Theory of Fast Flavor Conversion of Supernova Neutrinos : Linear and Nonlinear Analysis

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Content of The Talk

- A Brief Review
- Our Motivation
- Our Numerical Techniques
- Results for Nonlinear Analysis
- Results for Linear Analysis

Works Done

<u>Non-Linear Analysis :</u>

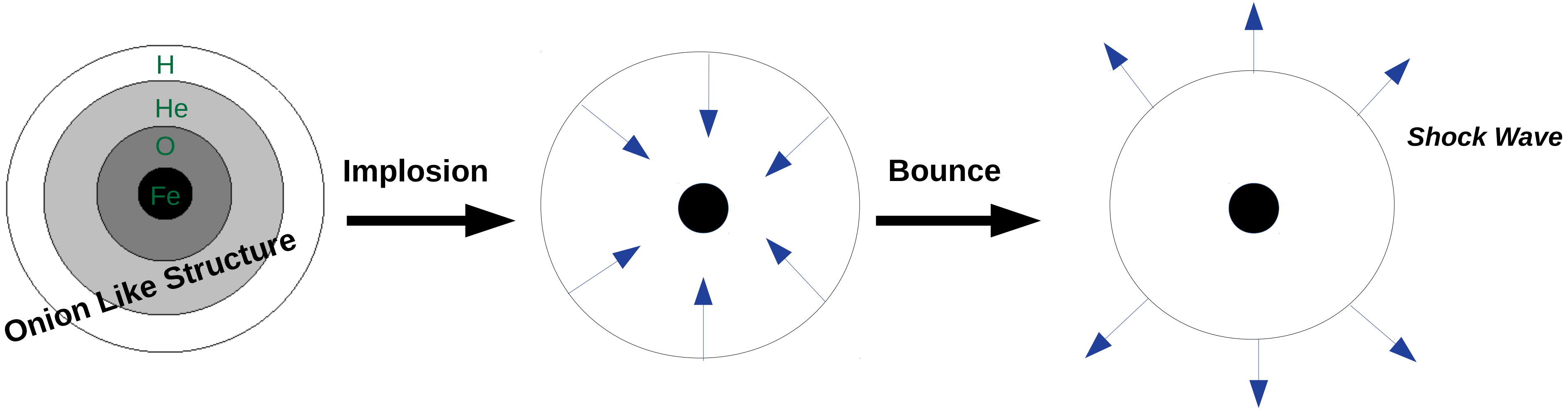
- **Late-time Behavior of Fast Neutrino Oscillations *(Phys.Rev.D 102 (2020) 6, 063018)***
- **Fast Flavor Depolarization of Supernova Neutrinos *(Phys.Rev.Lett. 126 (2021) 6, 061302)***
- **Elaborating the Depolarization of Fast Collective Neutrino Flavor Oscillations *(Manuscript in Preparation)***

<u>Linear Analysis :</u>

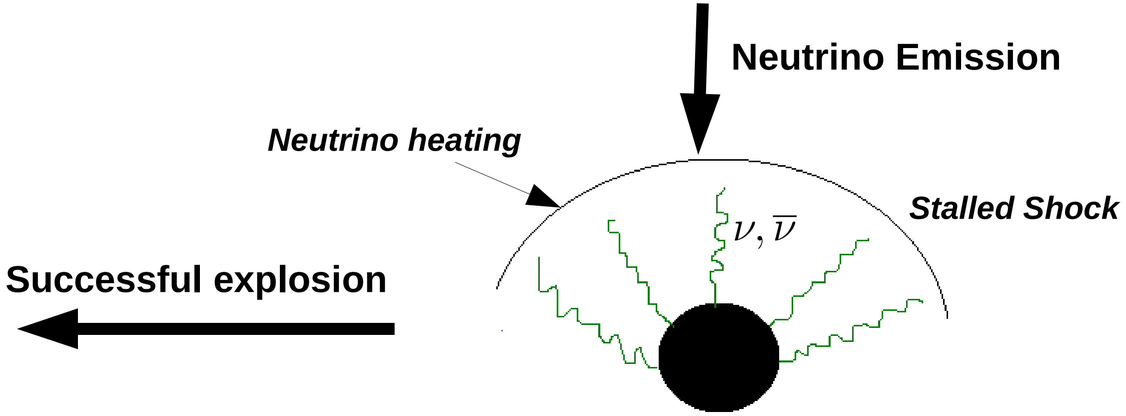
- **Fast Flavor Oscillations of Astrophysical Neutrinos with 1, 2, …, ∞ Crossings *(JCAP 07 (2021) 023)***

A Brief Review

Neutrinos from SN : A Brief Review



Georg G. Raffelt
“Stars as laboratories for fundamental physics”
Chicago, USA: Univ. Pr. (1996) 664 p (1996)



Flavor Conversion of SN Neutrinos

Ensemble of neutrinos → **density matrix** → $\rho_{\vec{p}}[\vec{x}, t]$ → $\begin{pmatrix} \rho_{\vec{p}}^{ee} & \rho_{\vec{p}}^{ex} \\ \rho_{\vec{p}}^{xe} & \rho_{\vec{p}}^{xx} \end{pmatrix}$

Survival Probability

Flavor Conversion

$$(\partial_t + \vec{p} \cdot \nabla_{\vec{x}})\, \rho_{\vec{p}} = \pm\omega \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \rho_{\vec{p}} \right] + \lambda \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \rho_{\vec{p}} \right] + \mu \int \frac{d^3\vec{q}}{(2\pi)^3} \,(1 - \cos\,\theta_{\vec{p}\vec{q}}) \left[\rho_{\vec{q}} - \overline{\rho}_{\vec{q}}, \rho_{\vec{p}} \right]$$

Vacuum Oscillations

$$\omega = \frac{\delta m^2}{4E}$$

Flavor Conversion of SN Neutrinos

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Matter Effects

$$\lambda = \sqrt{2} G_F n_e$$

Flavor Conversion of SN Neutrinos

Ensemble of neutrinos → **density matrix** → $\rho_{\vec{p}}[\vec{x}, t]$ → $\begin{pmatrix} \rho_{\vec{p}}^{ee} & \rho_{\vec{p}}^{ex} \\ \rho_{\vec{p}}^{xe} & \rho_{\vec{p}}^{xx} \end{pmatrix}$

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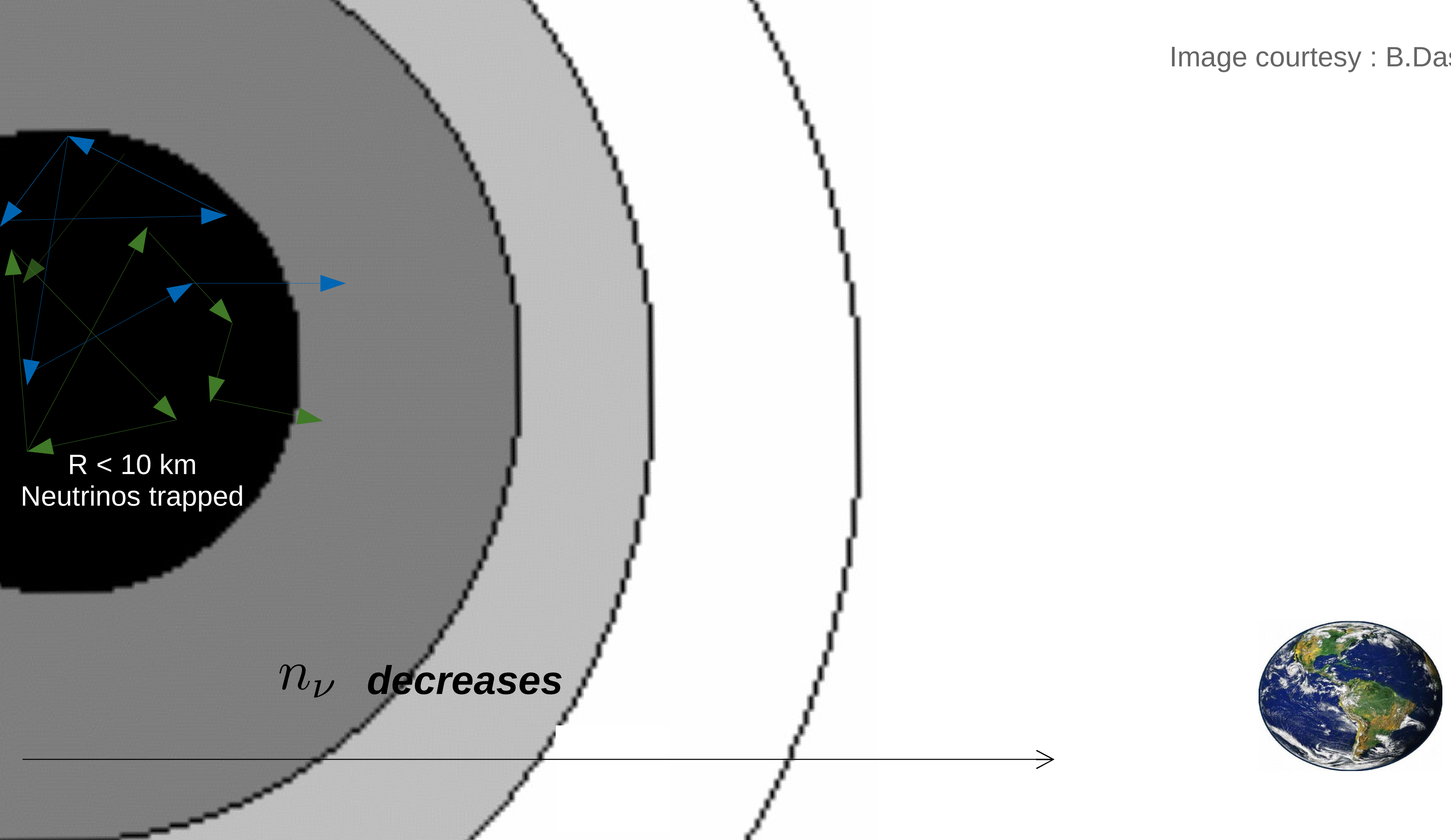
Neutrino Self-interaction

