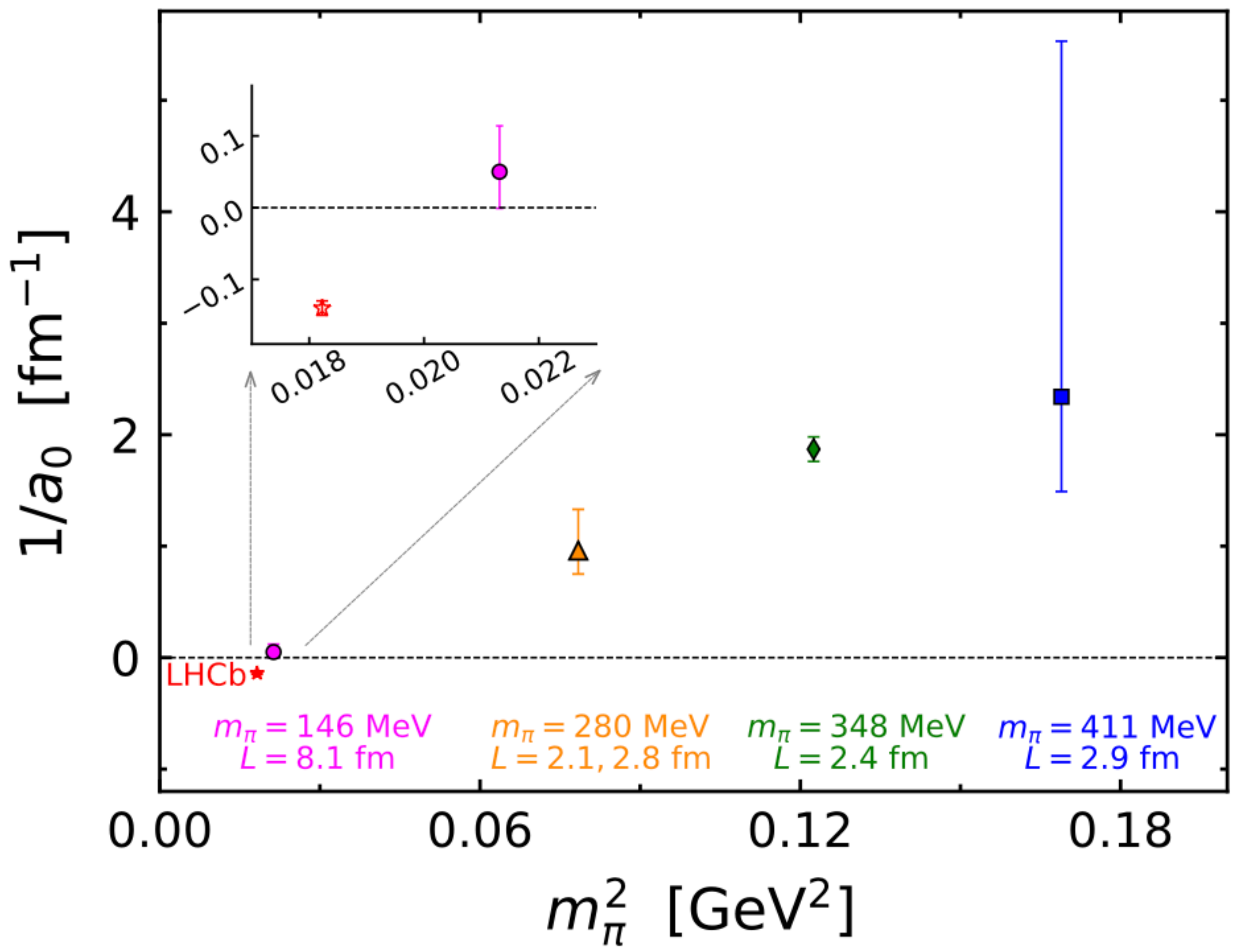
KN, $\Lambda(1405)$: Kotaro Murakami — Phys.Rev.D 113 (2026) 5, 054506

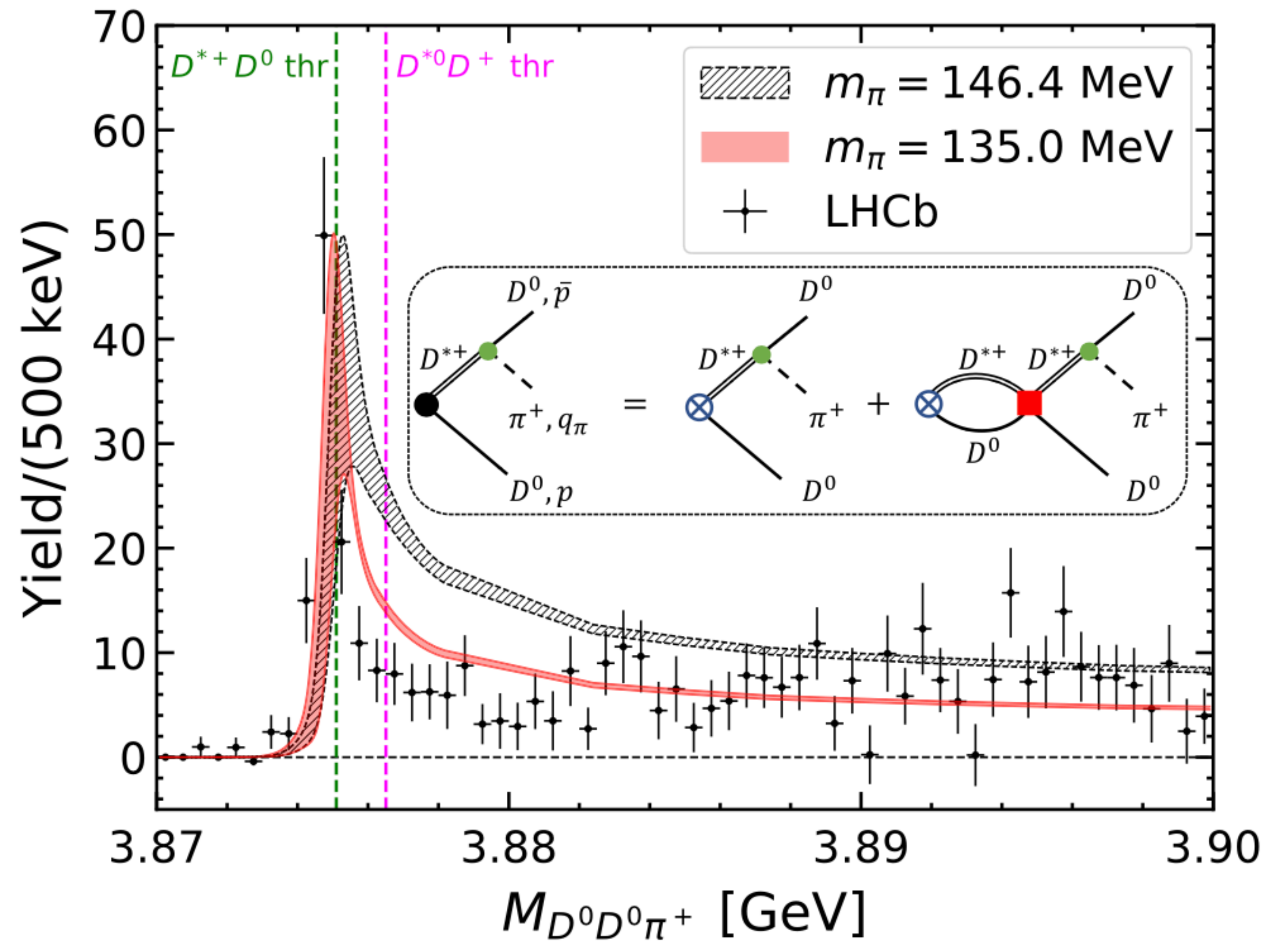
$\bar{D}N$: Wren Yamada — arXiv: 2603.12251

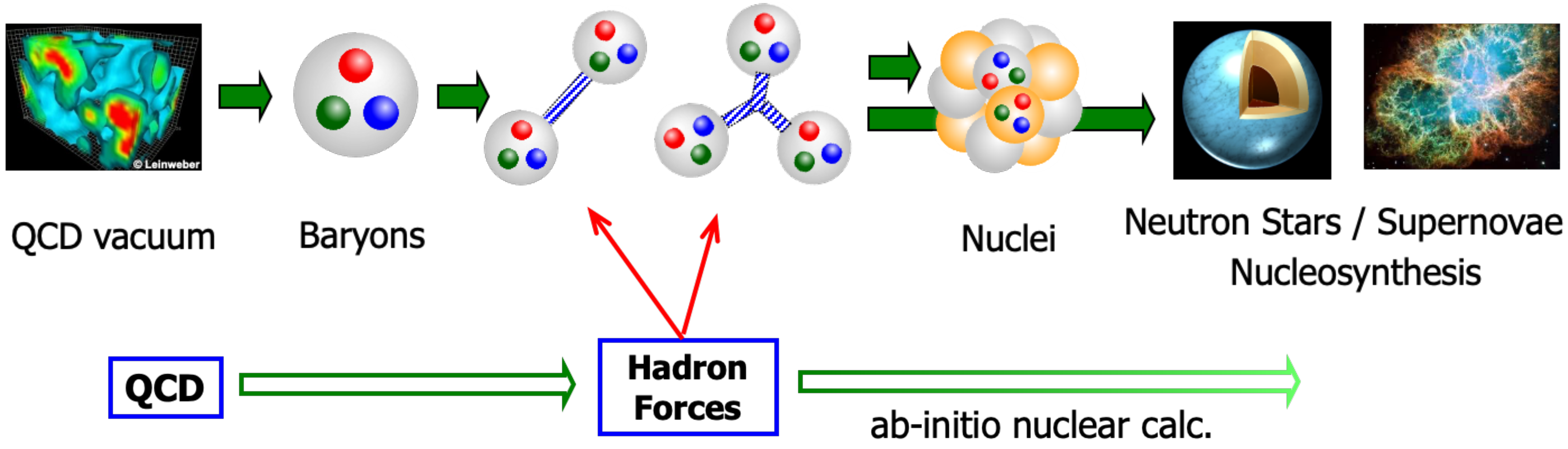
T_{bb} : Parry Junnarkar — On going

$N\Omega_{ccc}$,$N\Omega$: Liang Zhang — Phys.Lett.B 871 (2025) 139998; On going(N Omega, bound state?)

Also, all baryon-baryon interactions!







NN interaction (Yan Lyu)
Hyperon(sigma-N) (Murase)
H-dibaryon (K.Sasaki)

Spectral function reconstruction
Ryutaro Tsuji (KEK)

2. Nf=2 2color QCD at finite density

K. Iida, EI, K. Murakami, D. Suenaga, arXiv2606.13974 (To appear in JHEP)

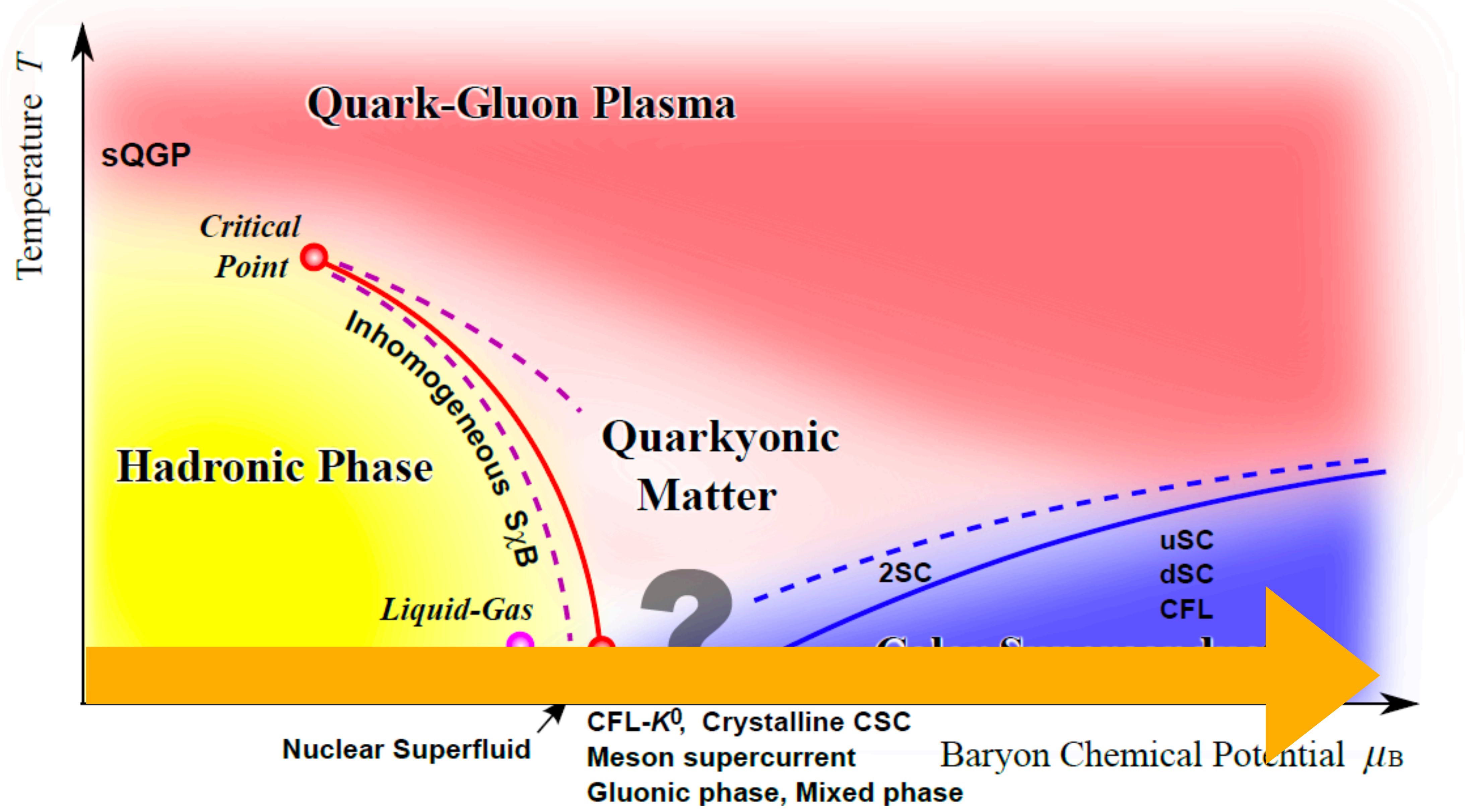


Phase diagram in T-μ plane

3 color QCD

expected phase diagram

Fukushima-Hatsuda (2010)



- Mass spectrum in low-T and high-density

- In the superfluid (color-super-conductor) phase, the chiral condensate decreases. Some hadron masses should change.

Brown-Rho (1991)

At $\mu \neq 0$, what is happened? (3color QCD)

- At $\mu = 0$

  (1) γ_5-hermiticity: $\gamma_5 D \gamma_5 = D^\dagger$

  (2) No disconnected diagram

  ➡ From the QCD inequality for $^\forall\Gamma$ **mesons ($\bar{q}\Gamma q$)**,

  PS meson (pion, $\Gamma = \gamma_5$) is the lightest!

- $\mu \neq 0$: γ_5-hermiticity is broken : $\gamma_5 D(\mu) \gamma_5 = D^\dagger(-\mu) \neq D^\dagger(\mu)$

  In dense-QCD, neither positivity nor inequality holds

  (cf:in high-density, color-flavor-locking phase, $m_\pi > m_{\eta'}$?) Son-Stepanov (1999)

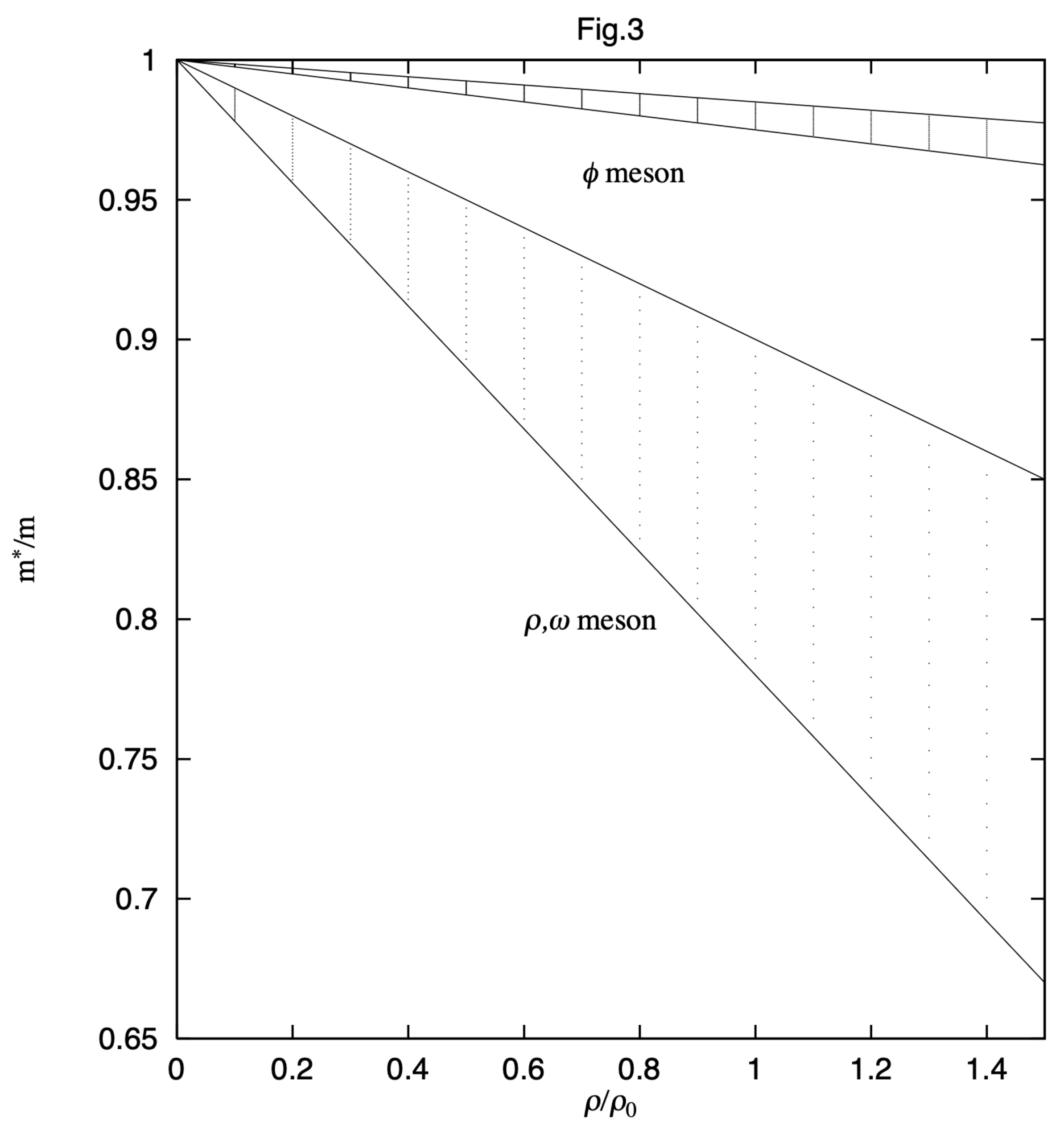
- According to QCD sum rules, Hatsuda and Lee, PRC 46(92)R34, PRC 52(95)3364 arXiv:nucl-th/9608037

  vector mesons become lighter?

  However, subsequent studies suggest

  ρ and ω masses do not necessarily become lighter.

  (ϕ can be said to do so). S. Zschocke, O.P. Pavlenko, B. Kämpfer, Eur. Phys. J. A 15 (2002) 529–534



- KEK-PS E325 and E16 experiment at J-PARC

  P. Gubler, M. Ichikawa, T. Song and E. Bratkovskaya, Phys. Rev. C 111 (2025) 034908.

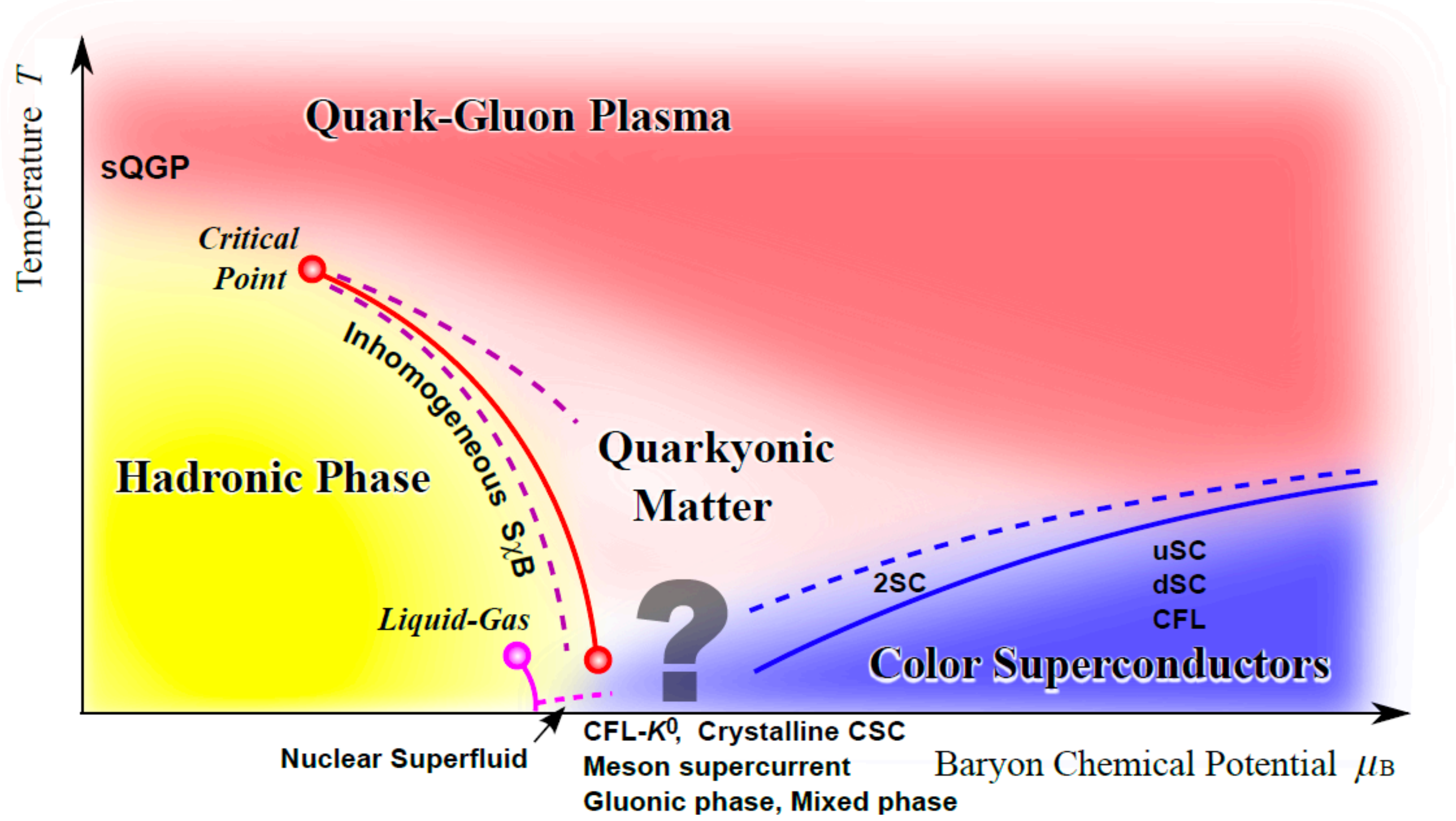
  JPARC-E-016 collaboration, Acta Phys.Polon. Supp. 16 (2023) 143.

2color QCD

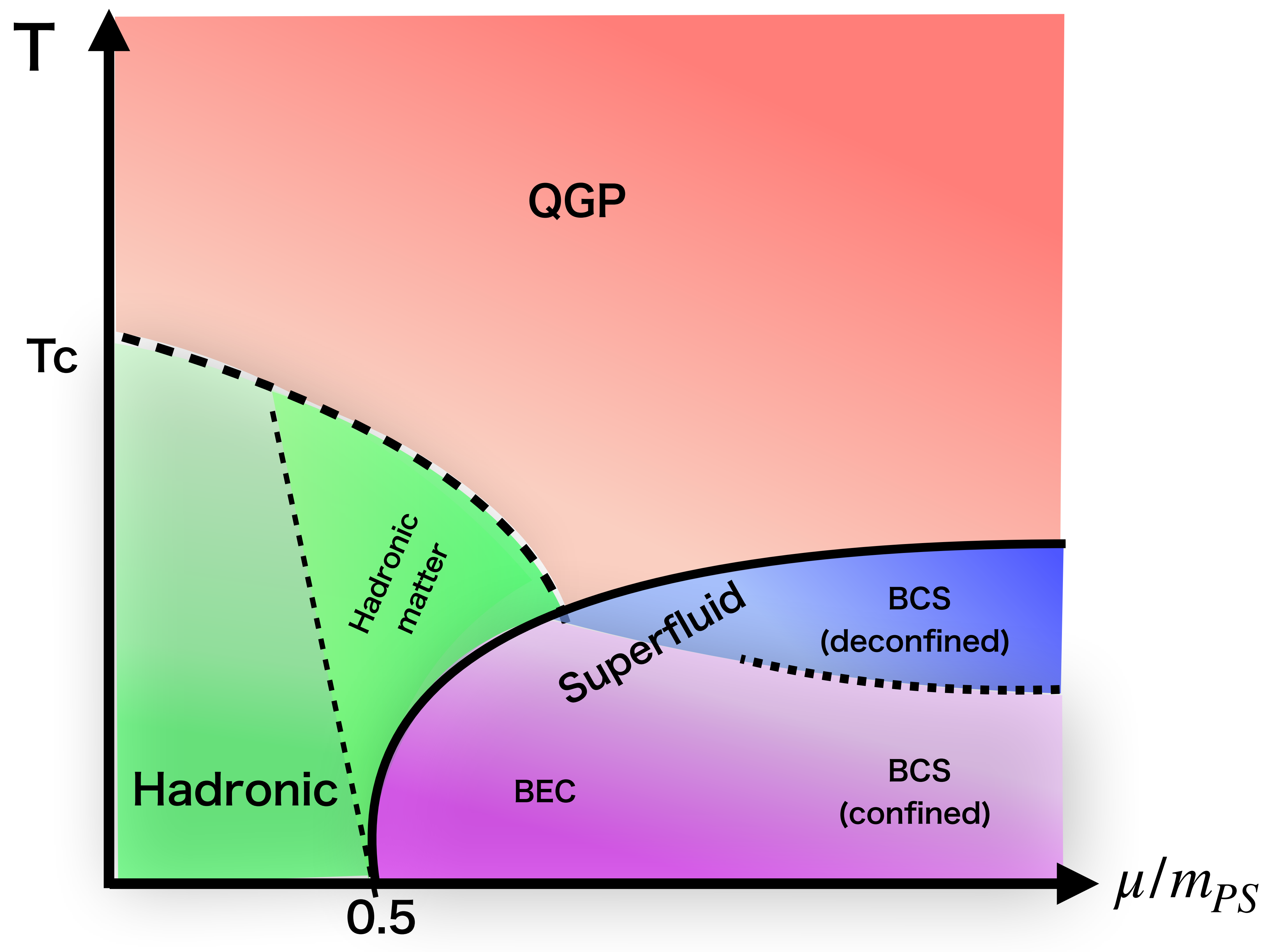
Phase diagram in T-μ plane

3 color QCD
expected phase diagram

Fukushima-Hatsuda (2010)

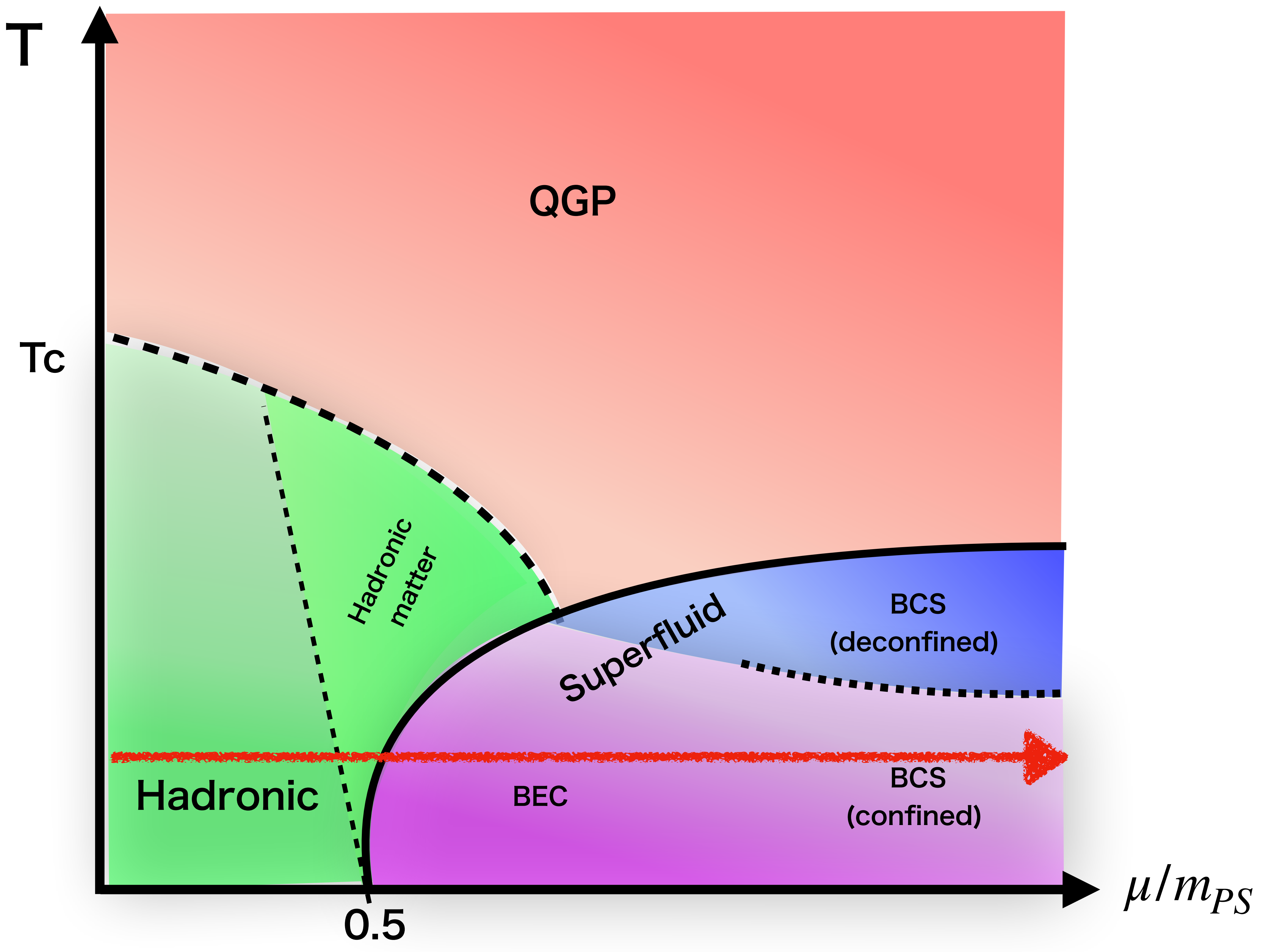


2 color QCD
numerically emerging phase diagram



Our model: 2color 2flavor dense-QCD

2color QCD Phase diagram



40MeV (32^4 lattices)

K.Iida, EI, T.-G. Lee: JHEP2001(2020)181
K.Iida, EI, K.Murakami, D.Suenage, JHEP 10 (2024) 022
Recent Review (Phase diagram and EoS):
EI "Lattice results for the equation of state in dense QCD-like theories" arXiv: 2508.03090

- In Superfluid phase, diquark cond.$\langle qq \rangle \neq 0$
  (color-singlet in 2color QCD)

- Bose-Einstein-Condensate (BEC)
  strong coupling regime
  Well explained by ChPT
  $\langle qq \rangle \propto (\mu - \mu_c)^{1/2}$, EoS

  Kogut et al., NPB 582 (2000) 477
  Son and Stephanov (2001)