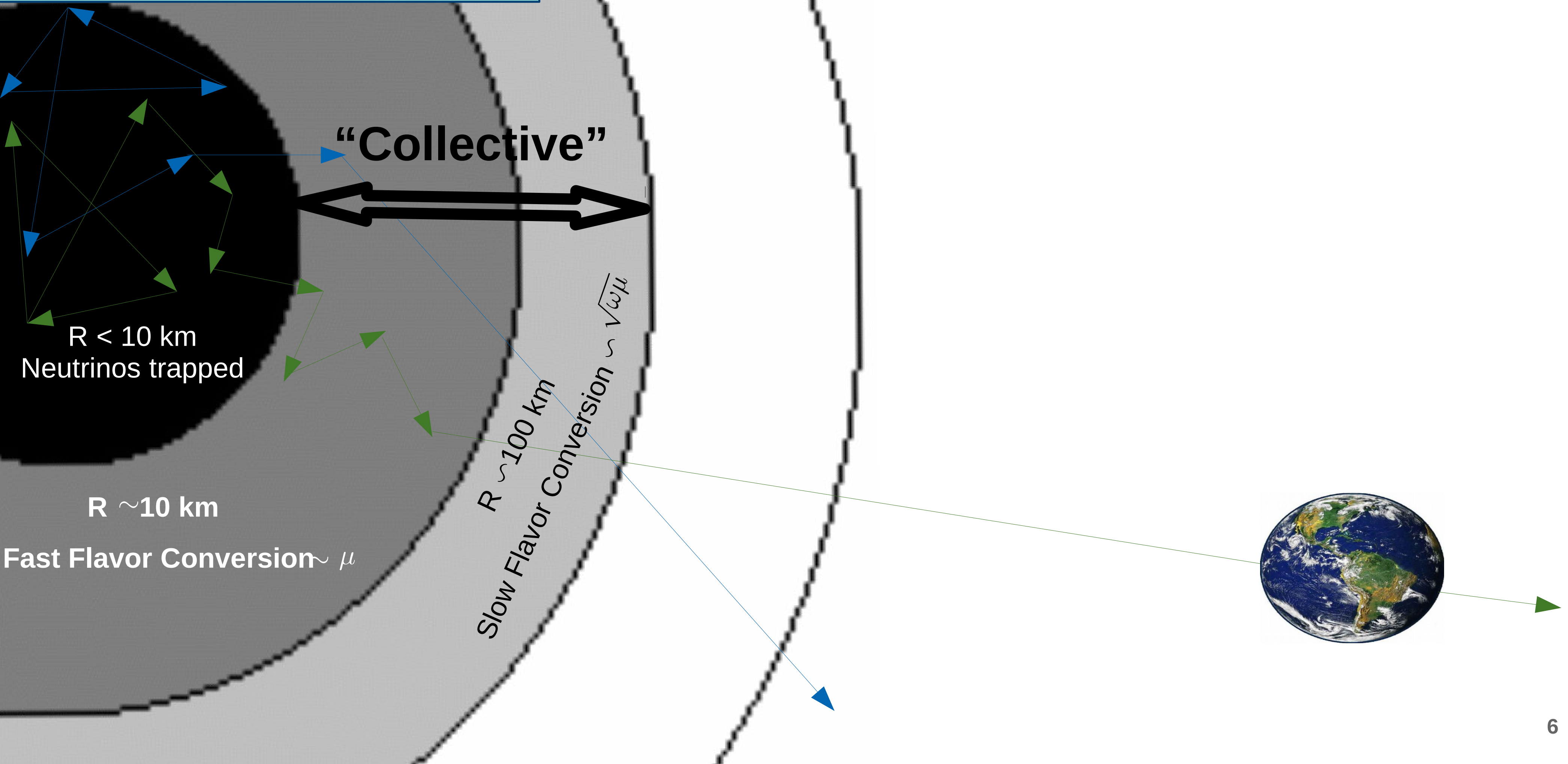
$$\mu = \sqrt{2}G_F n_\nu$$

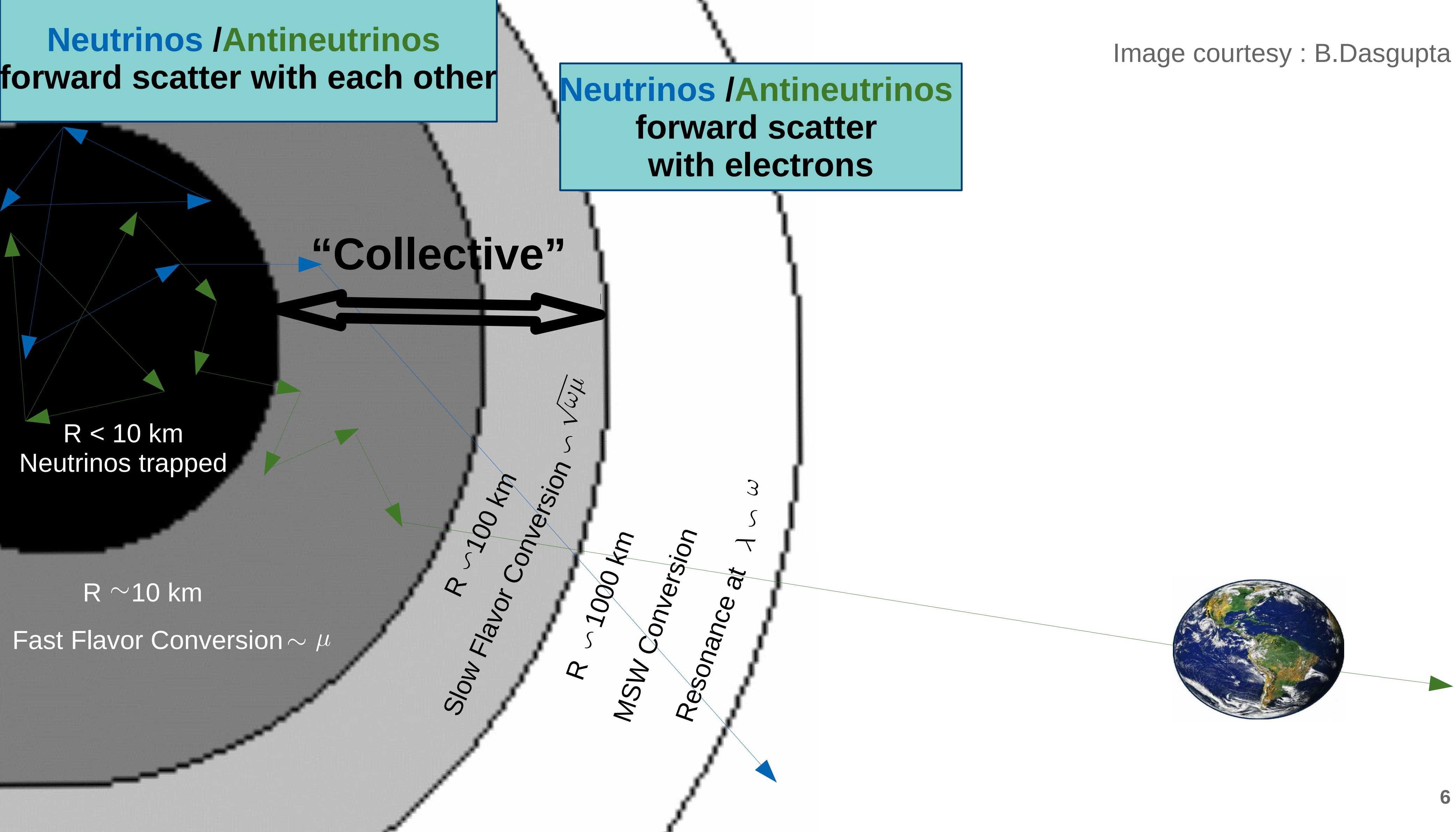
Image courtesy : B.Dasgupta

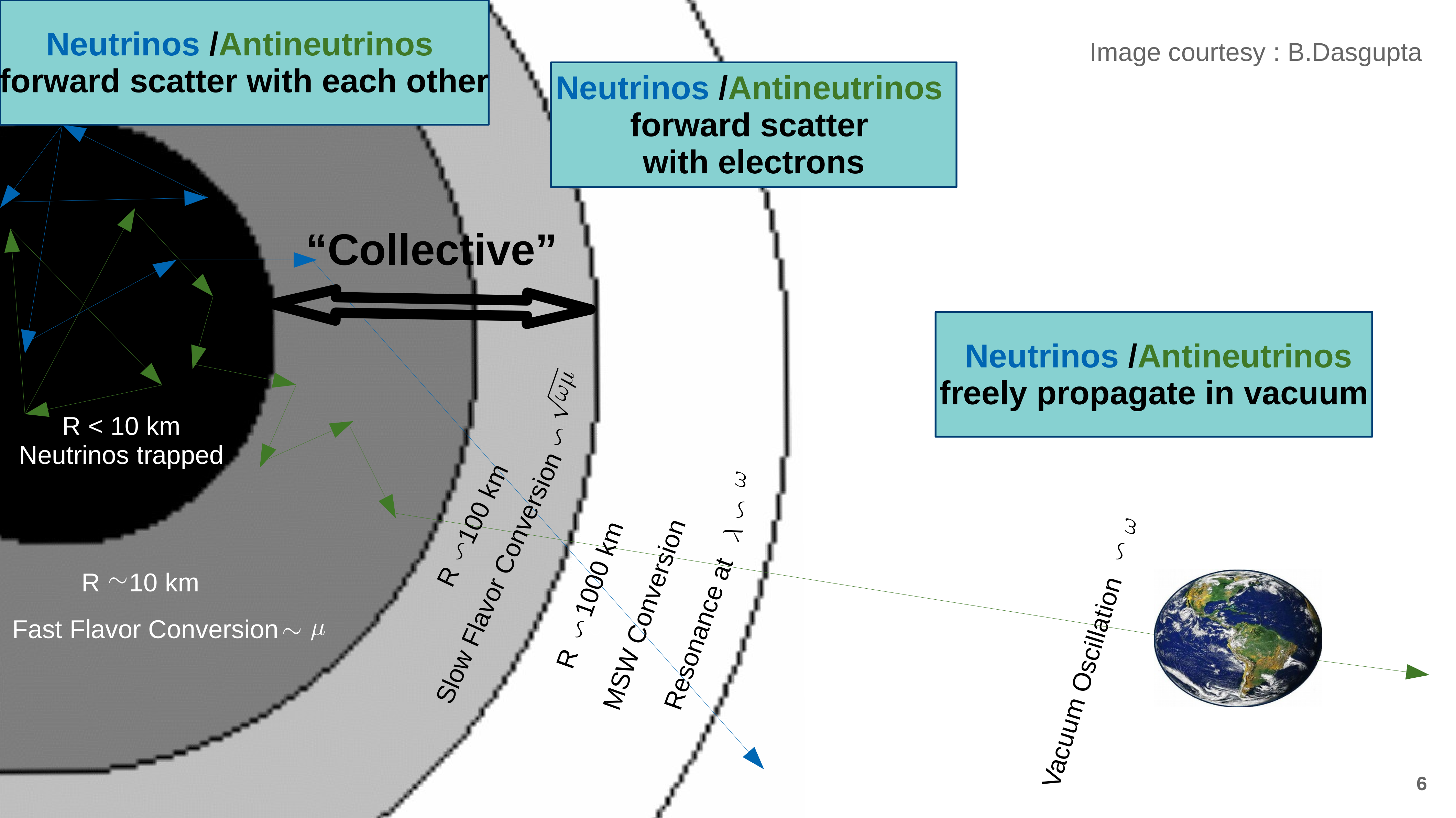


Neutrinos /Antineutrinos forward scatter with each other

Image courtesy : B.Dasgupta




Neutrinos /Antineutrinos
forward scatter with each other
Neutrinos /Antineutrinos
forward scatter
with electrons
"Collective"
R < 10 km
Neutrinos trapped
R $\sim$10 km
Fast Flavor Conversion $\sim \mu$
R $\sim$100 km
Slow Flavor Conversion $\sim \sqrt{\omega\mu}$
R $\sim$1000 km
MSW Conversion
Resonance at $\lambda \sim \omega$



Neutrinos /Antineutrinos
forward scatter with each other
Neutrinos /Antineutrinos
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Image courtesy : B.Dasgupta
"Collective"
Neutrinos /Antineutrinos
freely propagate in vacuum
R < 10 km
Neutrinos trapped
R ~10 km
Fast Flavor Conversion ~ μ
R ~100 km
Slow Flavor Conversion ~ $\sqrt{\omega\mu}$
R ~1000 km
MSW Conversion
Resonance at $\lambda \sim \omega$
Vacuum Oscillation ~ ω

Fast Flavor Conversion : A Review

- **Independent of mass hierarchy :** E **plays no role**

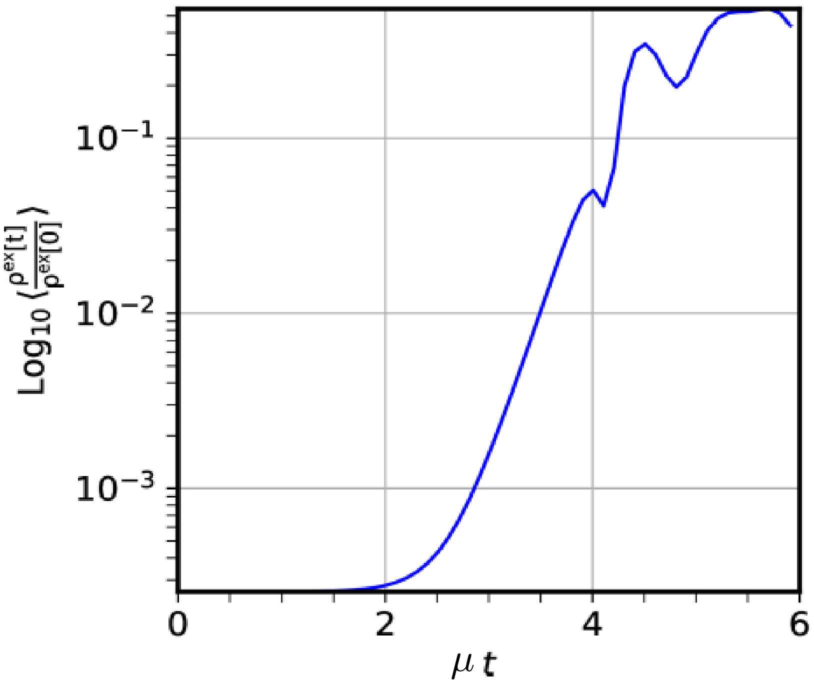
$$(\partial_t + \vec{p}\cdot\nabla_{\vec{x}})\,\rho_{\vec{p}} = \pm\omega\left[\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\rho_{\vec{p}}\right] + \lambda\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\rho_{\vec{p}}\right] + \mu\int d^3\vec{q}\,\left(1-\cos_{\vec{p}\vec{q}}\right)\left[\rho_{\vec{q}}-\bar{\rho}_{\vec{q}},\rho_{\vec{p}}\right]$$

(The ω and λ matrix terms are crossed out.)

Fast Flavor Conversion : A Review

- **Independent of mass hierarchy : E plays no role**
- **Rapid conversion**

$$|\rho^{ex}| \propto e^{\mu t}$$



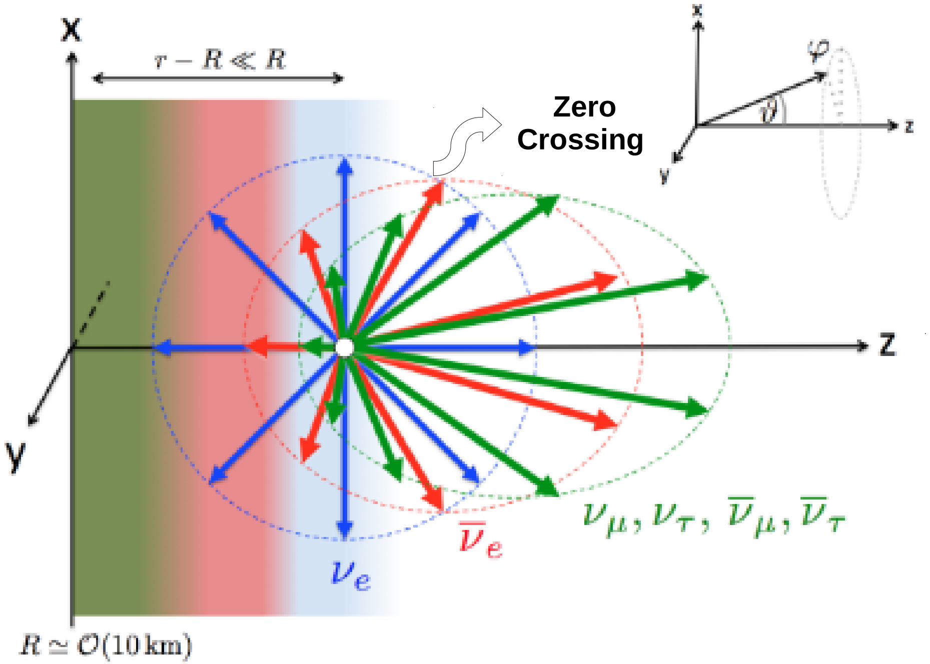
Fast Flavor Conversion : A Review

- **Independent of mass hierarchy :** E **plays no role**

- **Rapid conversion**

$$|\rho^{ex}| \propto e^{\mu t}$$

- **Requires "zero" crossing**

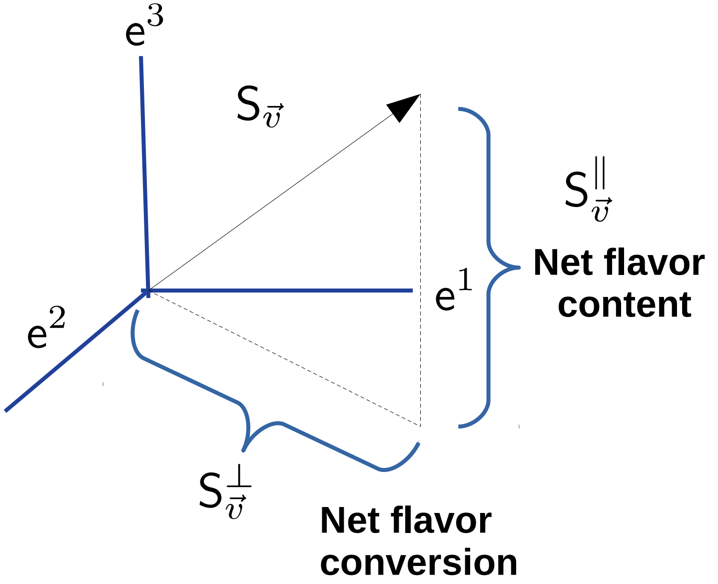


Fast Flavor Conversion : A Review

$$\rho_{\vec{p}} = \frac{Tr[\rho_{\vec{p}}]}{2}\mathbb{I}_{2\times 2} + \frac{g_{\omega,\vec{p}}}{2}\mathsf{S}\cdot\sigma$$

$$\Big\Downarrow \; \omega, \lambda \ll \mu$$

$$\left(\partial_t + \vec{v}.\vec{\nabla}\right)\mathsf{S}_{\vec{v}} = \mu \int d^3\vec{v}^{\,\prime}\, G_{\vec{v}^{\prime}} \left(1 - \vec{v}\cdot\vec{v}^{\,\prime}\right) \mathsf{S}_{\vec{v}^{\prime}} \times \mathsf{S}_{\vec{v}}$$



$$G_{\vec{v}} = \int d\omega\, g_{\omega,\vec{v}} \longrightarrow$$ **“Zero” Crossing in v**

Fast Flavor Conversion : A Review

$$\rho_{\vec{p}} = \frac{Tr[\rho_{\vec{p}}]}{2}\mathbb{I}_{2\times 2} + \frac{g_{\omega,\vec{p}}}{2}\mathsf{S}\cdot\sigma$$

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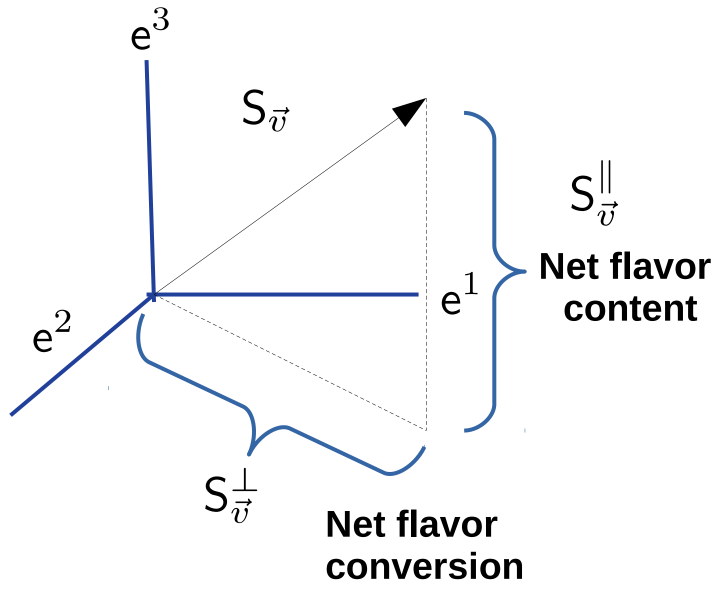
Coupled nonlinear P.D.E

Huge Phase space : 3 space + 3 momentum + 1 time



$$G_{\vec{v}} = \int d\omega\, g_{\omega,\vec{v}} \longrightarrow \text{“Zero” Crossing in v}$$

Coupled Nonlinear P.D.E : Previous attempts

Linear stability analysis

$\mathsf{S}^{\parallel} \approx 1$

Phys. Rev. Lett. 118 (2017) 021101

Phys. Rev. D 96 (2017)043016

$$i\left(\partial_t + \vec{v}.\vec{\nabla}\right)\mathsf{S}^{\perp}_{\vec{v}} = v^{\mu}\psi_{\mu}\mathsf{S}^{\perp}_{\vec{v}} - \int d^3\vec{v}^{'}\, v^{\mu}v^{'}_{\mu}G_{\vec{v}'}\,\mathsf{S}^{\perp}_{\vec{v}'}$$

$$\mathsf{S}_{\vec{v}}[t,\vec{r}] = \mathsf{Q}_{\vec{v}}[\omega,\vec{K}]e^{-i\Omega t + i\vec{K}\cdot\vec{r}}$$

$$det[\Pi^{\mu\nu}[\Omega,\vec{K}] = 0$$

Algebraic equation

$$\mathrm{Im}\,\Omega > 0$$

$$\mathsf{S}^{\perp}_{\vec{v}}[t,\vec{r}] \propto e^{\mathrm{Im}\,\Omega t}$$

Understood quite well !!

Coupled Nonlinear P.D.E : Previous attempts

Linear stability analysis
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$$i\left(\partial_t + \vec{v}.\vec{\nabla}\right)\mathsf{S}^{\perp}_{\vec{v}} = v^{\mu}\psi_{\mu}\mathsf{S}^{\perp}_{\vec{v}} - \int d^3\vec{v}^{\,\prime}\; v^{\mu}v^{\prime}_{\mu}G_{\vec{v}^{\prime}}\,\mathsf{S}^{\perp}_{\vec{v}^{\prime}}$$

$$\mathsf{S}_{\vec{v}}[t,\vec{r}] = \mathsf{Q}_{\vec{v}}[\omega,\vec{K}]e^{-i\Omega t + i\vec{K}\cdot\vec{r}}$$

$\rightarrow$ $det[\Pi^{\mu\nu}[\Omega,\vec{K}] = 0$

Algebraic equation

$$\mathrm{Im}\,\Omega > 0$$

$$\mathsf{S}^{\perp}_{\vec{v}}[t,\vec{r}] \propto e^{\mathrm{Im}\,\Omega t}$$

Understood quite well !!

Coupled Nonlinear P.D.E : Previous attempts

Nonlinear analysis

~~$S^{\parallel} \approx 1$~~

Phys. Rev. Lett. 118 (2017) 021101

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$$i(\partial_t + \vec{v}.\vec{\nabla})S^{\perp}_{\vec{v}} = v^{\mu}\psi_{\mu}S^{\perp}_{\vec{v}} - \int d^3\vec{v}^{\,\prime}\; v^{\mu}v^{\prime}_{\mu}G_{\vec{v}^{\prime}}S^{\perp}_{\vec{v}^{\prime}}$$

$$S_{\vec{v}}[t,\vec{r}] = Q_{\vec{v}}[\omega,\vec{K}]e^{-i\Omega t + i\vec{K}\cdot\vec{r}}$$

→ ~~$det[\Pi^{\mu\nu}[\Omega,\vec{K}] = 0$~~

~~**Algebraic equation**~~

$S^{\parallel}_{\vec{v}}|_{final}$??

$S^{\perp}_{\vec{v}}|_{final}$??

Not Understood quite well !!

Huge Phase Space : Previous attempts

Stationary

B.Dasgupta, A.Mirizzi and M.Sen
JCAP 2017 (2017) 019

$$(\cancel{\partial_t} + \vec{v}.\vec{\nabla})\mathsf{S}_{\vec{v}} = \mu \int d^3\vec{v}'\, G_{\vec{v}'} \left(1 - \vec{v}\cdot\vec{v}'\right) \mathsf{S}_{\vec{v}'} \times \mathsf{S}_{\vec{v}}$$

Lack of temporal information !!