- BCS phase
  weak coupling regime
  $\langle qq \rangle \propto \mu^2$: weak coupling analysis
  $\langle n_q \rangle \approx n_q^{free}$

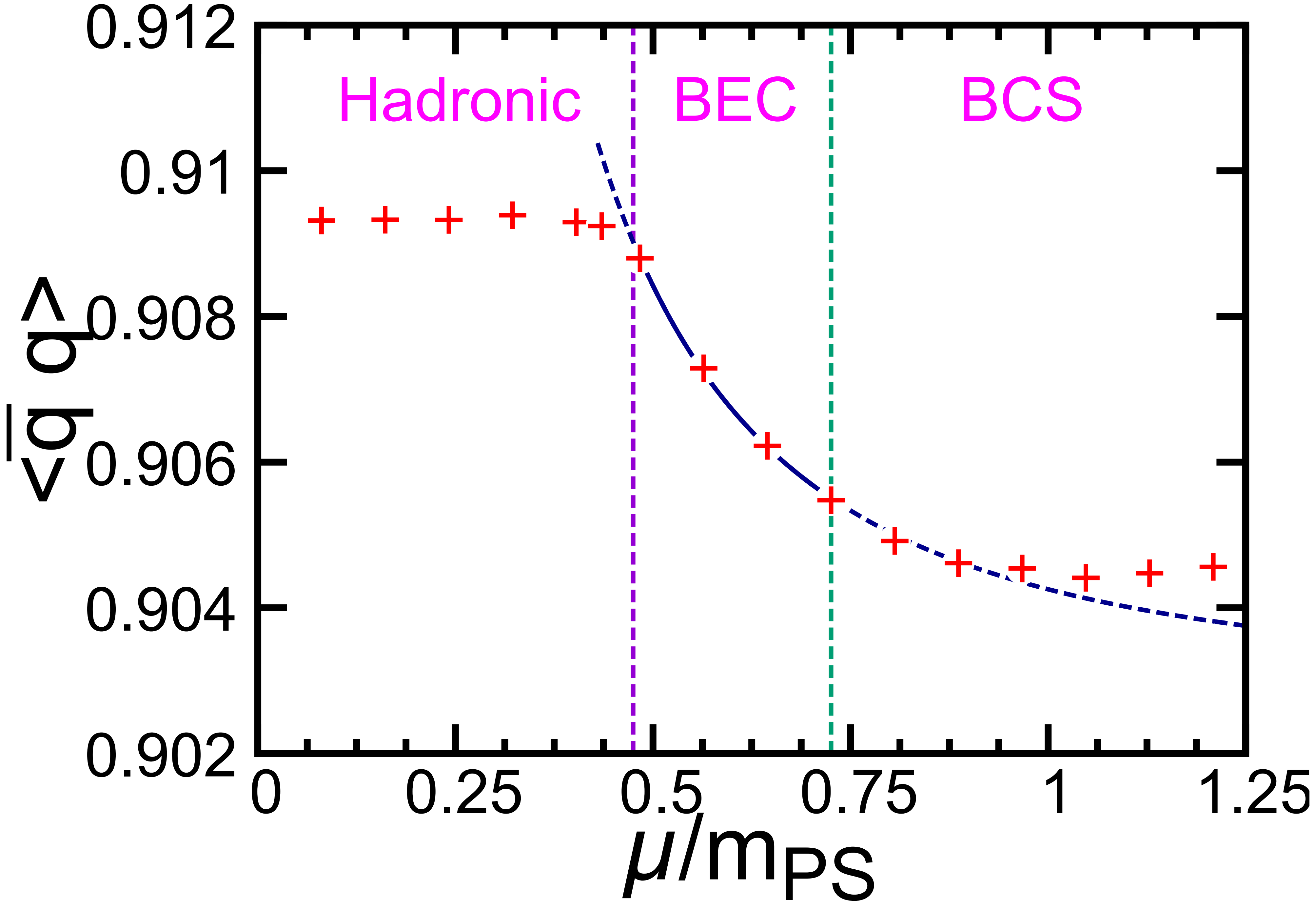
  T. Schaefer, NPB 575 (2000) 269

  confinement remains
  nontrivial instanton contributions
  Quarkyonic-matter (free-quark matter with confining gluons)
  Sound velocity exceeds the conformal bound
  ($c_s^2/c^2 > 1/3$)

Chiral symmetry and hadron spectra

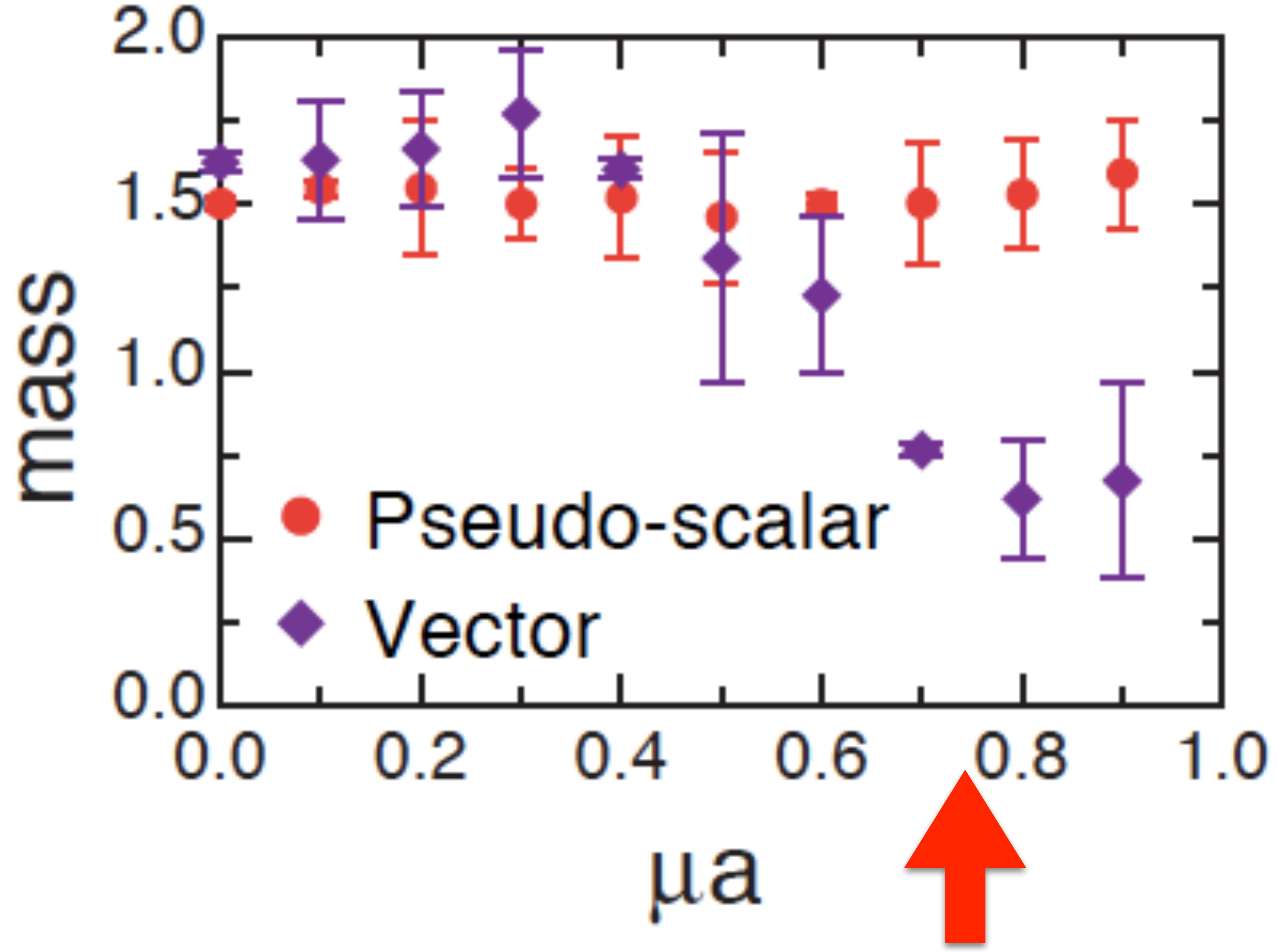
- In the superfluid phase,
  the chiral condensate decreases.

K.Iida et al., JHEP 10 (2024) 022



Muroya et al. Phys.Lett. B551 (2003) 305

2color QCD, Lattice size 4^3 x 8



$\mu/m_{PS} = 0.5$

Special properties in 2color QCD

- $\mu \neq 0$: γ_5-hermiticity is broken : $\gamma_5 D(\mu)\gamma_5 = D^\dagger(-\mu) \neq D^\dagger(\mu)$

- In dense 2color QCD, quarks take a pseudo-real reps.$\gamma_5 C\tau_2 D(\mu)\gamma_5 C\tau_2 = D^*(\mu)$, here C is charge conjugation

  From the QCD inequality, the iso-singlet scalar diquark (baryon) $M_{qq} = \psi^T C\tau_2\gamma_5\psi$ must be the lightest hadron

- In hadronic phase, violation of time-reversal symmetry ($\tau \leftrightarrow (N_\tau - \tau)$) of C(tau)

  $E(\vec{p},\mu) = \sqrt{\vec{p}^2 + m^2} - \mu n_O$, it can be regarded as a mass shift: $m(\mu) = m(\mu = 0) - n_O\mu$

- In the SuperFluid (SF) phase, meson-baryon mixing

  $U(1)_B$ is broken, no difference between mesons and baryons

  Technically, consider all possible contractions of $\bar{q}$-q and q-q.

$|0\rangle$: vacuum state at $\mu = 0$ ($\hat{N}|0\rangle = 0$)

(This is valid only for Hadronic phase)

On lattice, $\langle\Omega_i|\hat{N}|\Omega_i\rangle = 0$ for each conf. in $\mu < m_\pi/2$

K. Nagata, PPNP(2022), A.Alexandru et al., JHEP(2016)

Flavor symmetry and its breaking in Nc=Nf=2

$\boxed{SU(4) \text{ at } m = \mu = 0}$

$\downarrow$ $m > 0$

$Sp(4) \simeq SO(5)$

5 pseudo-Goldstone bosons
$3\pi, qq, \bar{q}\bar{q}$

$\downarrow$ $0 < \mu < \mu_c$

$SU(2)_V \times U(1)_B$
meson-baryon sym.

diquark condensation $\downarrow$ $\mu_c < \mu$

$Sp(1)_V \simeq SU(2)_V$

- Extended multiplet: $\Psi \equiv \begin{pmatrix} \psi \\ K\bar{\psi}^T \end{pmatrix}$

  Lagrangian is invariant under $U \in SU(4)$;

  $\Psi \to U\Psi$

- pion $P^a(x)$ and I=0 S diquark $qq(x)$

  $\delta P^a(x) = i\epsilon[X, P^a(x)] = i\epsilon c^a(qq)$

  for broken generator $X^\alpha \in SU(4)/Sp(4)$,

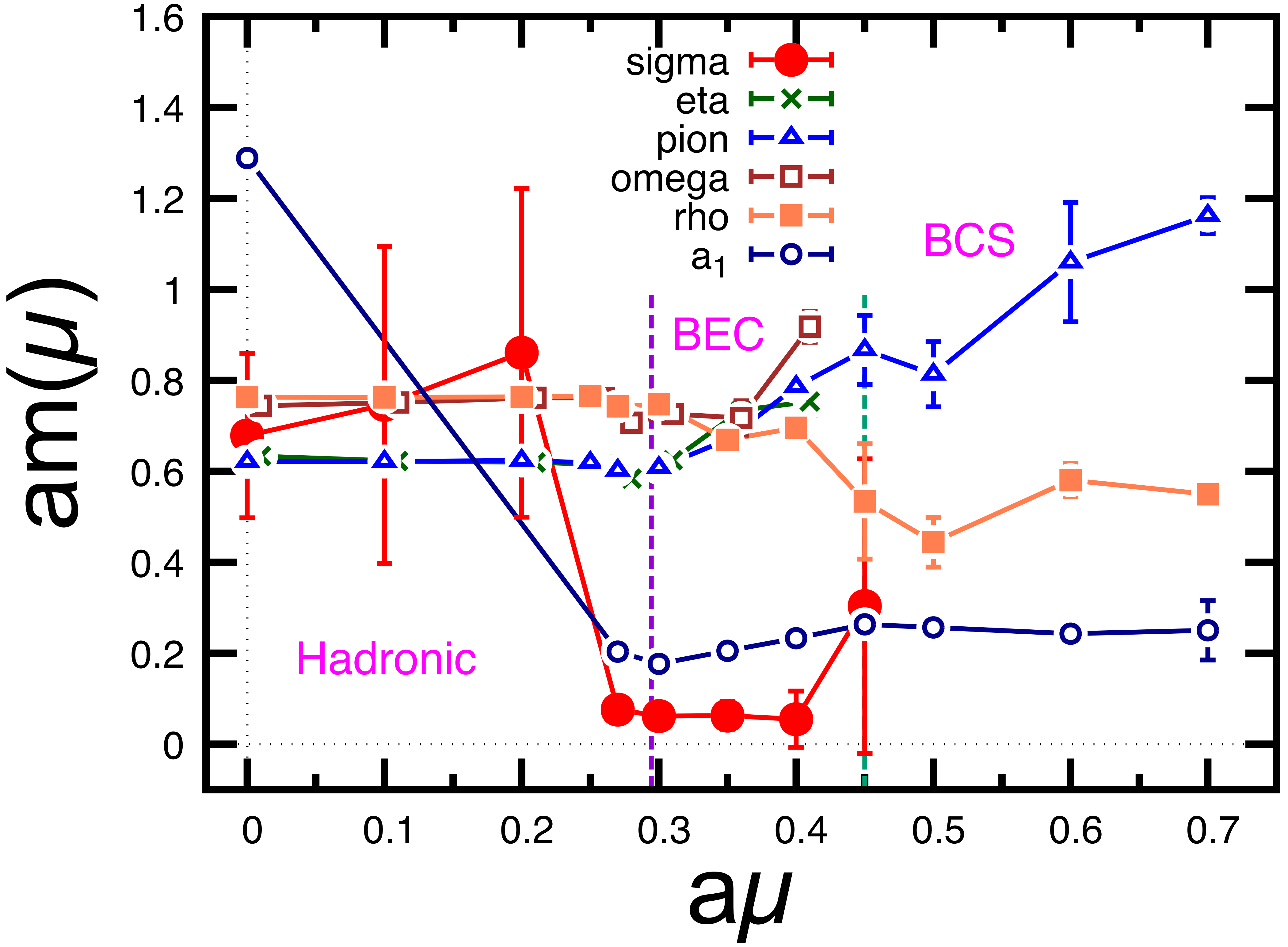
  $\alpha = 1,\cdots 5$ => 5-dim. multiplet $\{P^a(x), qq(x), \bar{q}\bar{q}(x)\}$