Huge Phase Space : Previous attempts

B.Dasgupta, A.Mirizzi and M.Sen
JCAP 2017 (2017) 019

B.Dasgupta and M.Sen
Phys.Rev.D 97 (2018)023017

S.Chakraborty, R.S Hansen, I.Izaguirre and G.Raffelt
JCAP(2016) 042

Homogeneous

$$(\partial_t + \vec{v}.\vec{\nabla})\mathsf{S}_{\vec{v}} = \mu \int d^3\vec{v}^{\,\prime}\, G_{\vec{v}^{\,\prime}} \left(1 - \vec{v}\cdot\vec{v}^{\,\prime}\right) \mathsf{S}_{\vec{v}^{\,\prime}} \times \mathsf{S}_{\vec{v}}$$

Lack of Spatial information !!

Huge Phase Space : Previous attempts

B.Dasgupta, A.Mirizzi and M.Sen
Phys. Rev. D 98 (2018) 103001

1+1+1 D

$$\left(\underline{\partial_t} + \underline{\vec{v}.\vec{\nabla}}\right)\mathsf{S}_{\vec{v}} = \mu \int \underline{d^3\vec{v}'}\, G_{\vec{v}'} \left(1 - \vec{v}\cdot\vec{v}'\right) \mathsf{S}_{\vec{v}'} \times \mathsf{S}_{\vec{v}}$$

1 1 1

Magnitude of neutrino velocity not fixed to unity !!

Our Motivation

Our Motivation

Nonlinear Analysis

- Theory of FFC in nonlinear regime
- Developing a numerical code
- Final neutrino signal

Phys.Rev.D 102 (2020) 6, 063018
Phys.Rev.Lett. 126 (2021) 6, 061302
+ 1 in preparation

Phase space (Linear analysis)

- 2 space + 2 vel + 1 time
- Dependence on angular distribution
- Dispersion relation solver

JCAP 07 (2021) 023

Our Motivation

Nonlinear Analysis

- Theory of FFC in nonlinear regime
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Phys.Rev.D 102 (2020) 6, 063018
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Phase space (Linear analysis)

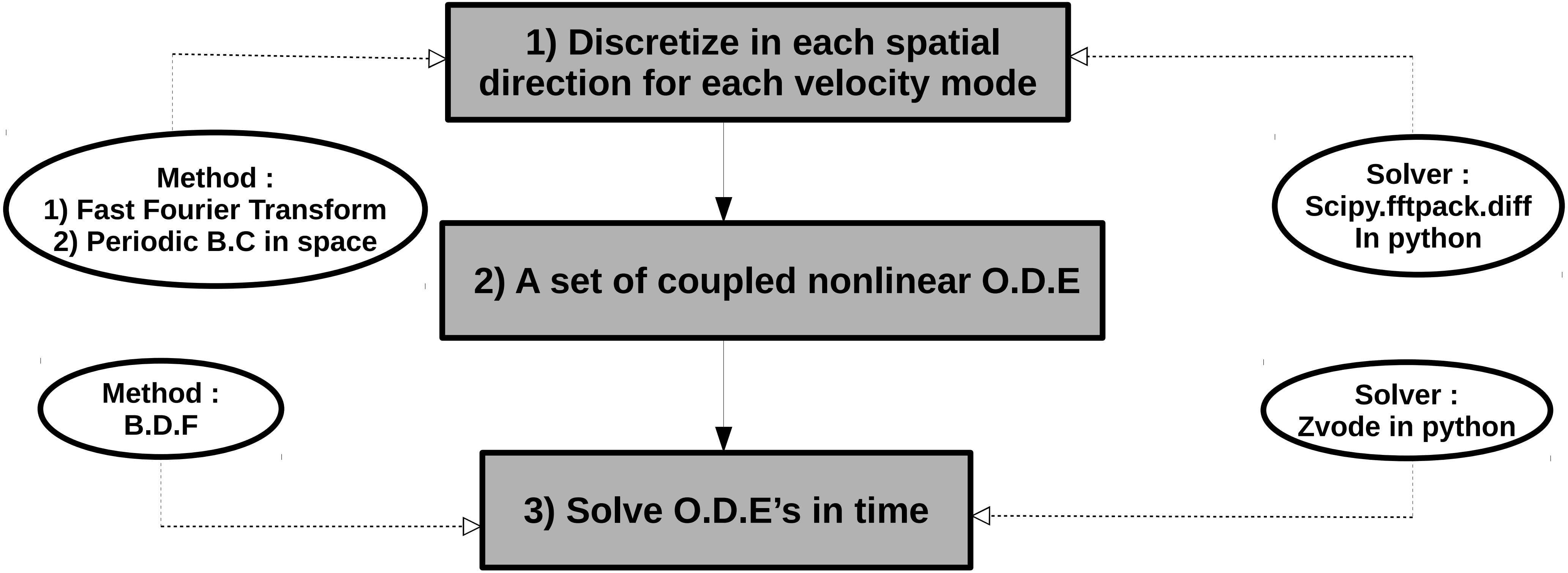
$x - y$ $\quad v_x^2 + v_y^2 = 1$

- 2 space + 2 vel + 1 time (1 vel)
- Dependence on angular distribution
- Dispersion relation solver

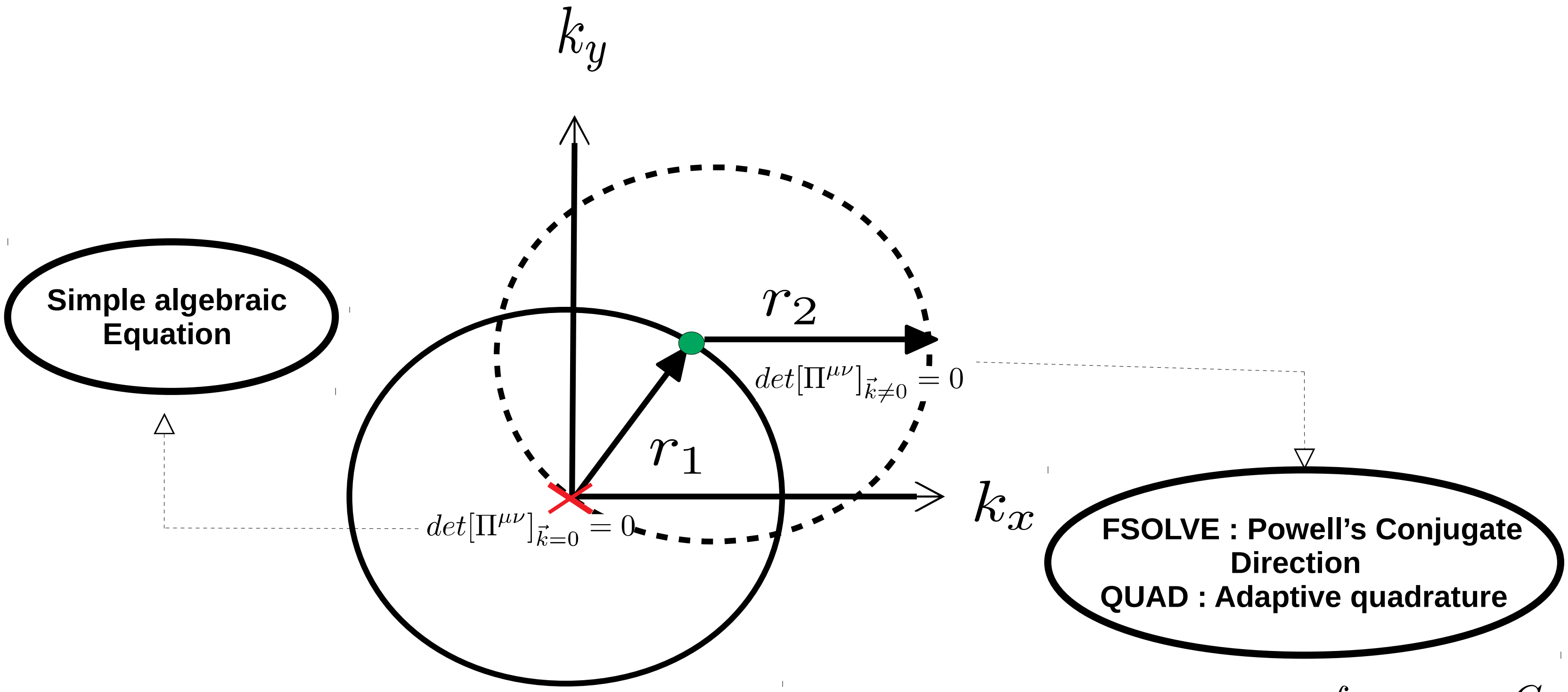
JCAP 07 (2021) 023

Numerical Techniques

P.D.E Solver


1) Discretize in each spatial direction for each velocity mode
Method :
1) Fast Fourier Transform
2) Periodic B.C in space
Solver :
Scipy.fftpack.diff
In python
2) A set of coupled nonlinear O.D.E
Method :
B.D.F
Solver :
Zvode in python
3) Solve O.D.E’s in time

Dispersion relation solver



$$\Pi^{\mu\nu}[\vec{k}, \omega] = \eta^{\mu\nu} + \int d^3\vec{v}\, v^\mu v^\nu \frac{G_{\vec{v}}}{k^l v_l}$$

Results : Nonlinear Analysis

Nonlinear Analysis : Toy Model

Phase Space : 1 space + 1 vel + 1 time dimension

$$(\partial_t + v\partial_x)\mathsf{S}_v = \mu \int_{-1}^{1} dv' G_{v'} \left(1 - vv'\right) \mathsf{S}_{v'} \times \mathsf{S}_v$$

Neutrino angular distribution :

$$\int_{-1}^{1} G_v dv = A$$
$$0 < A < 1$$
$$-1 < v_c < 1$$

Linear

$$G'_v = 6v, \quad A = 0$$

$$G^L_v[v_c] = \begin{cases} 2\,(v - v_c)\,, & \text{if } v > v_c\,, \\ 2\frac{(1-v_c)^2 - A}{(1+v_c)^2}\,(v - v_c)\,, & \text{if } v < v_c\,, \end{cases}$$

Piecewise constant

$$G^B_v = \begin{cases} 1, & \text{if } v > 0\,, \\ A - 1, & \text{if } v < 0\,, \end{cases}$$

Cubic

$$G''_v = 12\,(v - 0.2)^3 \quad A < 0$$
$$G'''_v = 12\,(v + 0.2)^3 \quad A > 0$$

Initial Condition :

$$\mathsf{S}^{\parallel}_v[x, 0] = 1 \forall v$$
$$\mathsf{S}^{\perp}_v[x, 0] = O(10^{-6})$$

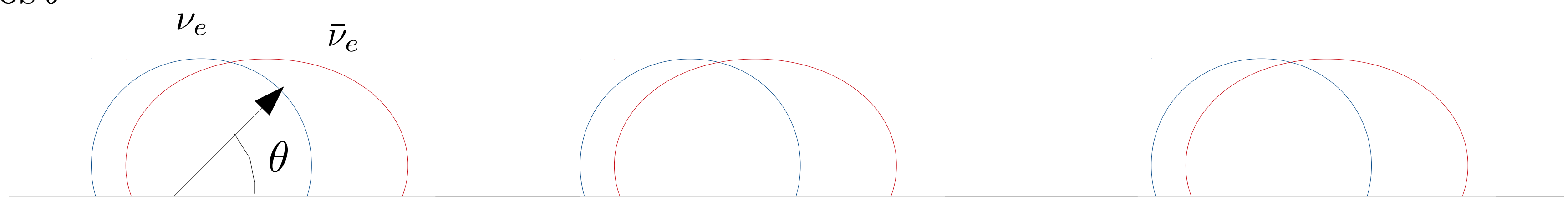
P.D.E Solver → $t = 50$

Extreme nonlinearity

Nonlinear Analysis : Toy Model

Phase Space : 1 space + 1 vel + 1 time dimension

$v = \cos\theta$



1 dimensional box of lenght L with periodic boundary conditions

Neutrinos emitted from each point in the box with velocity $v = \cos\theta$

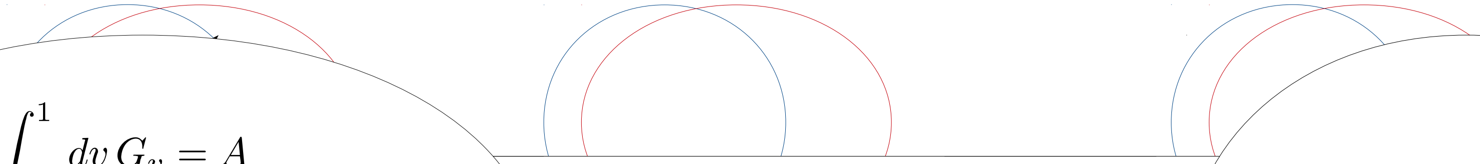
ELN distribution : $G_v = \sqrt{2} G_F \int_0^\infty \frac{dE E^2}{2\pi^2} [f_{\nu_e}(E, v) - f_{\bar{\nu}_e}(E, v)]$

Solve this equation : $(\partial_t + v\partial_x)\mathsf{S}_v = \mu \int_{-1}^{1} dv' G_{v'} \left(1 - vv'\right) \mathsf{S}_{v'} \times \mathsf{S}_v$

Nonlinear Analysis : Toy Model

Phase Space : 1 space + 1 vel + 1 time dimension

$$v = \cos\theta$$



$$\int_{-1}^{1} dv\, G_v = A$$

G_v **as a function of v crossing** v_c

Vary v_c A

lenght L with periodic boundary co

rom each point in the box with velo

$$= \sqrt{2}G_F \int_0^\infty \frac{dE E^2}{2\pi^2}[f_{\nu_e}(E,v) - f_{\bar{\nu}_e}(E,v)]$$

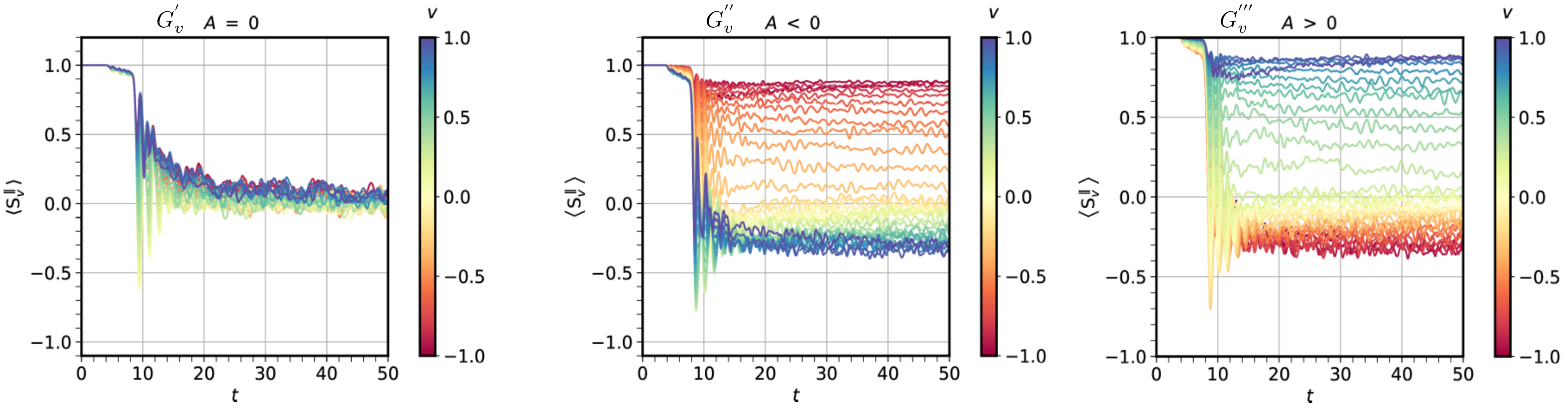
External perturbation

$\mathsf{S}^{\perp}$

Solve this equation : $(\partial_t + v\partial_z)\mathsf{S}_v = \mu \int_{-1}^{1} dv' G_{v'} \left(1 - vv'\right) \mathsf{S}_{v'} \times \mathsf{S}_v$

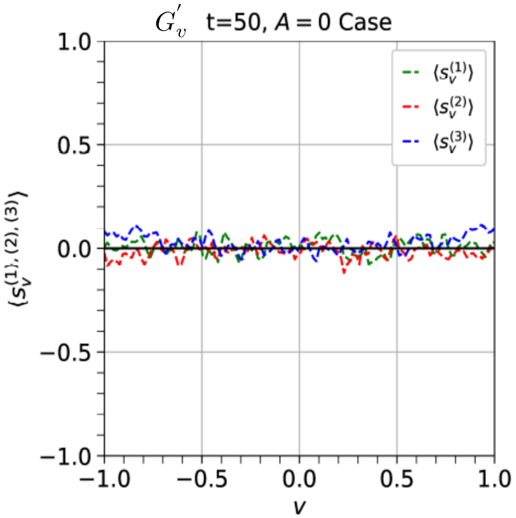
Nonlinear Analysis : Nature of nonlinear solution

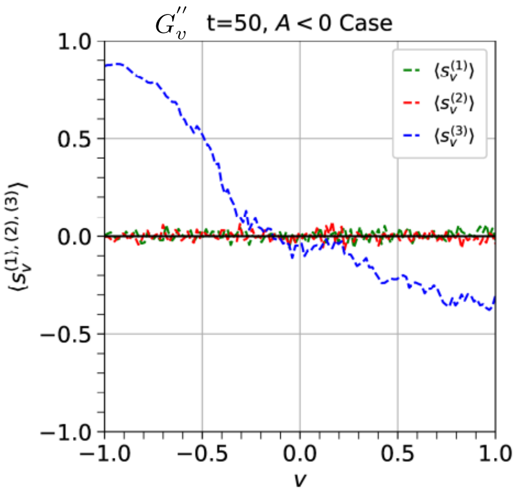
- **System shows irreversibility (steady state) in time**
- **The bloch vector shows tendency to flip over**

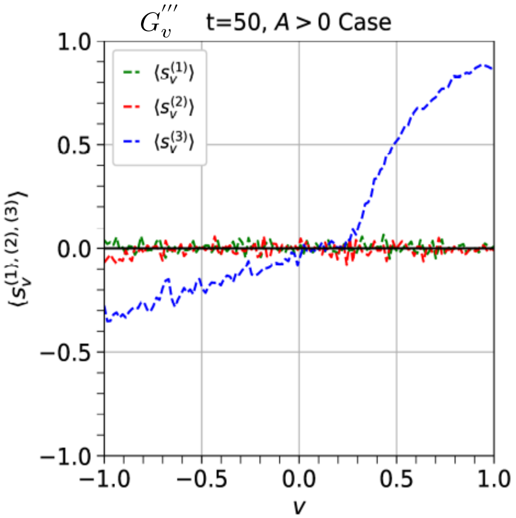


Nonlinear Analysis : Nature of nonlinear solution

- **Length of the bloch vector shrinks when spatially averaged**
- **Final solution shows flavor depolarization depending on the nature of v and A**
- **A = 0 ⟶ Full flavor depolarization for all v** ⟶ $\mathsf{S}_v = 0 \, \forall v$
- **A $\neq$ 0 ⟶ Partial flavor depolarization and the range of fully flavor depolarized modes depend on size and sign of A**

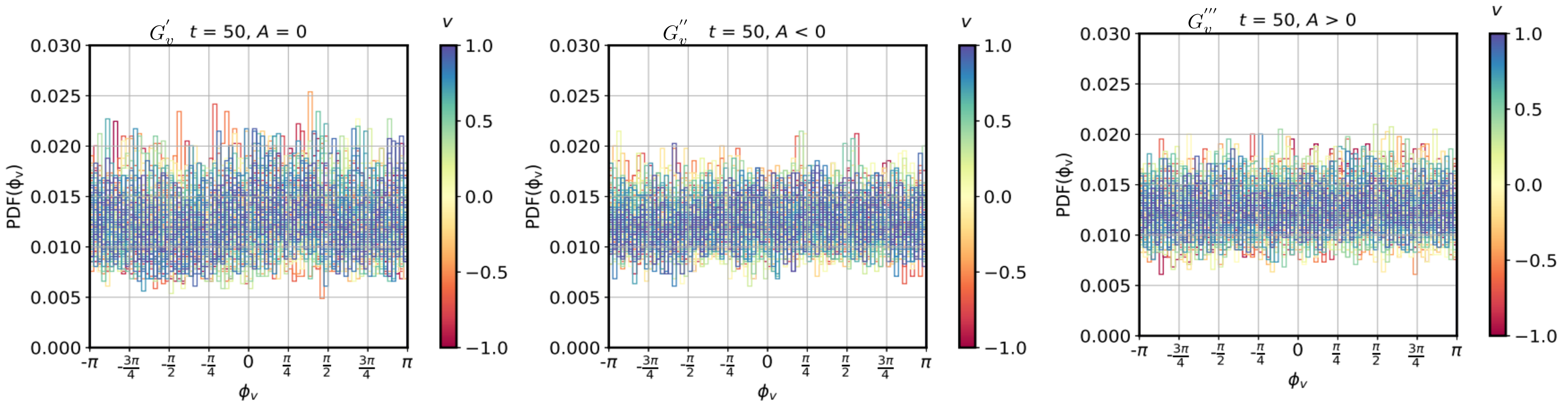






Nonlinear Analysis : Nature of nonlinear solution

- **The phase relationship between the transverse components of the bloch vector becomes randomized all over the space independent of the nature of v and A at late times**



Nonlinear Analysis : Theory of extreme nonlinear behaviour

Main physics aspects governing late-time nonlinear behaviour :

- ***Multipole cascade***
- ***Irreversible pendulum motion (for low n multipoles)***
- ***Transverse relaxation***
- ***Spatial coarse-graining***

Nonlinear Analysis : Theory of extreme nonlinear behaviour

Multipole Diffusion :

Georg G. Raffelt and Günter Sigl
Phys. Rev. D 75, 083002

Bhattacharyya and Dasgupta (2020)
Phys.Rev.Lett. 126 (2021) 6, 061302

Set of coupled nonlinear P.D.E

$$\mathsf{M}_n = \int_{-1}^{1} dv\, G_v L_n[v] \mathsf{S}_v$$

a) Multipole space
b) Spatial coarse-graining
c) Periodic bondary condition
d) $2n+1 \gg 1$

$$\partial_t \langle M_n \rangle = \frac{\langle M_1 \rangle}{2} \left(\partial_n^2 \langle M_n \rangle + \frac{1}{n} \partial_n \langle M_n \rangle \right)$$

$n \to an$

$t \to a^2 t$

$$\eta = \frac{n^2}{t}$$

$$\langle M_n[t] \rangle \sim \langle M[\frac{n^2}{t}] \rangle$$

$$2 d_\eta^2 \langle M_n \rangle + \left(\frac{1}{\langle M_1 \rangle} + \frac{2}{\eta} \right) d_\eta \langle M_n \rangle = 0$$

Nonlinear Analysis : Theory of extreme nonlinear behaviour

Multipole Diffusion :

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Bhattacharyya, Dasgupta (2020)
Phys.Rev.Lett. 126 (2021) 6, 061302

Set of coupled nonlinear P.D.E

a) Multipole space
b) Spatial coarse-graining
c) Periodic bondary condition
d) $2n+1 \gg 1$

$$\partial_t \langle M_n \rangle = \frac{\langle M_1 \rangle}{2} \left(\partial_n^2 \langle M_n \rangle + \frac{1}{n} \partial_n \langle M_n \rangle \right)$$

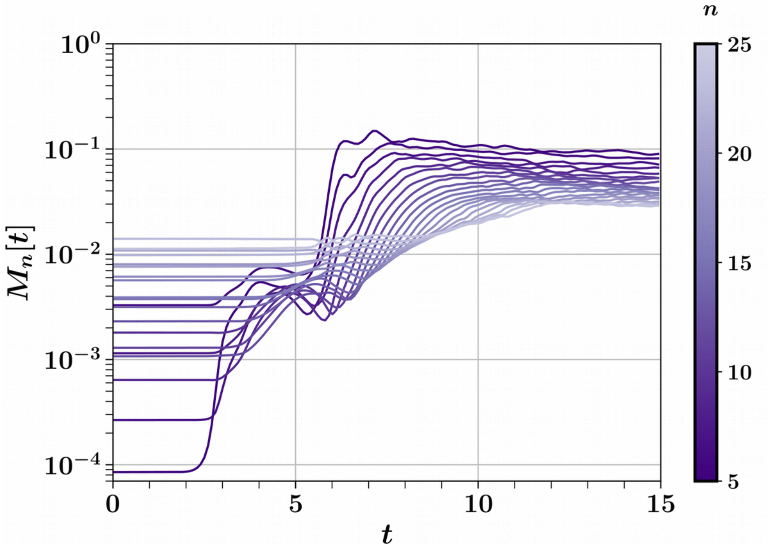
Single decoupled O.D.E

$$2 d_\eta^2 \langle M_n \rangle + \left(\frac{1}{\langle M_1 \rangle} + \frac{2}{\eta} \right) d_\eta \langle M_n \rangle = 0$$

Diffusion from low to high n multipoles

Irreversibility !!