- Rho $V_i^a(x)$ and I=1 AV diquark $D_{AV}^{a,i}(x)$

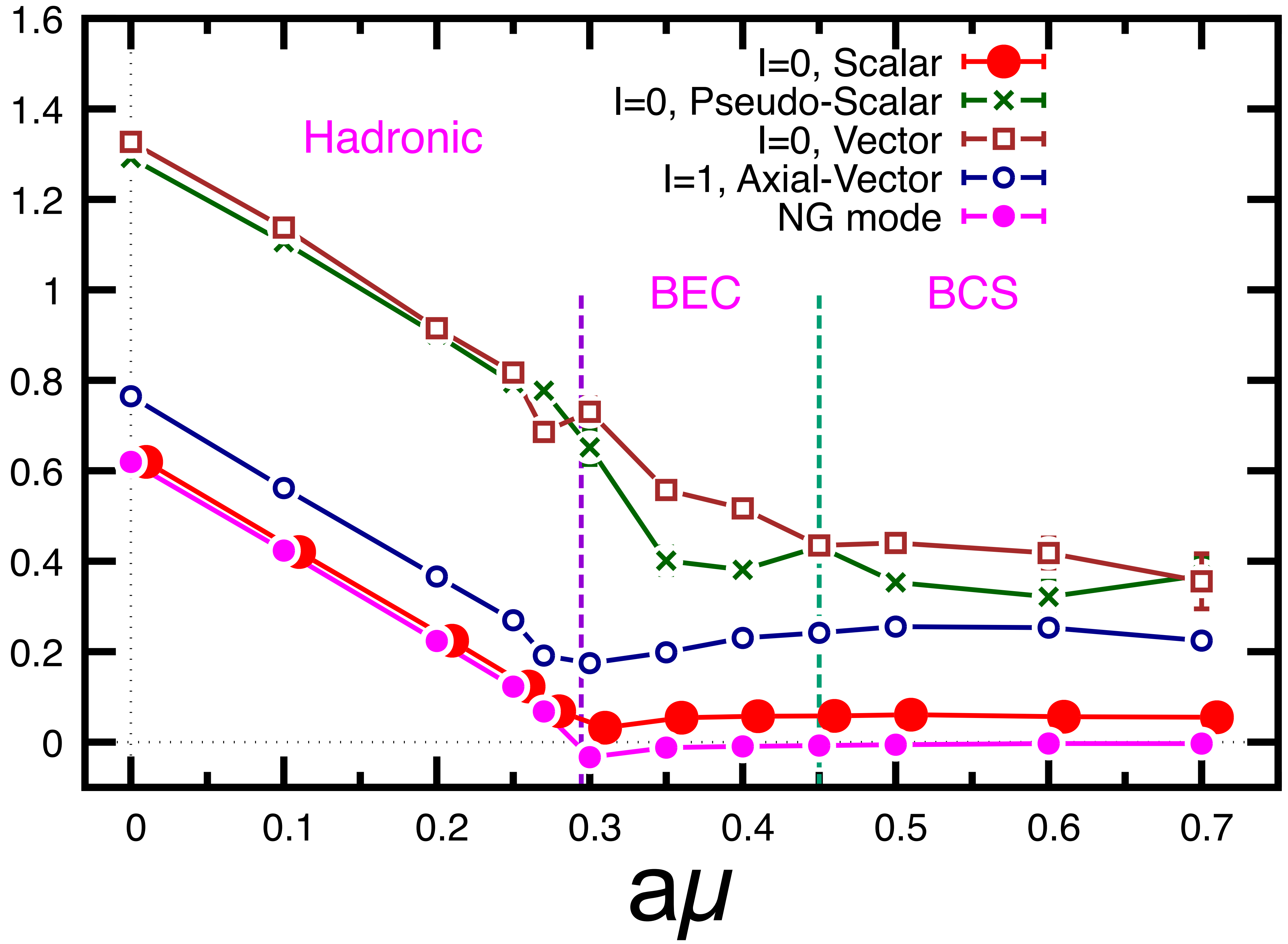
  same multiplet (3dim.) of unbroken $Sp(4)$

Summary of mass spectra

Mesons



Diquarks



In Hadronic phase, $m_\pi \lesssim m_\eta < m_\sigma(\text{noisy}) < m_\rho \sim m_\omega \ll m_{a_1}$

(same with 3color QCD vacua)

In SF phase, $m_\sigma(\text{noisy}) < m_{a_1} < m_\rho < m_\pi \sim m_\eta(\text{noisy}) \ll m_\omega(\text{noisy})$

In both phases, $m_{NG} \lesssim m_{I=0,S} < m_{I=1,AV} < m_{I=0,PS} \lesssim m_{I=0,V}$

(In our notation($\psi_1^T K \Gamma \psi_2$, $K \equiv C\gamma_5\tau_2$), at $\mu = 0$

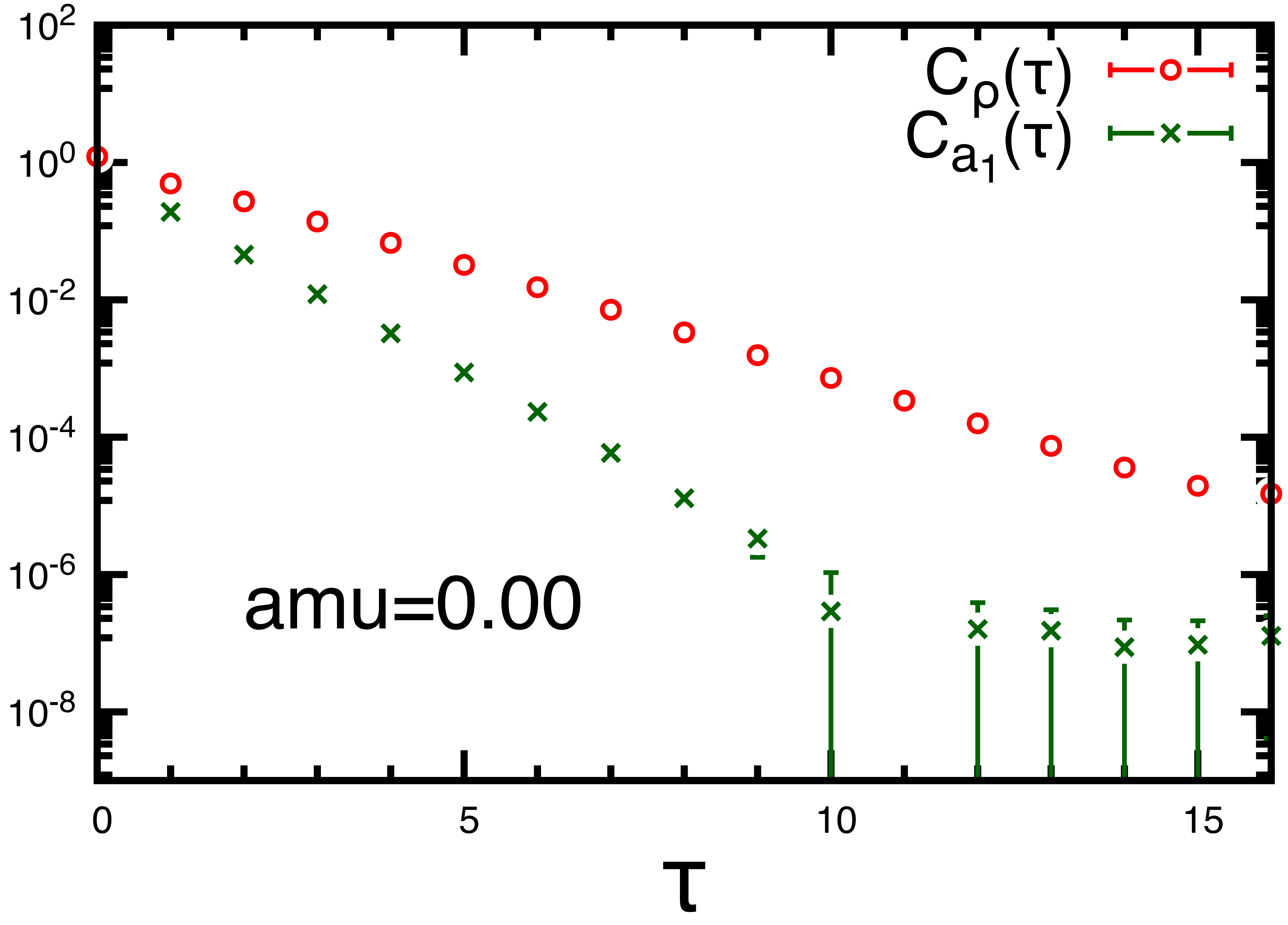
$\pi \leftrightarrow D_{I=0,S}$, $\rho \leftrightarrow D_{I=1,AV}$ are same multiplets)

Chiral symmetry and 2pt fn.

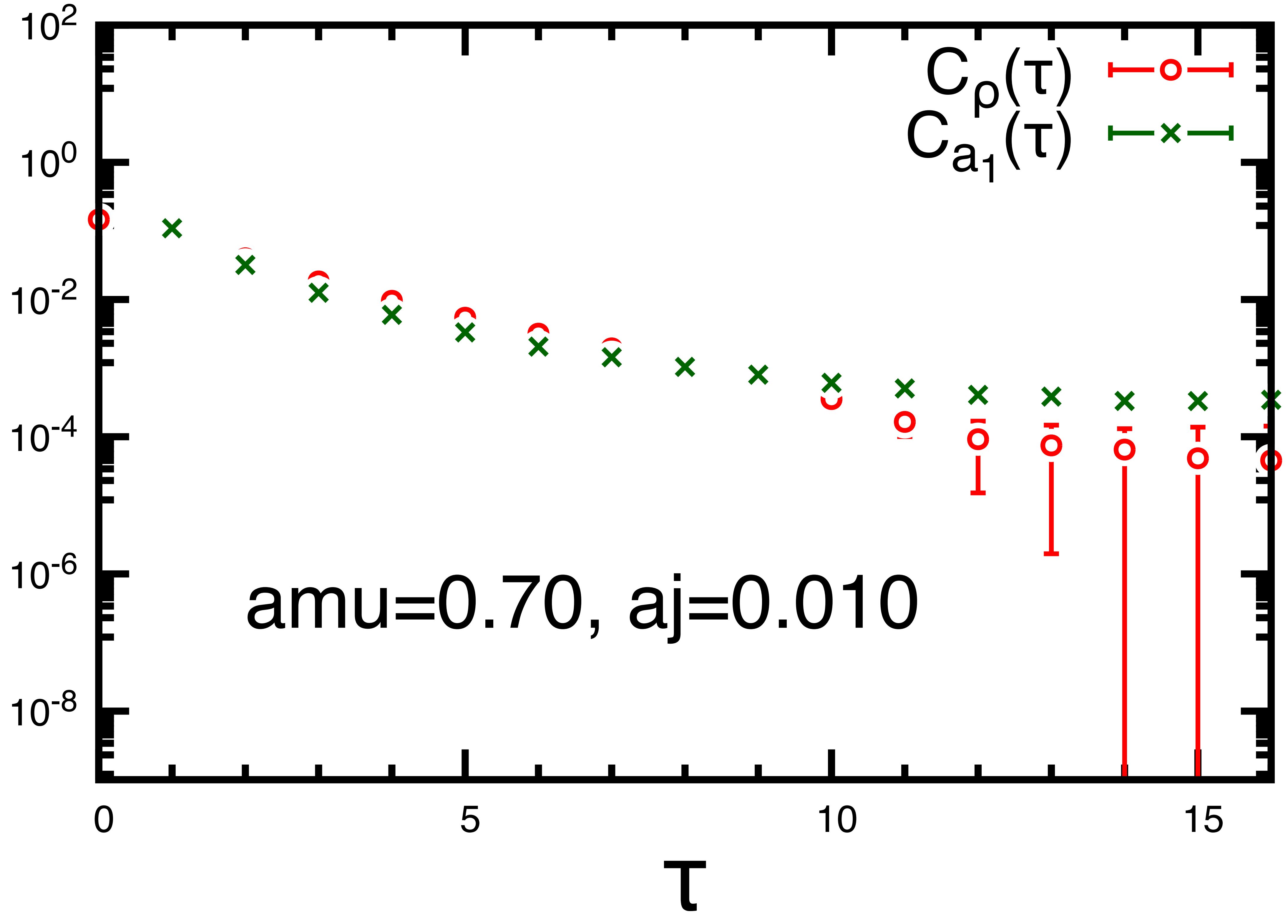
2pt fn. of rho and a1 mesons

We take $\mu = 1$ for V_μ^a and A_μ^a current op. to extract J=1 component.

Hadronic phase (low-density)



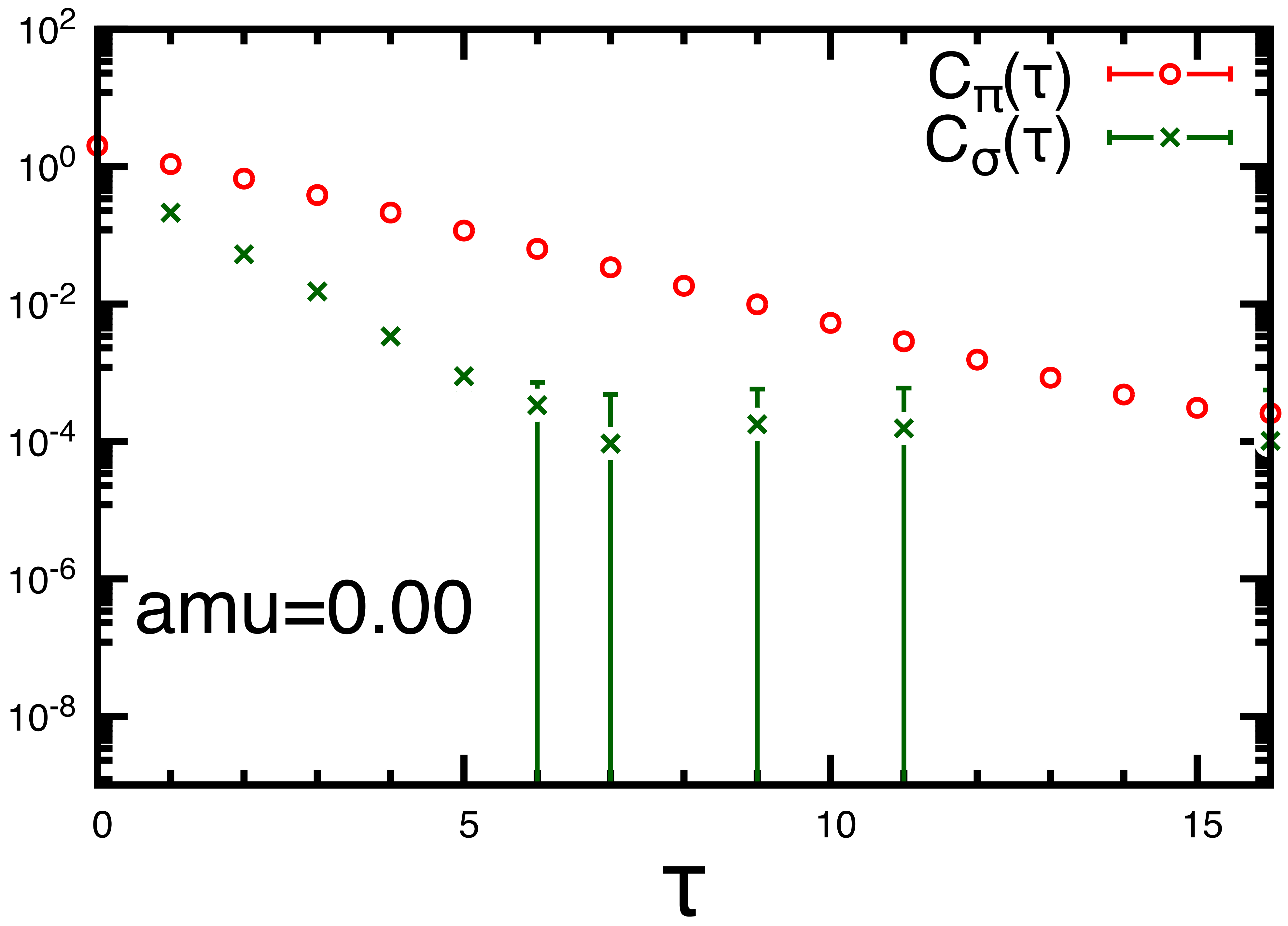
Superfluid phase (high-density)



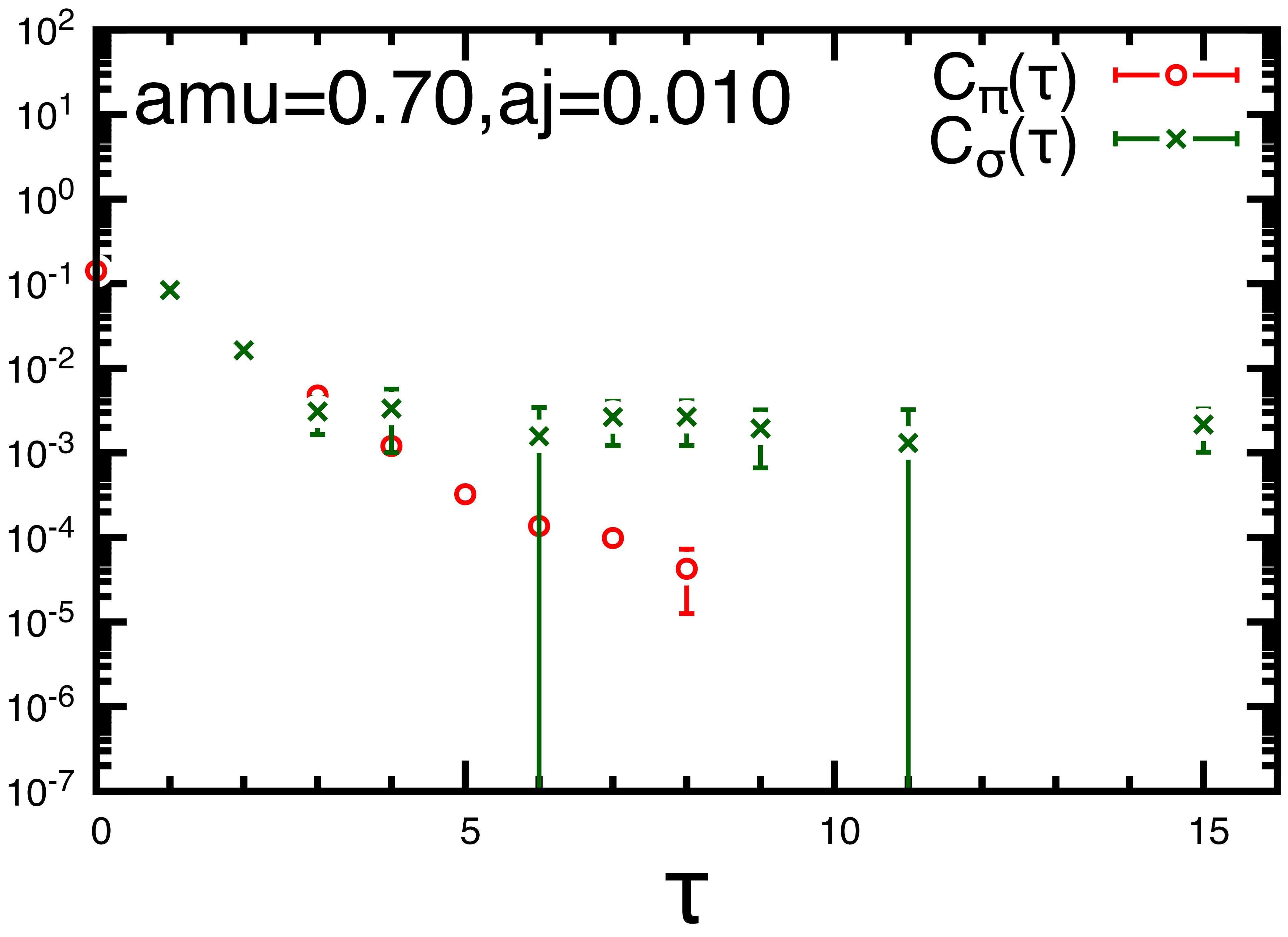
At high density, two 2pt-fns. appear to degenerate.

2pt fn. of pion and sigma meson

Hadronic phase (low-density)



Superfluid phase (high-density)

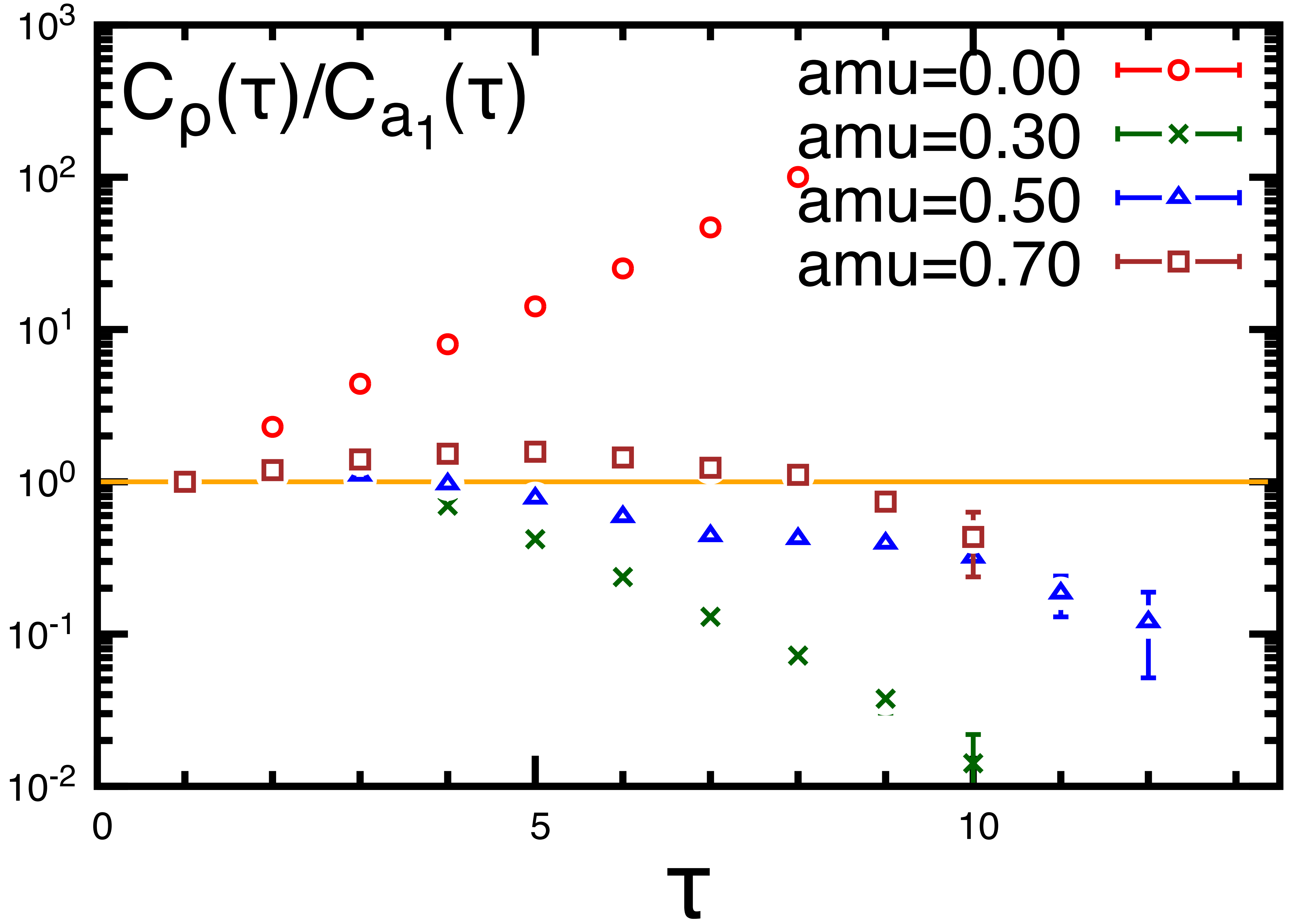


At high density, two 2pt-fns. appear to degenerate.

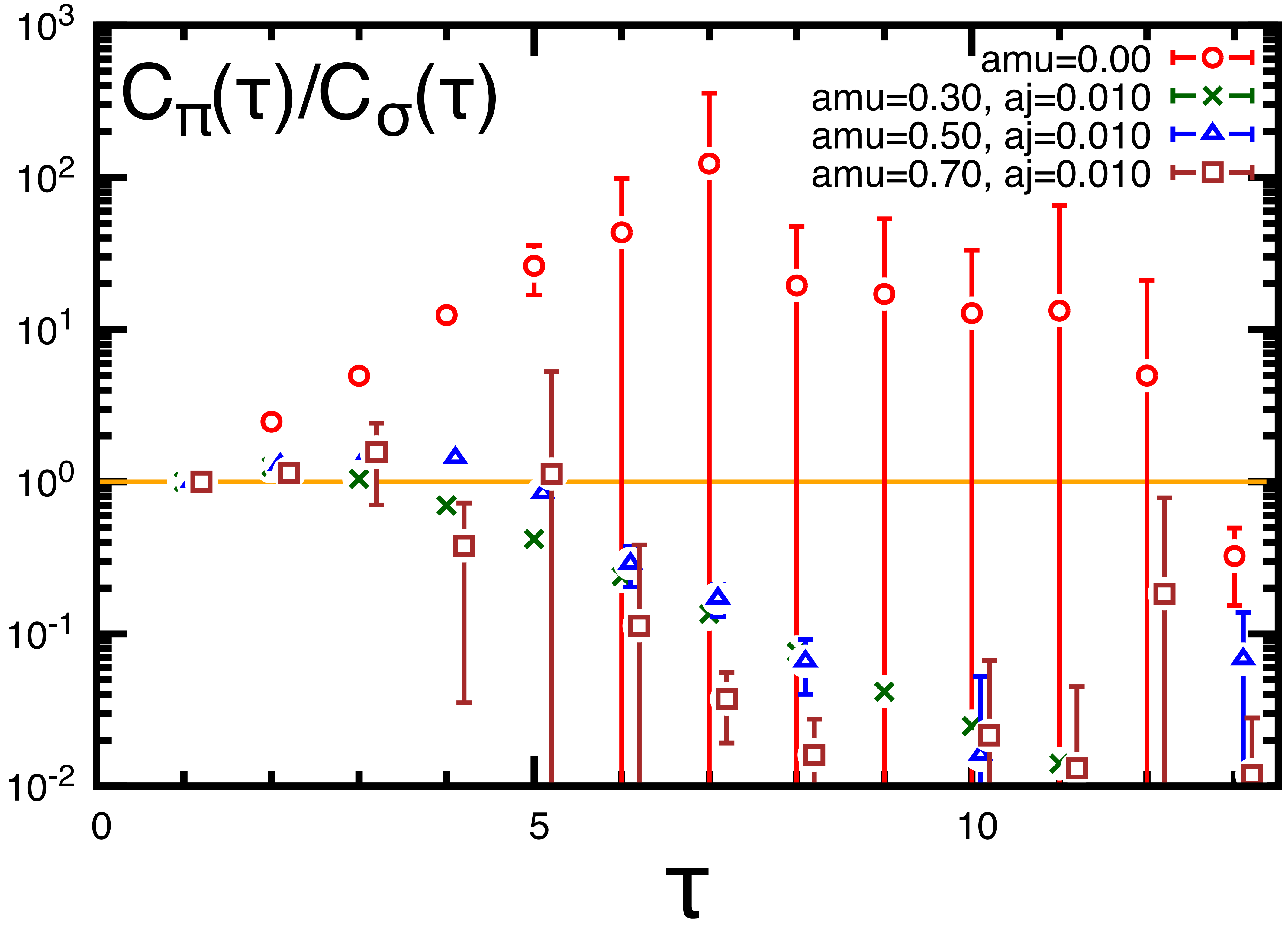
(but noisy…)

Ratio of 2pt fn. bw chiral partners

Rho and a1



Pion and sigma



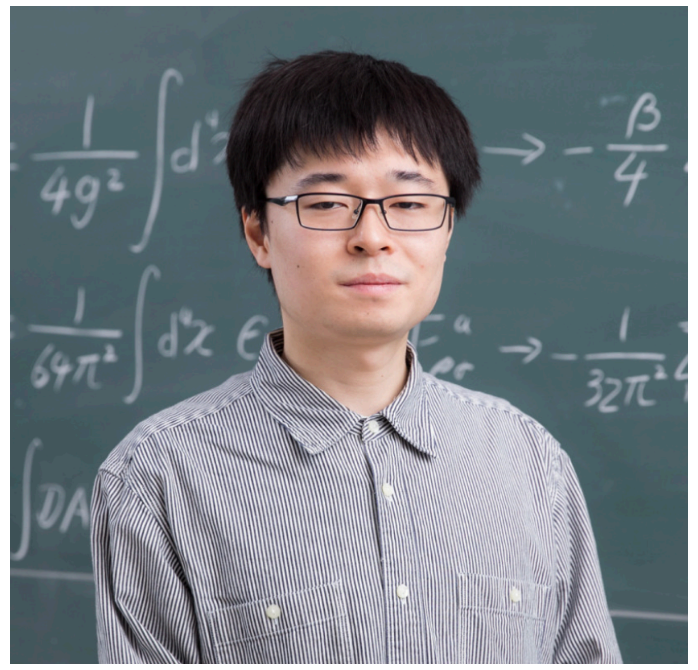
Here, the ratio is normalized to 1 with $\tau = 1$.

A region close to 1 extends in higher density.

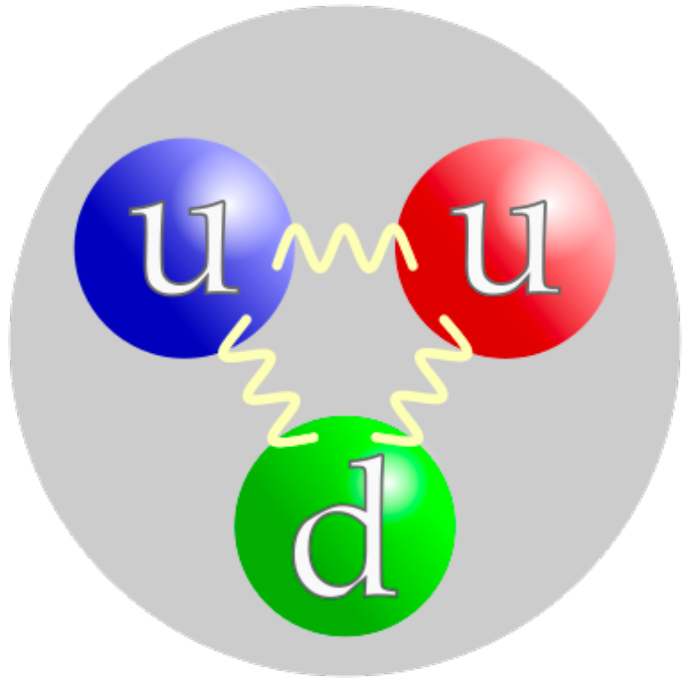
It suggests the recovery of chiral symmetry.

3. (1+1)-dim. Nf=2 Schwinger model using Tensor Network

E.I., Akira Matsumoto, Yuya Tanizaki, JHEP 11 (2023) 231
E.I., Akira Matsumoto, Yuya Tanizaki, JHEP 09 (2024) 155



Conventional calculation method: Lattice QCD



- So far, path integral in Lagrangian formalism

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\phi \mathcal{O} e^{-S[\phi]}, \quad \text{Euclidean action : } S = \int d^4x \mathscr{L},$$

- Generate configurations using $e^{-S[\phi]}$ as the Boltzmann weight

  Hybrid Monte Carlo algorithm

  S. Duane, A. Kennedy, B. Pendleton, D. Roweth, PLB195(1987) 216

- Sign problem if $e^{-S[\phi]}$ takes complex value

  Sign problem is NP-hard (Troyer and Wiese: 2005)

- Reformulate the models/calculation methods in Hamiltonian formalism

  real-time evolution / finite density QCD / topological θ term

  New calculation method expands the range of theories to be explored

Schwinger model + θ term (Nf=1)

Kogut and Susskind (1975)
Shaw et al. Quantum 4, 306 (2020)

- Lagrangian

$$\mathscr{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{photon propagation}} + \boxed{\frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}} + \underbrace{i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi}_{\text{Kinetic/mass terms of electron}}$$

non-zero θ_0: Sign problem in conventional method

Legendre transformation
Gauge fixing
Gauss' law
Open BC
Jourdan-Winger trans.

- Hamiltonian by spin variables

$$H = J\sum_{n=0}^{N-2}\left[\underbrace{\sum_{i=0}^{n}\frac{Z_i + (-1)^i}{2}}_{\text{kinetic term of electric field}} + \boxed{\frac{\theta}{2\pi}}\right]^2 + \underbrace{\frac{w}{2}\sum_{n=0}^{N-2}\left[X_nX_{n+1} + Y_nY_{n+1}\right] + \frac{m_{\text{lat.}}}{2}\sum_{n=0}^{N-1}(-1)^nZ_n}_{\text{kinetic/mass terms of electron}}$$

- θ is constant shift of electric field

- all-to-all interaction of Z
  Gaped system (even in massless fermion for Nf=1)

$$J = g^2a/2,\ w = 1/2a,\ m_{\text{lat}} = m - \frac{g^2a}{8}$$

R.Dempsey et al. PRR 4 (2022) 043133

"Hadron" state in Nf=2 Schwinger model

- Prediction by analytical study (Coleman, 1976) at $\theta = 0$

(1)pion (Iso-triplet pseudo-scalar meson)

$$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$$

$$J^{PG} = 1^{-+}\ (J_z = -1,0,1)$$

(2)sigma(Iso-singlet scalar meson)

$$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2,\ J^{PG} = 0^{++}$$

(3)eta(Iso-singlet pseudo-scalar meson)

$$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right),\ J^{PG} = 0^{--}$$

Quantum numbers:

$\mathbf{J}^2, J_z$ Isospin

associate with SU(2) flavor sym.