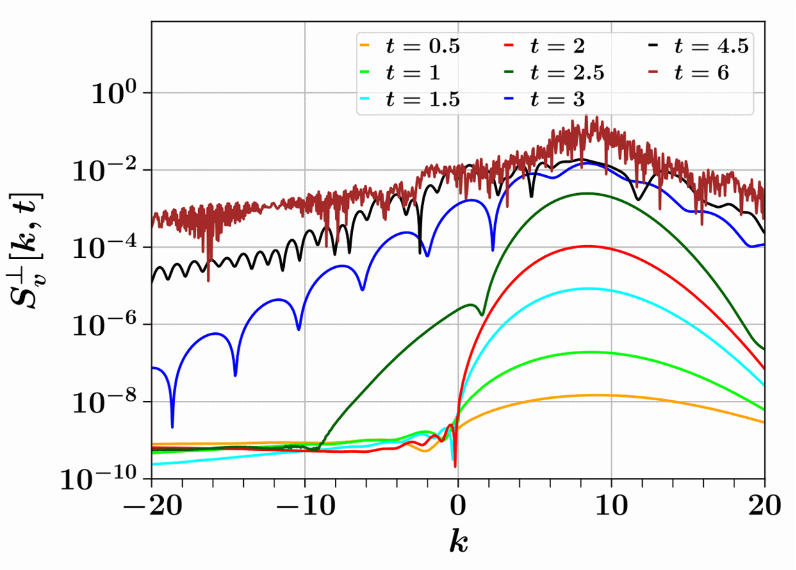
$$\langle M_n[t] \rangle = c_1 \, \mathrm{Ei}\left[-n^2 / \left(2 \langle M_1 \rangle t\right) \right] + c_2$$



Nonlinear Analysis : Theory of extreme nonlinear behaviour

Breaking of flavor waves:

$$(\partial_t + ivk)\, \mathsf{S}_v^{\perp}[k,t] = i\mu_0 \int_{-\infty}^{+\infty} dk' \int_{-1}^{+1} dv' G_{v'}\, (1 - vv')$$
$$\left(-\mathsf{S}_v^{\perp}[k', t]\mathsf{S}_{v'}^{\parallel}[k-k', t] + \mathsf{S}_v^{\parallel}[k', t]\mathsf{S}_{v'}^{\perp}[k-k', t]\right) .$$



Nonlinear Analysis : Theory of extreme nonlinear behaviour

Irreversible Pendulum Dynamics

$$d_t \mathsf{S}_v = \mathsf{H}_v \times \mathsf{S}_v = \left(-\frac{A}{3} - v\mathsf{M}_1\right) \times \mathsf{S}_v$$

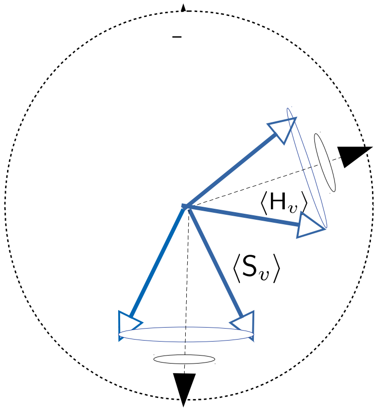
$$\mathsf{M}_1 \times d_t^2 \mathsf{M}_1 + \left(\mathsf{D}.\mathsf{M}_1\right) d_t \mathsf{M}_1 = |\mathsf{M}_1|^2 \mathsf{B} \times \mathsf{M}_1$$

$$d_t \mathsf{B} = -\frac{3A}{35}\,\hat{\mathsf{e}}_3 \times \mathsf{M}_1$$

$$\mathsf{D} = \frac{\mathsf{M}_0}{3} + \frac{2\mathsf{M}_2}{3}$$

$$\mathsf{B} = \frac{2\mathsf{M}_3}{5} - \frac{9\mathsf{M}_1}{35}$$

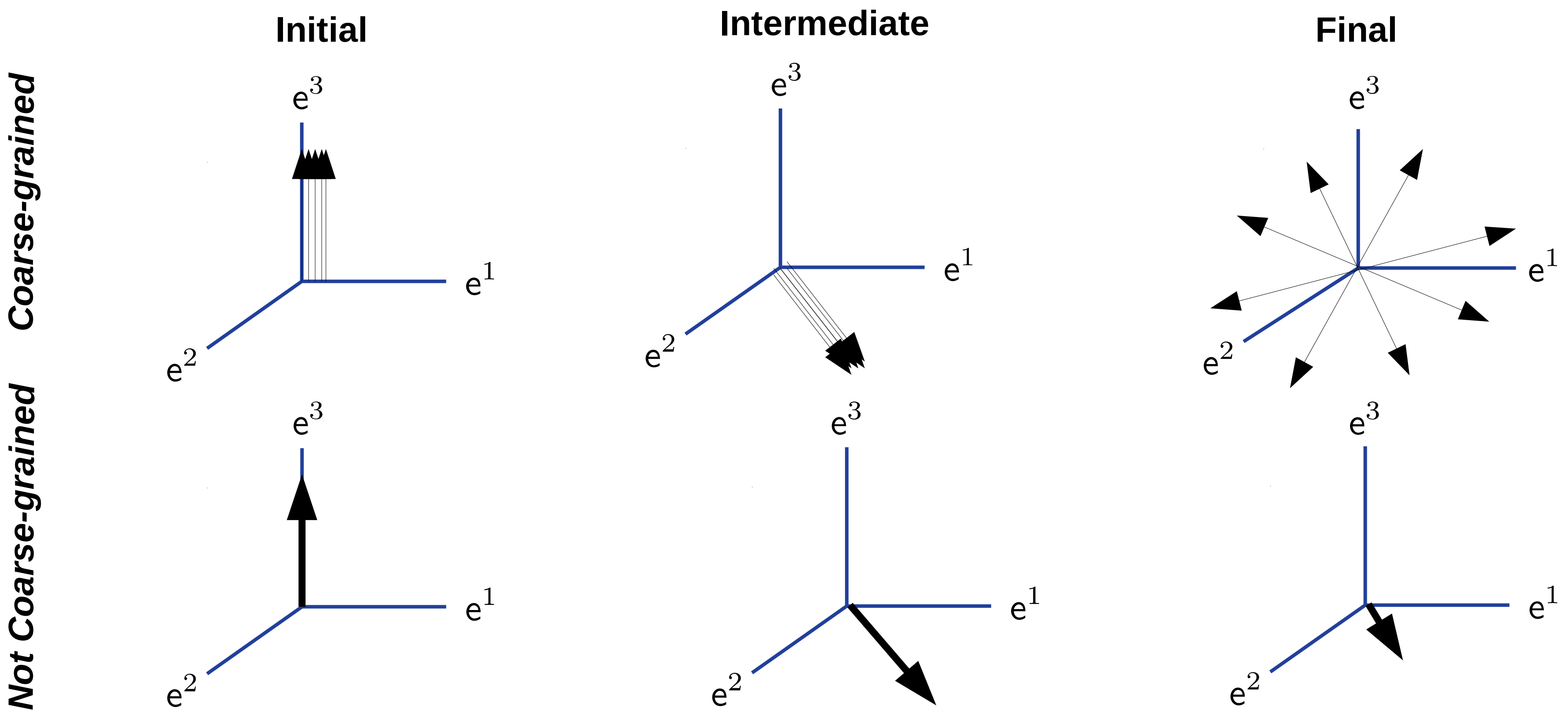
“Coarse-grained” Picture



Nonlinear Analysis : Theory of extreme nonlinear behaviour

T2 relaxation in pictures :

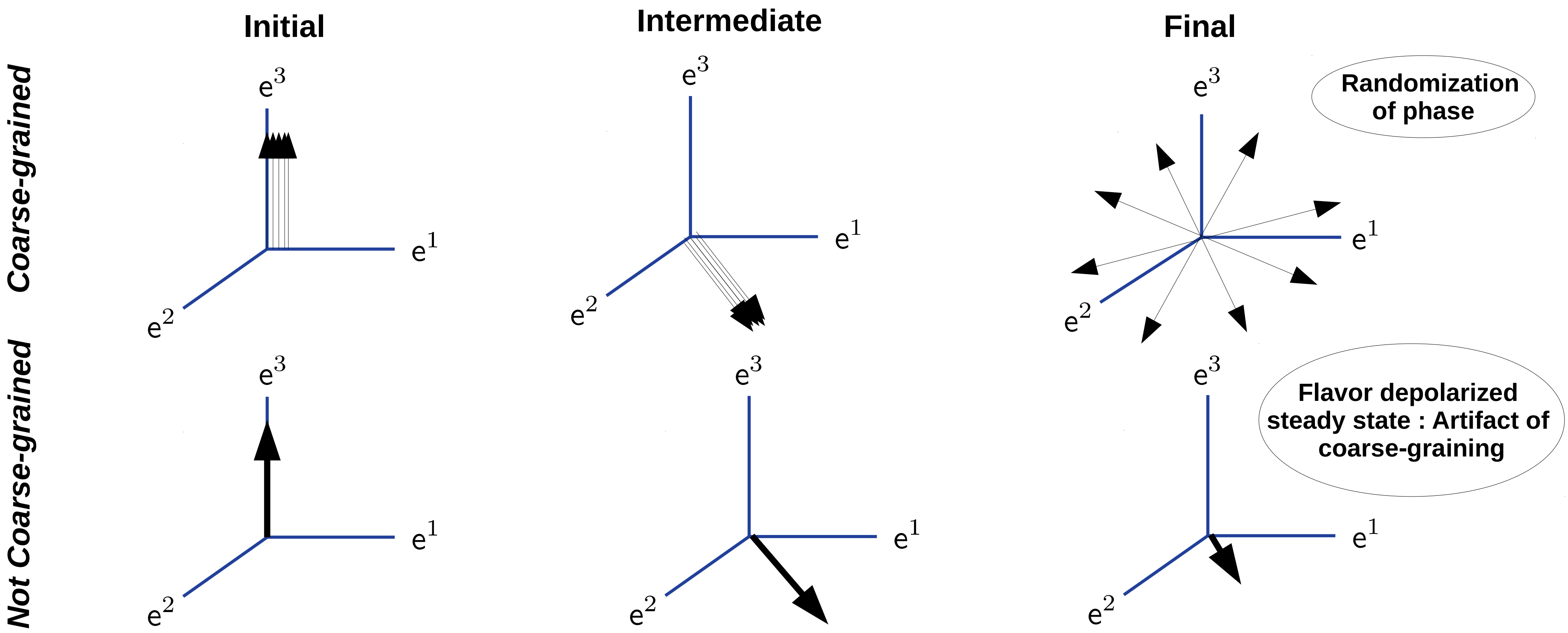
Image courtesy : B.Dasgupta



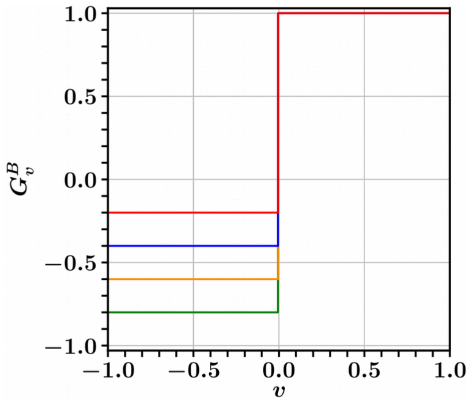
Nonlinear Analysis : Theory of extreme nonlinear behaviour