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \mathcal{U}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

P: Parity

G-parity (generalized C.C.)

Three calculation methods (at $\theta = 0$)

E.I., Akira Matsumoto, Yuya Tanizaki, JHEP 11 (2023) 231

(1) (Spatial) correlation-function scheme (conventional method)

$$\langle \mathcal{O}(r)\mathcal{O}(0)\rangle \propto K_0(Mr) \sim \frac{1}{\sqrt{r}}e^{-Mr}$$

ex.) $\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$

(2) One-point-function scheme

Calculate $\langle \mathcal{O}(x)\rangle = \langle \mathrm{Vac.}\,|\,\mathcal{O}(x)\,|\,\mathrm{Bdry}\rangle \sim e^{-Mx}$

one-point fn. = correlation fn. with edge state

By tuning the b.c. and value of θ , we obtain the desired meson state

(3) Dispersion-relation scheme

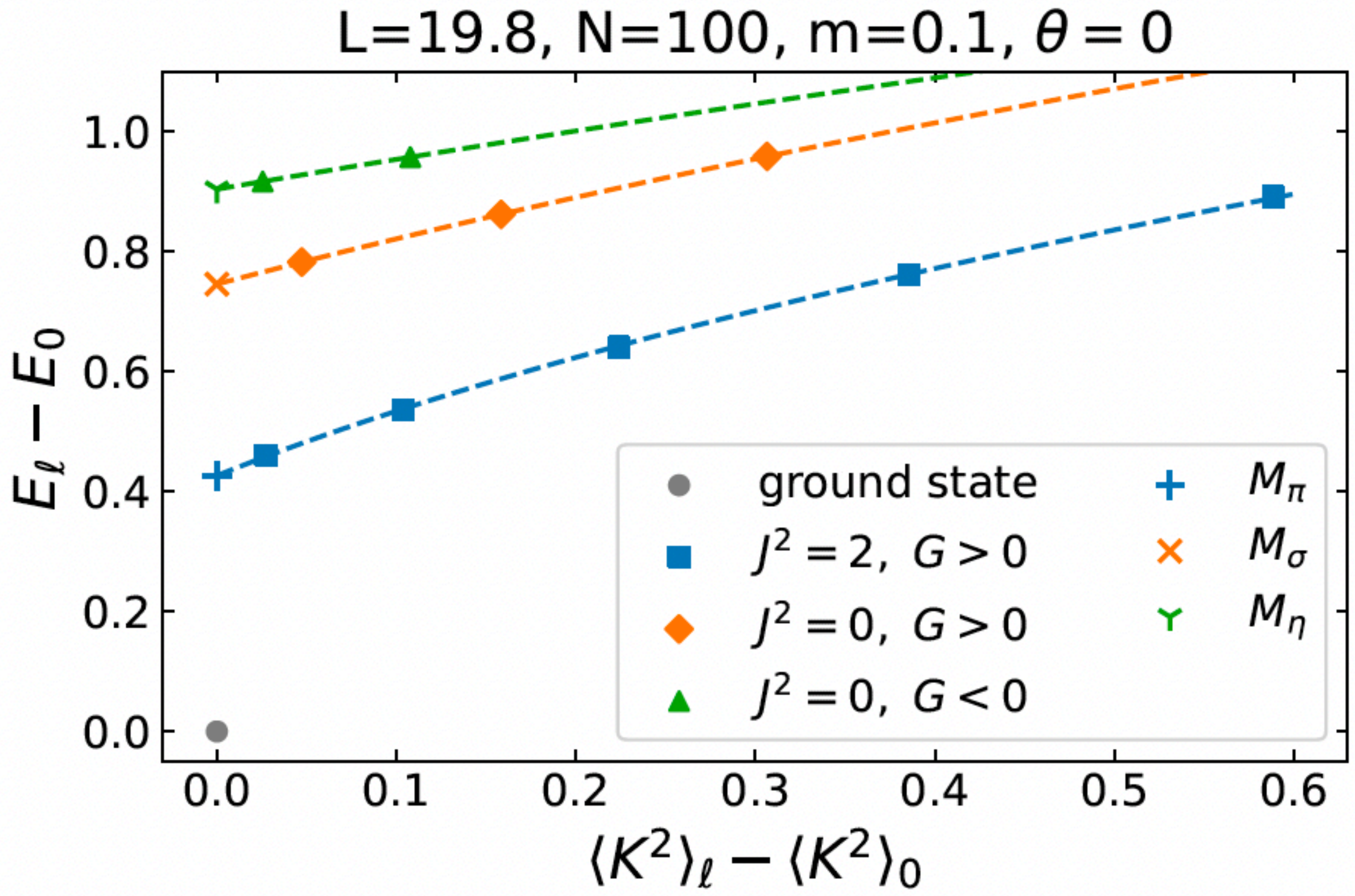
Construct excited states and measure energy, momentum and quantum numbers

excited state calc.: M.C. Banuls, K. Cichy, J.I. Cirac and K. Jansen (2013)

(3)Results of dispersion-relation scheme

E.I., Akira Matsumoto, Yuya Tanizaki, JHEP 11 (2023) 231

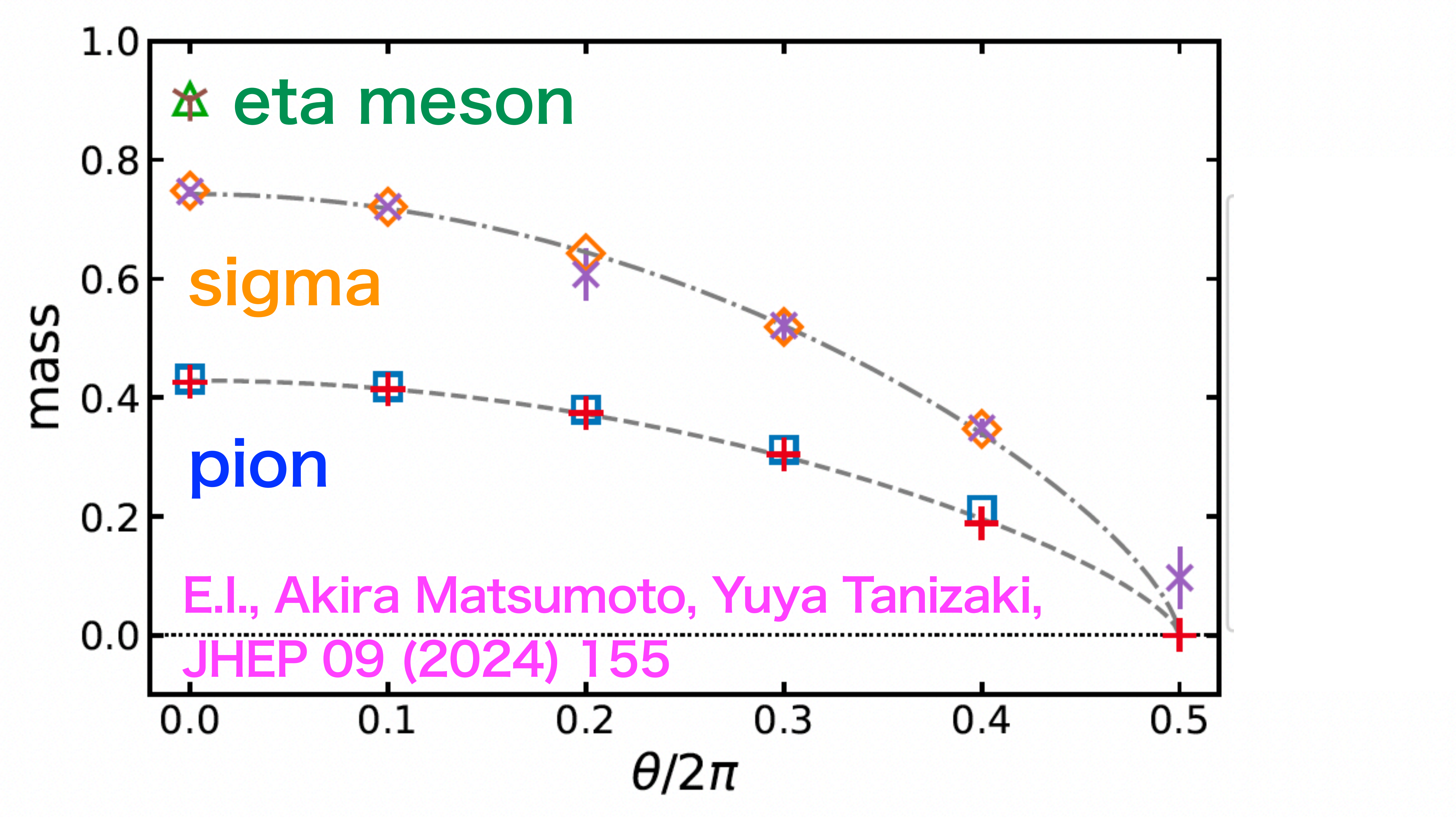
Plot ΔE_ℓ against ΔK_ℓ^2 for each meson and fit the data using $\Delta E = \sqrt{M^2 + b^2 \Delta K^2}$



excited state calc.: M.C. Banuls, K. Cichy, J.I. Cirac and K. Jansen (2013)
T. Schwägerl, K. Jansen, S. Kühn (2025)

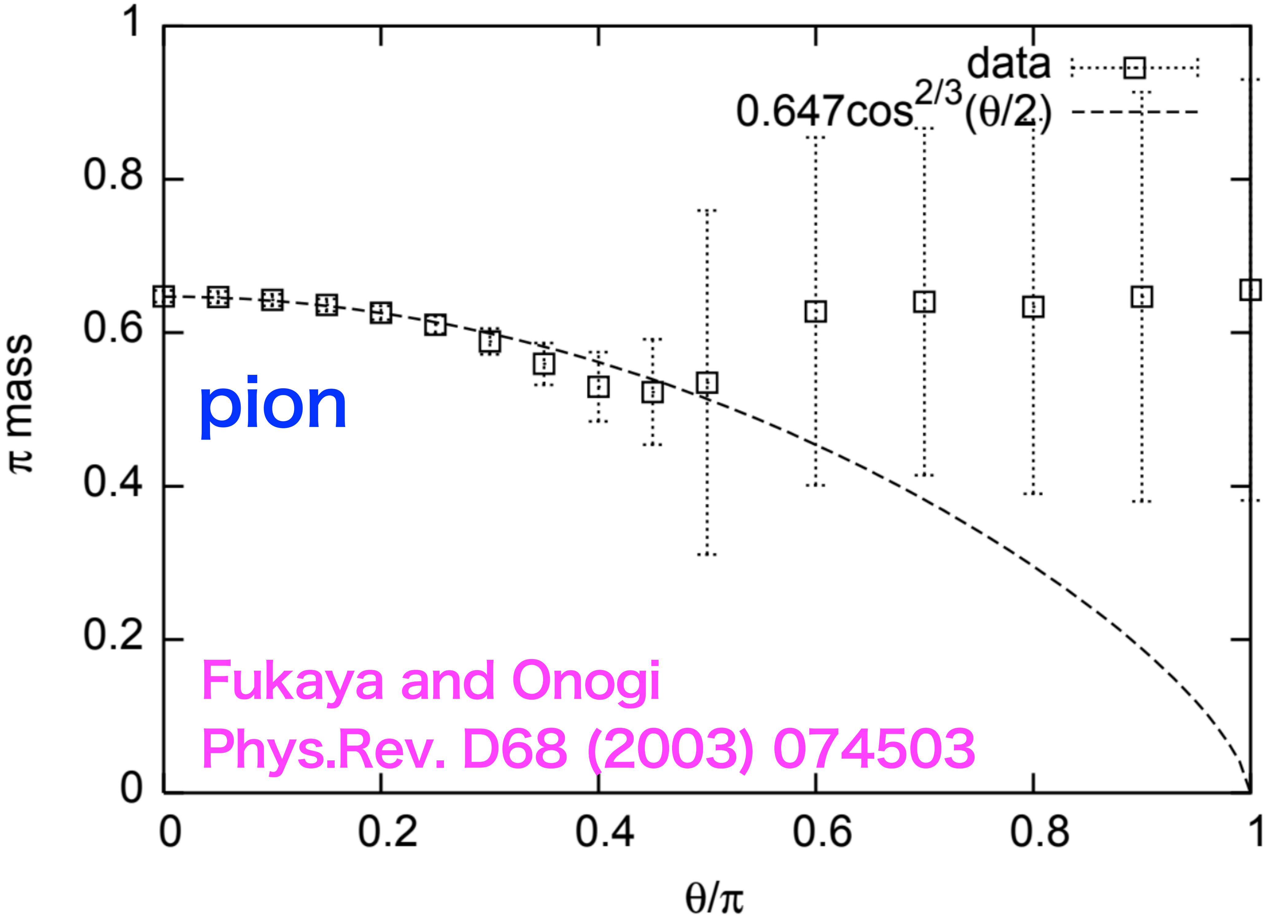
Comparison w/ previous work in Lattice MC

Tensor network based on Hamiltonian



- Straightforwardly apply to $\theta = \pi$ regime (even near CFT)
- Higher states are heuristically found
- Consistent with several theoretical predictions

Monte Carlo based on Lagrangian (w/ improvement techniques)



- In large θ, the signal is very noisy because of the sign problem
- Difficult to find a heavy η-meson and σ-meson

cf.) Gabriel Cuomo et al., “The two-flavor Schwinger model at 50: Solving Coleman's puzzles”, arXiv:2605.08042

Summary

Summary

- (4+0)-dim. Nf=2+1 QCD at physical point
  The HAL QCD Coll. generated high-statistics configs at the physical point.
  Studies of hadron interactions, exotic hadrons, and femtoscopy are ongoing.

- (4+0)-dim. Nf=2 2color QCD at finite density
  We showed that the ordering of the hadron spectrum changes at high density.
  A connection to 3-color QCD is expected.

- (1+1)-dim. Nf=2 QED (Schwinger model) at nonzero topological theta regime
  In Hamiltonian-formalism TN calculations, a new method for calculating spectra using dispersion relations has proven useful.
  Although the calculations are still limited to lower-dimensional theories, they have already revealed new physics. We are currently extending the method to higher dimensions.

Backup

Calculation strategy (2 methods)

- local definition for quark and meson masses

$$m_{\rm eff} = \log \frac{C(\tau)}{C(\tau+1)}, \quad C(\tau) = \text{<PP> and <}A_4\text{P>}$$

Note: the derivative of A_4 ($m_f^{\rm AWI} + m_g^{\rm AWI} = \frac{\langle 0 | \partial_4 A_4 | PS \rangle}{\langle 0 | P | PS \rangle}$) is estimated by the 4th-order symmetric difference

- simultaneous-fit between <PP> and <A_4 P> for quark and meson masses and decay const. (Central analysis)

$$\langle P(\tau) P^\dagger(0) \rangle \simeq N_s^3 C_{PP} W(m_{\rm PS} T) \times \left[\exp(-m_{\rm PS}\tau) + \exp(-m_{\rm PS}(T-\tau))\right]$$

$$\langle A_4(\tau) P^\dagger(0) \rangle \simeq N_s^3 C_{AP} W(m_{\rm PS} T) \times \left[\exp(-m_{\rm PS}\tau) - \exp(-m_{\rm PS}(T-\tau))\right]$$

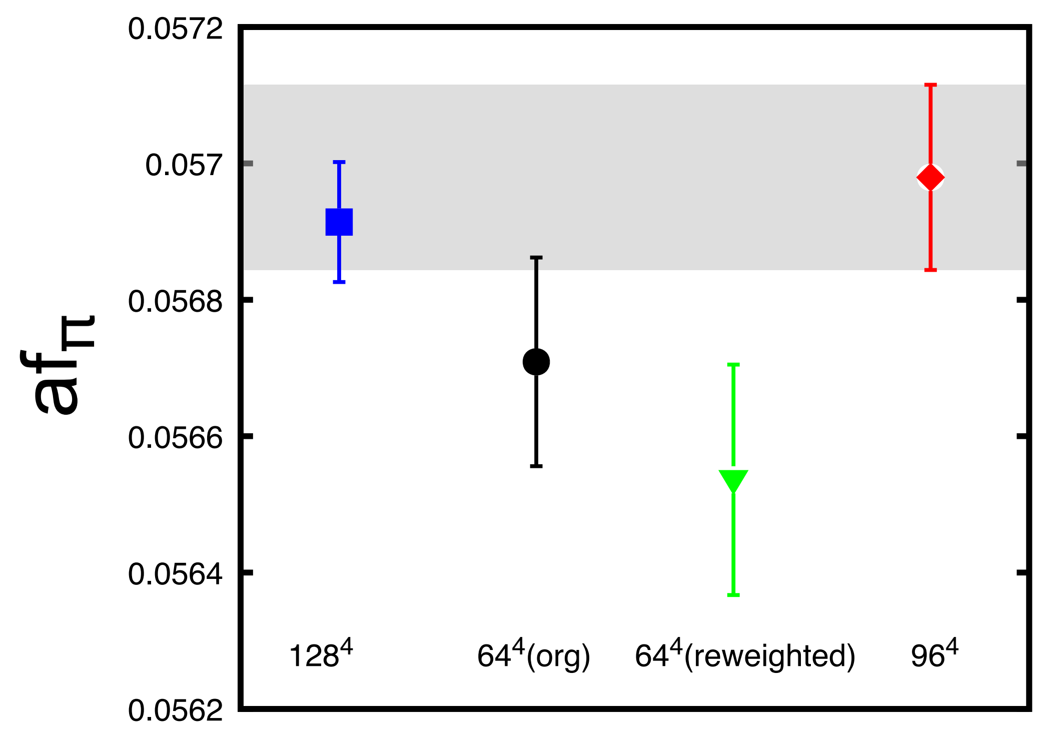
fit parameters:

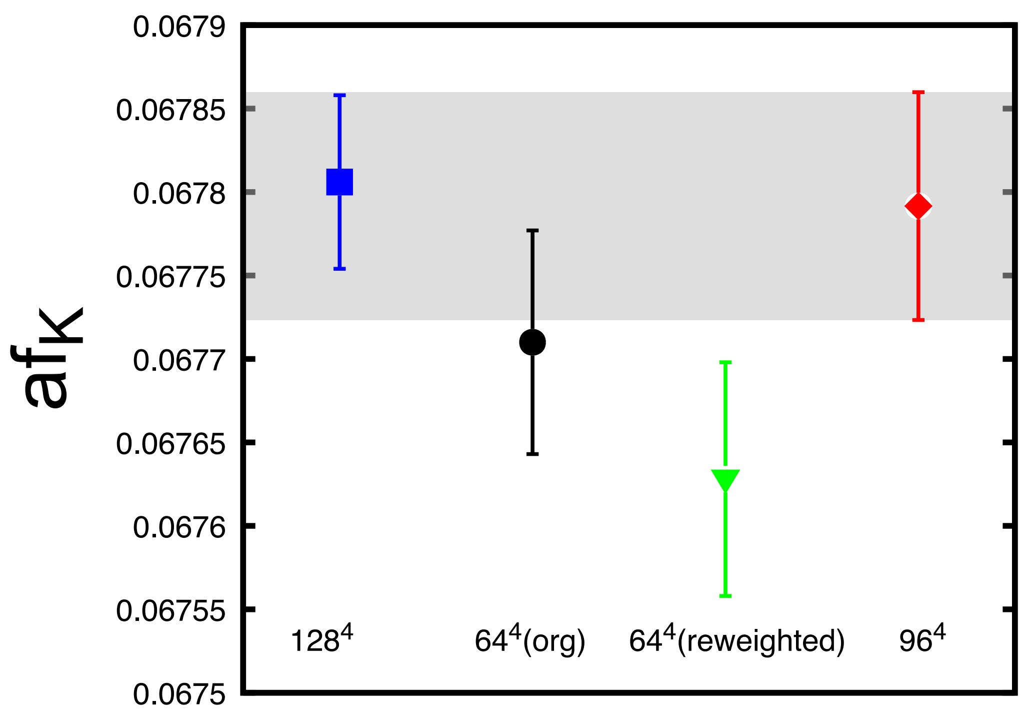
$m_{\rm PS}, C_{PP}, C_{AP}$

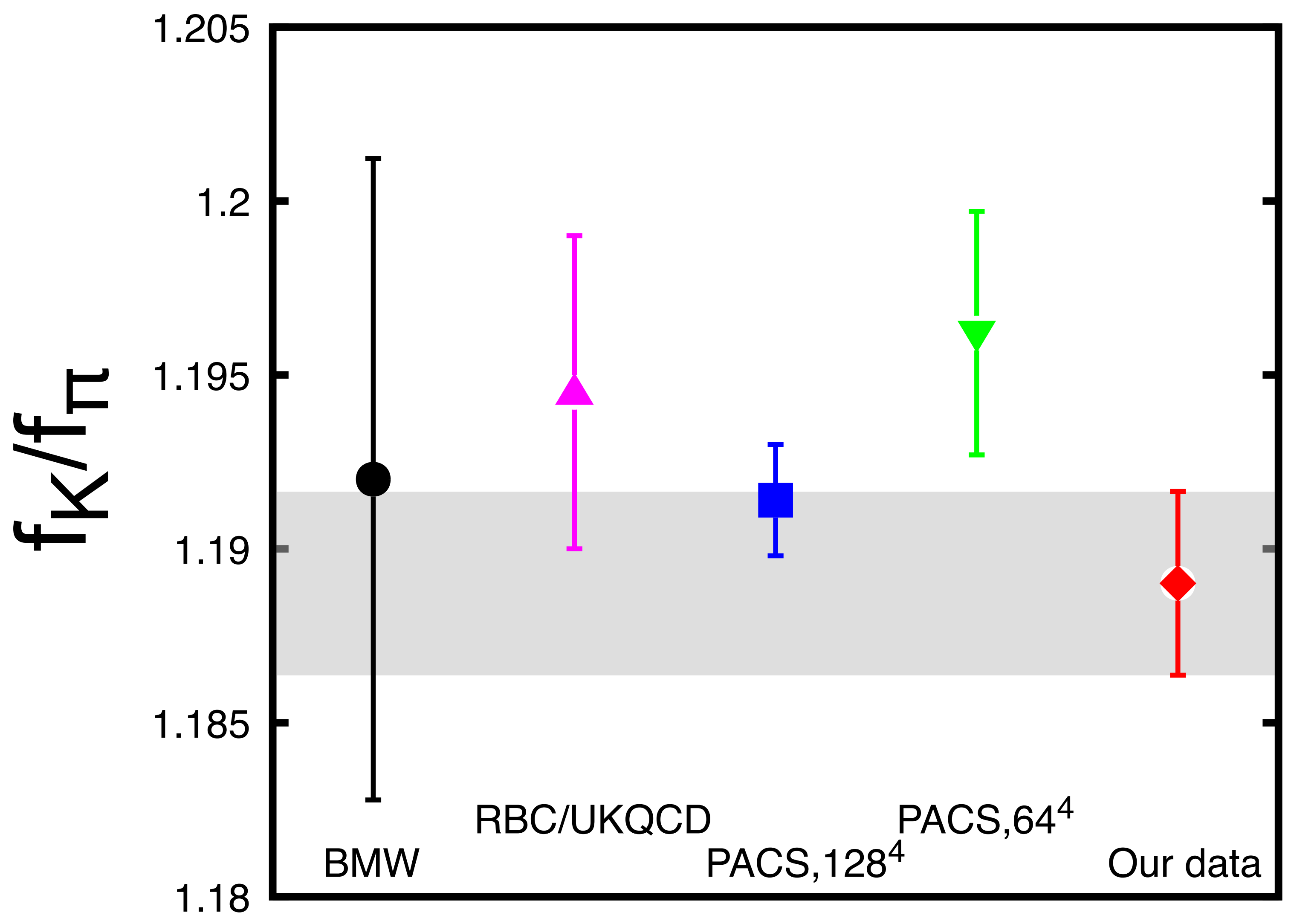
- $$m_f^{\rm AWI} + m_g^{\rm AWI} = m_{\rm PS} \left| \frac{C_{AP}}{C_{PP}} \right|, \quad f^{\rm bare}_{{\rm PS},fg} = \sqrt{2\kappa_f}\sqrt{2\kappa_g} \frac{\sqrt{2}\,|C_{AP}|}{\sqrt{m_{\rm PS}\,|C_{PP}|}}$$

=> $m_{fg}^{\overline{\rm MS}} = \frac{Z_A}{Z_P} m_{fg}^{\rm imp.}, \quad f_{{\rm PS},fg} = Z_A f^{\rm imp.}_{{\rm PS},fg}$ w/ renormalization factor and improved m and f

Decay constants







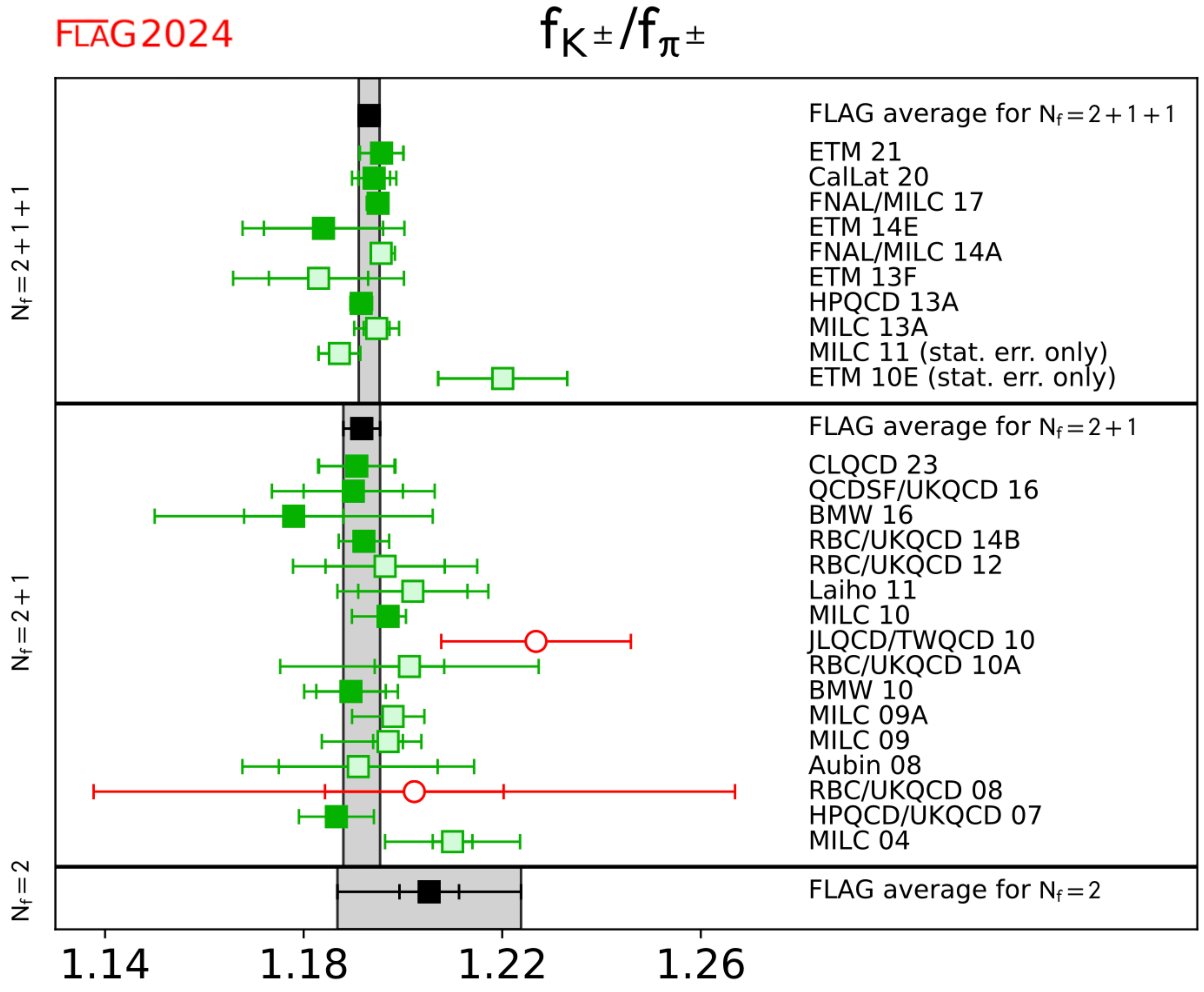
- Results:

af_π	af_K	f_K/f_π
$0.056980(136)(^{+4}_{-6})(690)$	$0.067792(68)(^{+5}_{-5})(822)$	$1.1898(26)(^{+0}_{-1})$

- strong-isospin correction based on NLO SU(3) ChPT: $\dfrac{f_{K^\pm}}{f_{\pi^\pm}} = \dfrac{f_K}{f_\pi}\sqrt{1+\delta_{\mathrm{SU(2)}}}$

HAL : $\dfrac{f_{K^\pm}}{f_{\pi^\pm}} = 1.1875(26)(^{+0}_{-1})(23)$

(new) FLAG average of $f_{K^\pm}/f_{\pi^\pm}$ (Nf=2+1)



- [FLAG 2021]

  $f_{K^\pm}/f_{\pi^\pm}$=1.1917(37)

- [FLAG 2023 on Web]

  $f_{K^\pm}/f_{\pi^\pm}$=1.1899(26) (w/ HAL)

- [FLAG 2024]

  $f_{K^\pm}/f_{\pi^\pm}$=1.1916(34) (w/o HAL)

  direct simulation

Calculation strategy (baryon, basic)

- $$C(\tau) = \sum_{\vec{x}} \langle B^{(\mathrm{sink})}(\vec{x}, \tau)\overline{B}^{(\mathrm{src})}(\vec{y},0)\rangle$$

- Octet baryons w/ spin 1/2 and decuplet baryons with spin 3/2
  perform the spin-parity projection to $C(\tau)$

- Source of correlation fn.: (after benchmark of several possibilities...)
  point-sink and wall-source as main result

- # of measurement:
  rotational sym. (4) x temporal source locations (96) x fw-bw average (2),
  1600conf x 4 x 96 x 2 = 1,228,800 measurements

2pt fn. formula using quark propagators

Mesons: $\bar{\psi}\Gamma\psi$

- iso-single (I=0) or iso-triplet (I=1)
- $\Gamma = \{1, \gamma^5, \gamma^1, i\gamma^5\gamma^1\}$={S, P, V, AV}

$$
\begin{aligned}
\langle M^0(x) M^{0\dagger}(0)\rangle_F &= -2\mathrm{tr}\langle t,\vec{x};c,\beta|S_N\Gamma|t,\vec{x};c,\beta\rangle \cdot \mathrm{tr}\langle 0,\vec{0};c,\beta|S_N\bar{\Gamma}|0,\vec{0};c,\beta\rangle \\
&+\mathrm{tr}\langle t,\vec{x};a,\rho|S_N\bar{\Gamma}|0,\vec{0},c;\beta\rangle \left(\langle t,\vec{x};a,\rho|\Gamma^\dagger S_N^\dagger|0,\vec{0};c,\beta\rangle\right)^* \\
&+\mathrm{tr}\langle t,\vec{x};a,\rho|S_A\bar{\Gamma}^T|0,\vec{0},c;\beta\rangle \left(\langle t,\vec{x};a,\rho|\Gamma^\dagger \bar{S}_A^\dagger|0,\vec{0};c,\beta\rangle\right)^* \\
\langle M^1(x) M^{1\dagger}(0)\rangle_F &= \mathrm{tr}\langle t,\vec{x};a,\rho|S_N\bar{\Gamma}|0,\vec{0},c;\beta\rangle \left(\langle t,\vec{x};a,\rho|\Gamma^\dagger S_N^\dagger|0,\vec{0};c,\beta\rangle\right)^* \\
&-\mathrm{tr}\langle t,\vec{x};a,\rho|S_A\bar{\Gamma}^T|0,\vec{0},c;\beta\rangle \left(\langle t,\vec{x};a,\rho|\Gamma^\dagger \bar{S}_A^\dagger|0,\vec{0};c,\beta\rangle\right)^*
\end{aligned}
$$