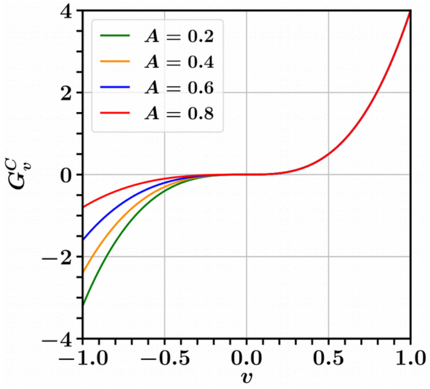
T2 relaxation in pictures :

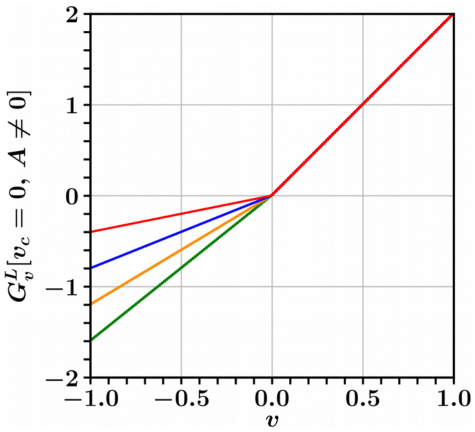
Image courtesy : B.Dasgupta

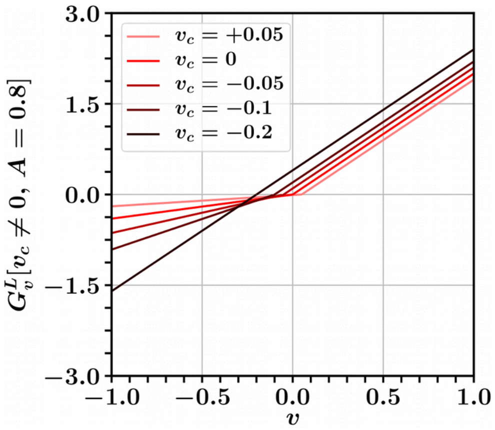


Nonlinear Analysis : Theory of extreme nonlinear behaviour



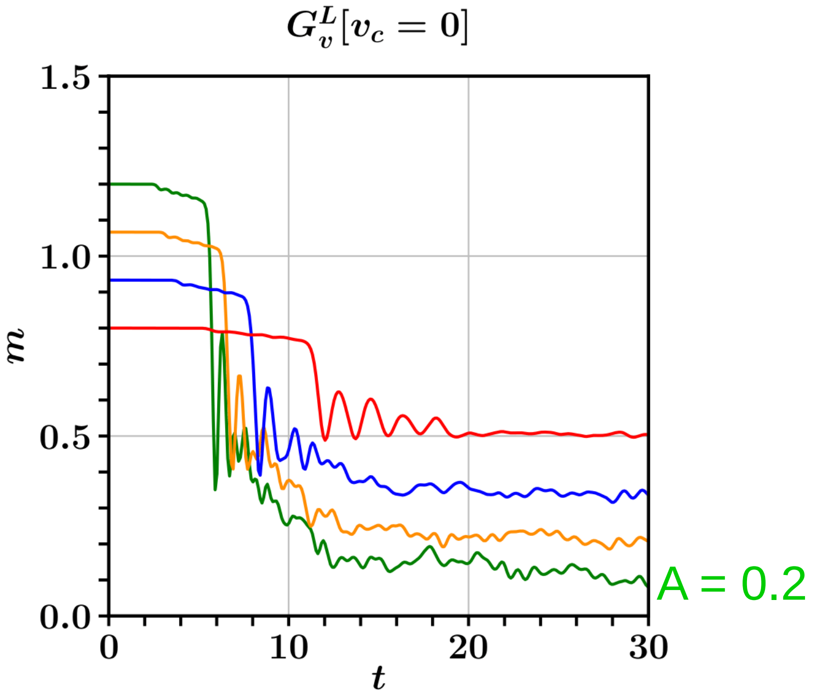


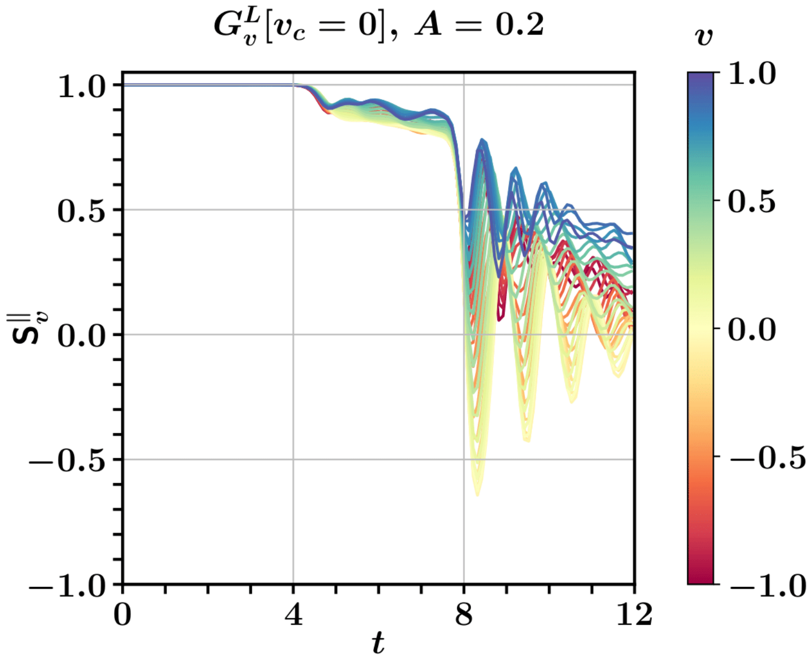




Nonlinear Analysis : Theory of extreme nonlinear behaviour

Flavor depolarized steady state !!

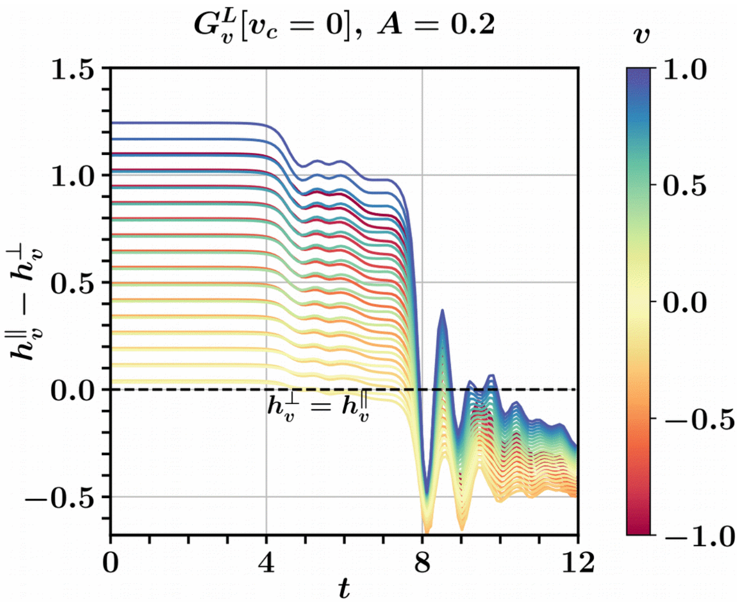


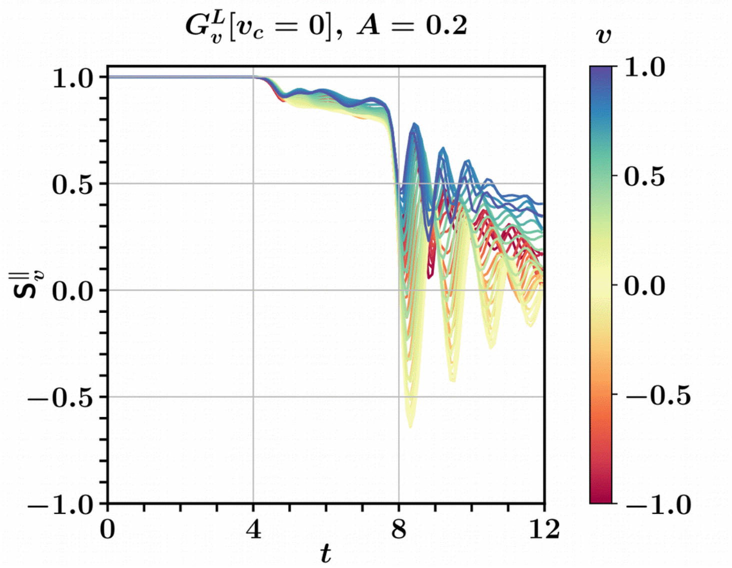


Nonlinear Analysis : Theory of extreme nonlinear behaviour

$$h_v^{\parallel} = \frac{1}{L}\int_0^L dz\, |-A/3 - 2/3\,\mathsf{M}_2^{\parallel} - v\mathsf{M}_1^{\parallel}| \qquad h_v^{\perp} = \frac{1}{L}\int_0^L dz\, |\mathsf{M}_0^{\perp} - v\mathsf{M}_1^{\perp}|$$

$$h_v^{\perp} \geq h_v^{\parallel}$$



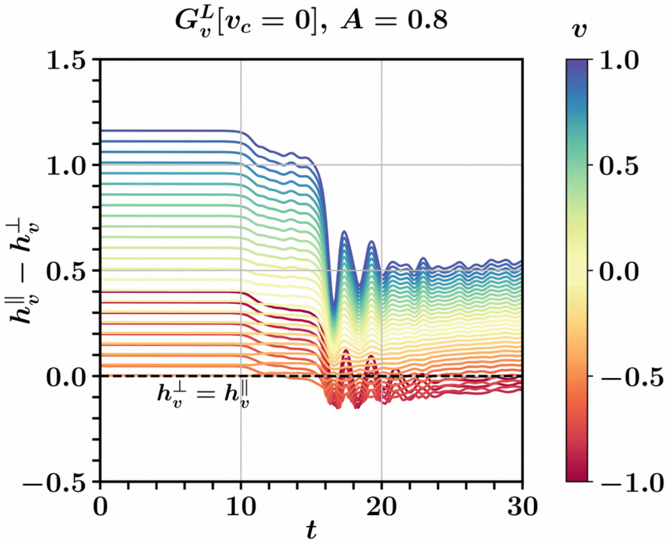


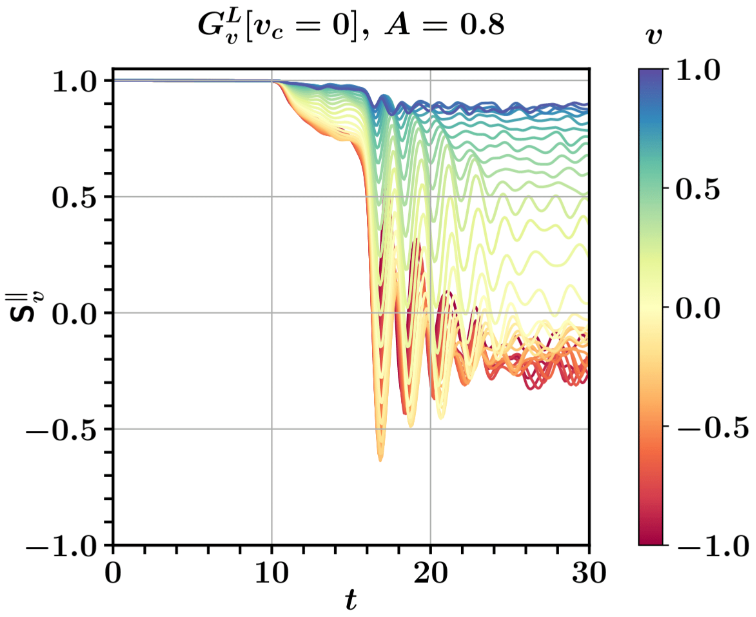
Nonlinear Analysis : Theory of extreme nonlinear behaviour

$$h_v^{\parallel} = \frac{1}{L}\int_0^L dz\, | - A/3 - 2/3\,\mathsf{M}_2^{\parallel} - v\mathsf{M}_1^{\parallel}|$$

$$h_v^{\perp} = \frac{1}{L}\int_0^L dz\, |\mathsf{M}_0^{\perp} - v\mathsf{M}_1^{\perp}|$$