Diquarks: $\psi_1^T K\Gamma\psi_2$, $K \equiv C\gamma_5\tau_2$

- iso-single (I=0) or iso-triplet (I=1)
- $\Gamma = \{1, \gamma^5, \gamma^1, i\gamma^5\gamma^1\}$={S, P, V, AV}

$$
\begin{aligned}
\langle D^0(x) D^{0\dagger}(0)\rangle_F &= -2\mathrm{tr}\langle t,\vec{x};c,\beta|\bar{S}_A\Gamma K|t,\vec{x};c,\beta\rangle \cdot \mathrm{tr}\langle 0,\vec{0};c,\beta|S_A K\bar{\Gamma}|0,\vec{0};c,\beta\rangle \\
&+\mathrm{tr}\langle t,\vec{x};a,\rho|S_N\Gamma K|0,\vec{0},c;\beta\rangle \left(\langle t,\vec{x};a,\rho|\bar{\Gamma}^\dagger K\bar{S}_N^\dagger|0,\vec{0};c,\beta\rangle\right)^* \\
&+\mathrm{tr}\langle t,\vec{x};a,\rho|S_N K\Gamma^T|0,\vec{0};c,\beta\rangle \left(\langle t,\vec{x};a,\rho|\bar{\Gamma}^\dagger K\bar{S}_N^\dagger|0,\vec{0};c,\beta\rangle\right)^* \\
\langle D^1(x) D^{1\dagger}(0)\rangle_F &= -\mathrm{tr}\langle t,\vec{x};a,\rho|S_N\Gamma K|0,\vec{0},c;\beta\rangle \left(\langle t,\vec{x};a,\rho|\bar{\Gamma}^\dagger K\bar{S}_N^\dagger|0,\vec{0};c,\beta\rangle\right)^* \\
&+\mathrm{tr}\langle t,\vec{x};a,\rho|S_N K\Gamma^T|0,\vec{0};c,\beta\rangle \left(\langle t,\vec{x};a,\rho|\bar{\Gamma}^\dagger K\bar{S}_N^\dagger|0,\vec{0};c,\beta\rangle\right)^*
\end{aligned}
$$

Updated points from Hands et al. (PLB 2007) and Murakami et al. (2023, Proceeding of Lattice 2023)

Disconnected contribution is included

$\mathrm{Tr}_{\vec{x},\mathrm{color},\mathrm{spinor}}[O\langle\tau,\vec{x}\,|\,\tau,\vec{x}\rangle] - \langle \sum_\tau \mathrm{Tr}_{\vec{x},\mathrm{color},\mathrm{spinor}} O(\tau,\vec{x})/N_\tau\rangle$ and its at $(\tau = 0, \vec{x} = \vec{0})$

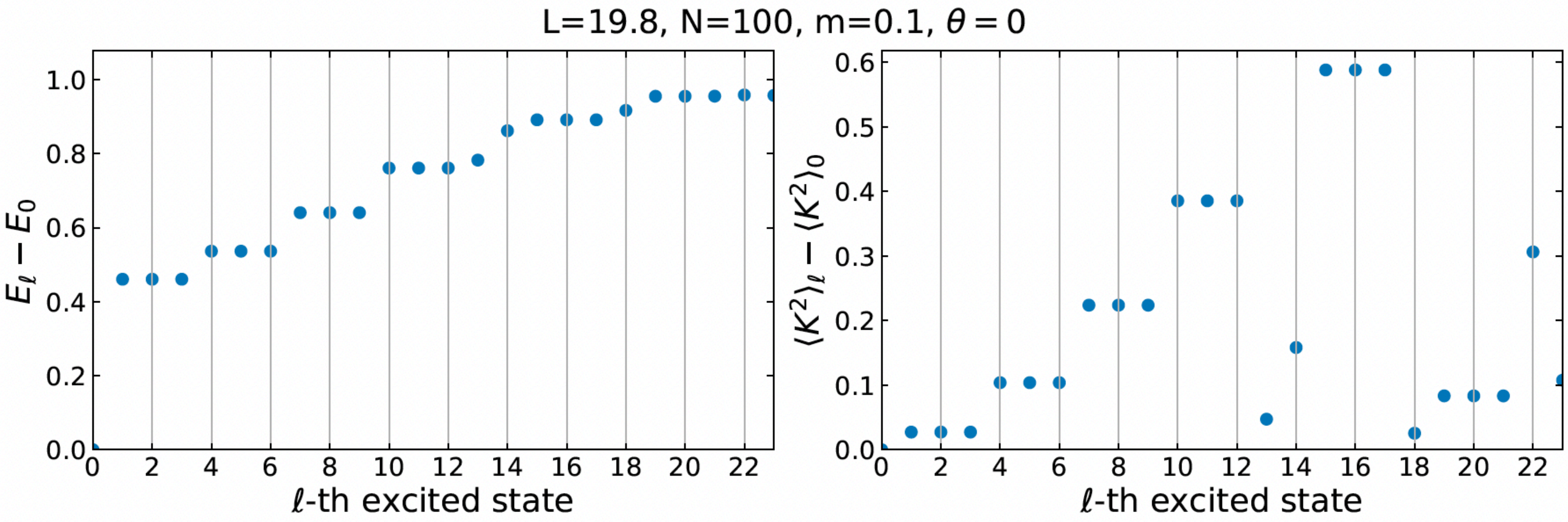
Mesons has 1pt fn. of $\mathrm{Tr}[S_N\Gamma]$, which are relatively large.

Diquark has the one of $\mathrm{Tr}[S_A\Gamma]$, which are proportional to $J^2 \ll 1$ and it is negrigible

(3) Dispersion-relation scheme

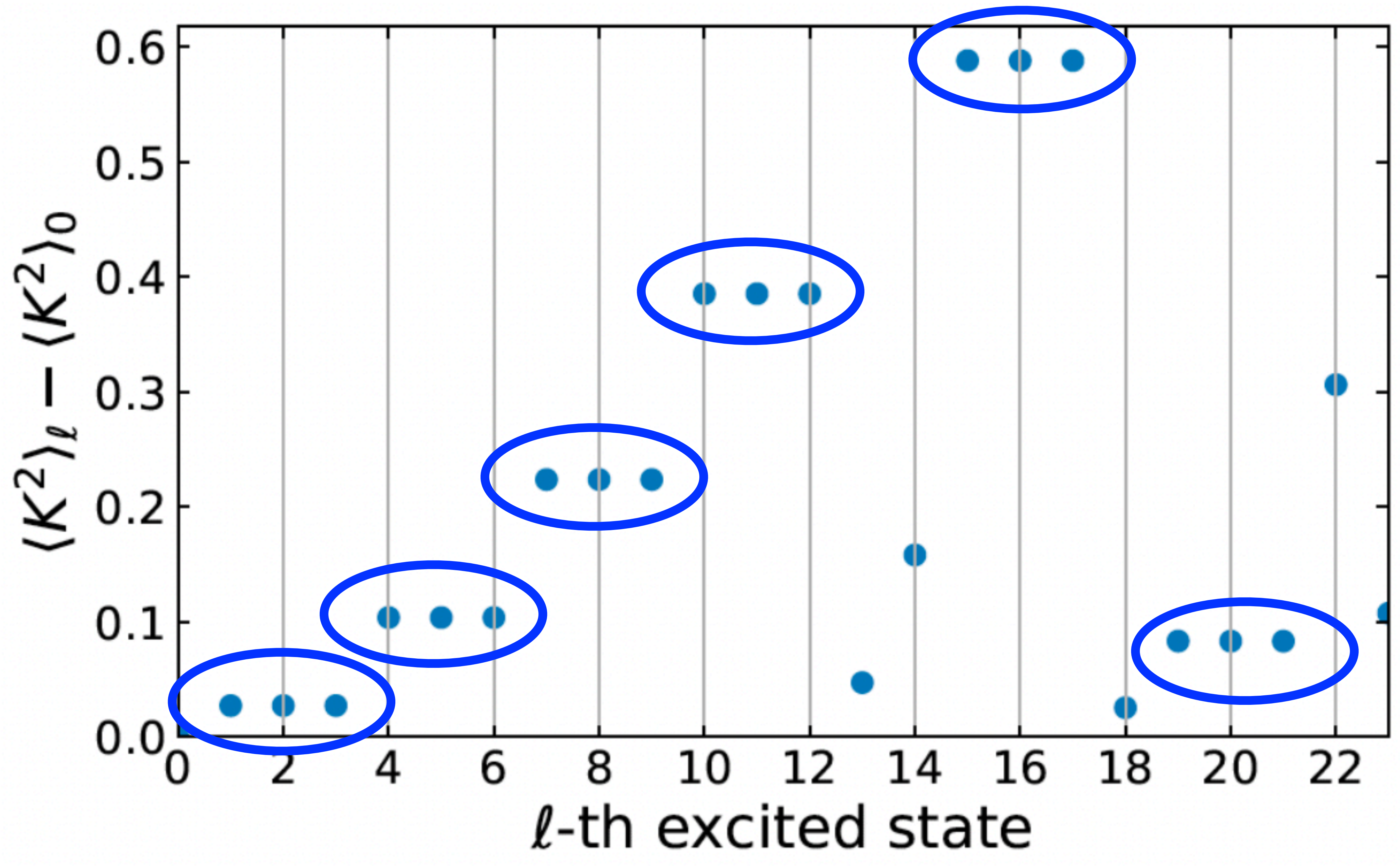
MPS for ℓ-th excited state is given by the modified cost fn.: $H_{eff} = H + \lambda \sum_{k=0}^{\ell-1} |\psi_k\rangle\langle\psi_k|$

Upto 23rd excited state



Measure the quantum number (Iso-spin, G-parity, Parity) of generated MPS to identify each meson

(3)Results: iso-triplet channel



ℓ	$\boldsymbol{J}^2$	J_z	G	P
1	2.00000004	0.99999997	0.27872443	-6.819×10^{-8}
2	2.00000012	-0.00000000	0.27872416	-6.819×10^{-8}
3	2.00000004	-0.99999996	0.27872443	-6.819×10^{-8}
4	2.00000007	0.99999999	0.27736066	7.850×10^{-8}
5	2.00000006	0.00000000	0.27736104	7.850×10^{-8}
6	2.00000009	-0.99999998	0.27736066	7.850×10^{-8}
7	2.00000010	1.00000000	0.27536687	-8.838×10^{-8}
8	2.00000002	0.00000000	0.27536702	-8.837×10^{-8}
9	2.00000007	-0.99999998	0.27536687	-8.838×10^{-8}
10	2.00000007	0.99999998	0.27356274	9.856×10^{-8}
11	2.00000005	0.00000001	0.27356277	9.856×10^{-8}
12	2.00000007	-0.99999999	0.27356274	9.856×10^{-8}
15	1.99999942	0.99999966	0.27173470	-1.077×10^{-7}
16	2.00000052	0.00000000	0.27173482	-1.077×10^{-7}
17	2.00000015	-1.00000003	0.27173470	-1.077×10^{-7}
19	2.00009067	1.00004377	0.27717104	-3.022×10^{-8}
20	2.00002578	-0.00000004	0.27717020	-3.023×10^{-8}
21	2.00003465	-1.00001622	0.27717104	-3.023×10^{-8}

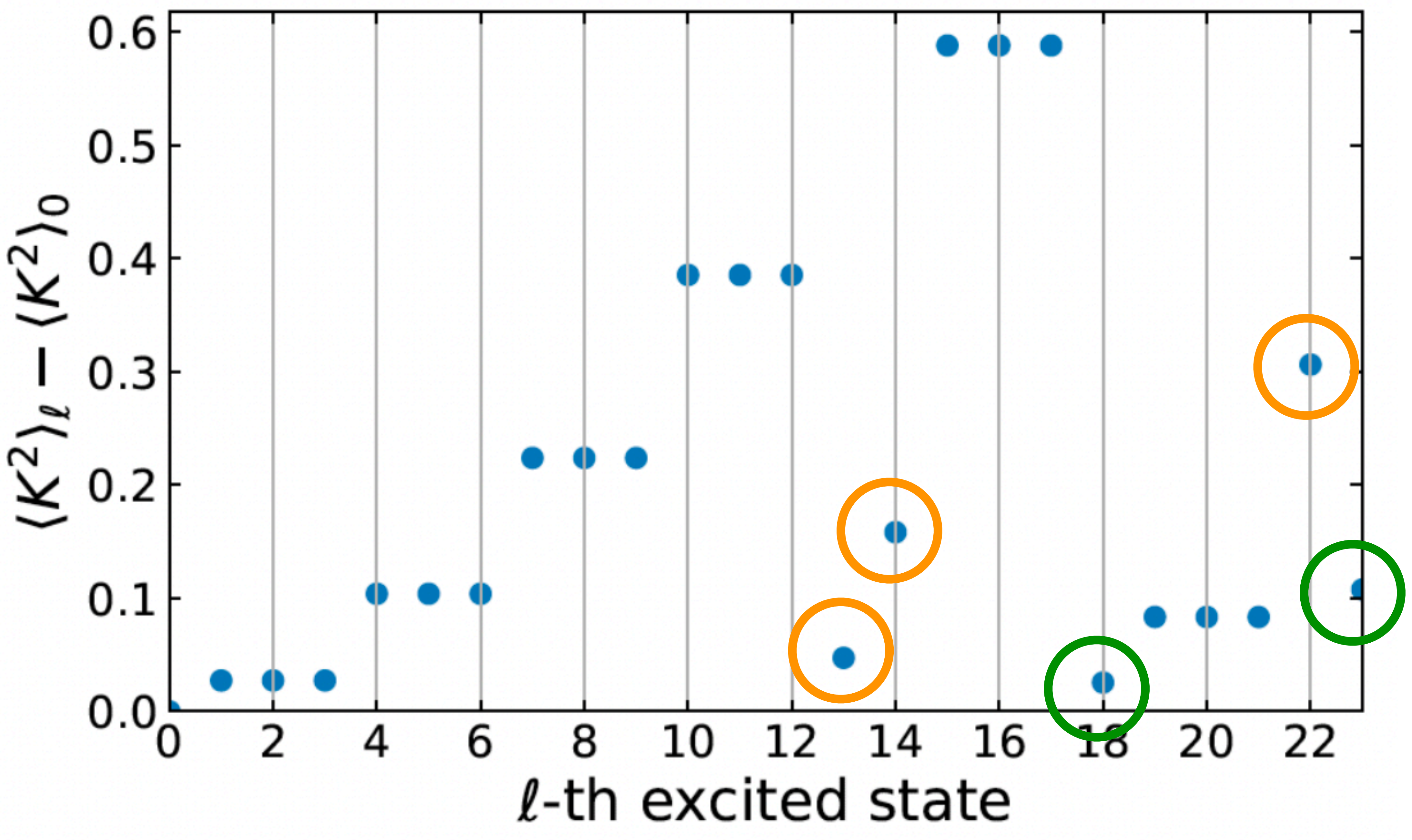
zero-mode

$P < 0$

J=1 $J_z = \pm 1$ $G > 0$ $P < 0$

pion : $J^{PG} = 1^{-+}$

(3)Results: iso-singlet channel



ℓ	$\boldsymbol{J}^2$	J_z	G	P
0	0.00000003	-0.00000000	0.27984227	3.896×10^{-7}
13	0.00000003	0.00000000	0.27865844	1.273×10^{-7}
14	0.00000003	0.00000000	0.27508176	-2.765×10^{-8}
18	0.00000028	0.00000006	-0.27390909	-6.372×10^{-7}
22	0.00001537	0.00000115	0.26678987	7.990×10^{-8}
23	0.00003607	-0.00000482	-0.27664779	5.715×10^{-7}

zero-mode

$P>0$

zero-mode

$P<0$

J=0 $J_z = 0$ $G>0$ $P>0$

sigma meson : $J^{PG} = 0^{++}$

J=0 $J_z = 0$ $G<0$ $P<0$

eta meson : $J^{PG} = 0^{--}$