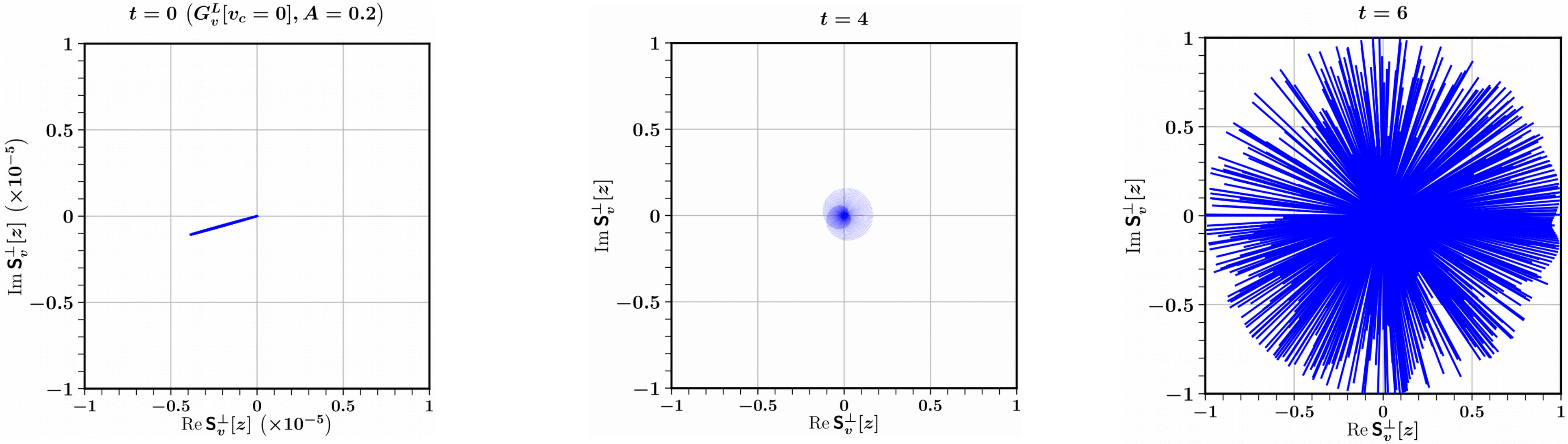
$$h_v^{\perp} \geq h_v^{\parallel}$$





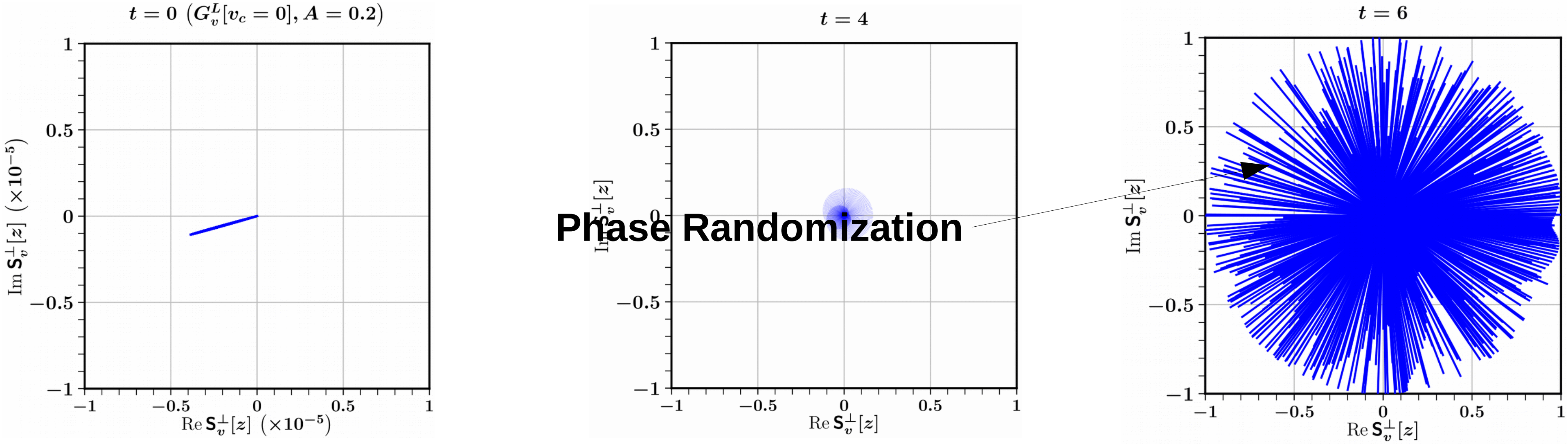
Nonlinear Analysis : Theory of extreme nonlinear behaviour

T2 relaxation in reality :



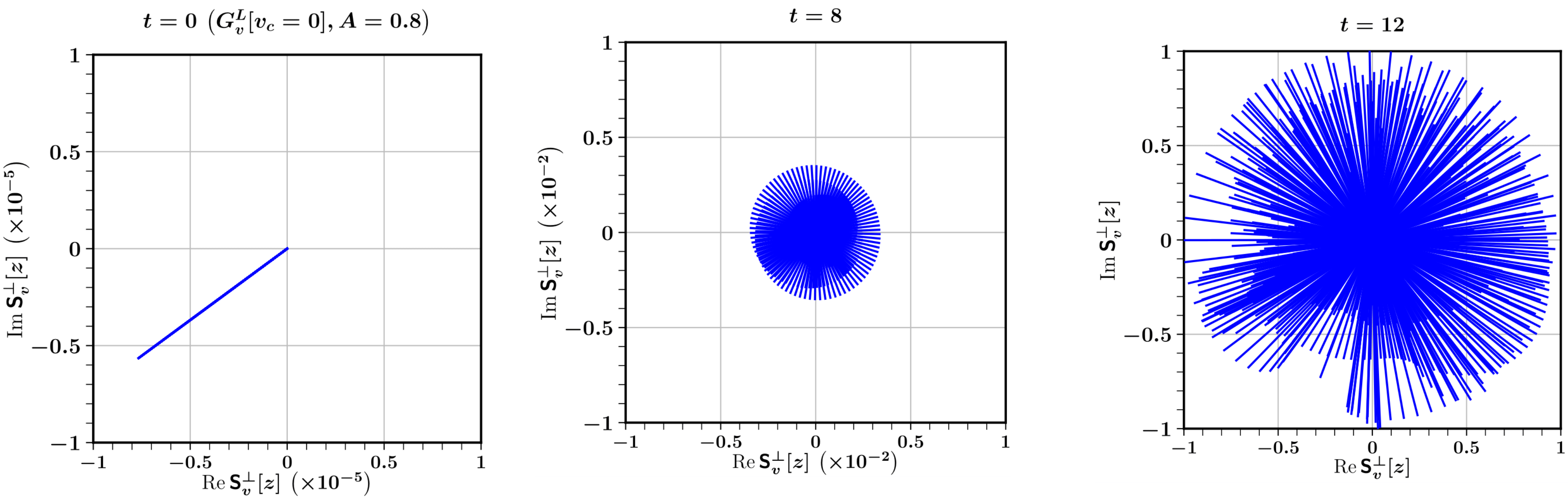
Nonlinear Analysis : Theory of extreme nonlinear behaviour

T2 relaxation in reality :



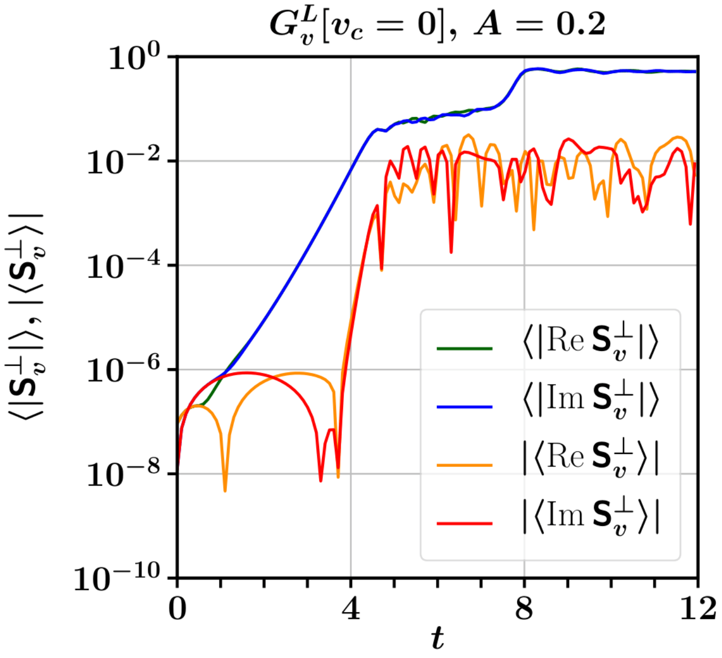
Nonlinear Analysis : Theory of extreme nonlinear behaviour

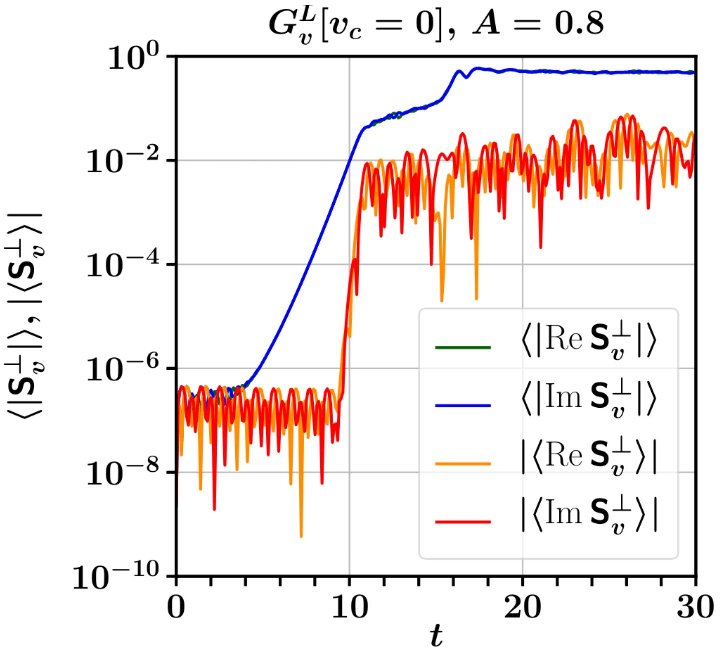
T2 relaxation in reality :



Nonlinear Analysis : Theory of extreme nonlinear behaviour

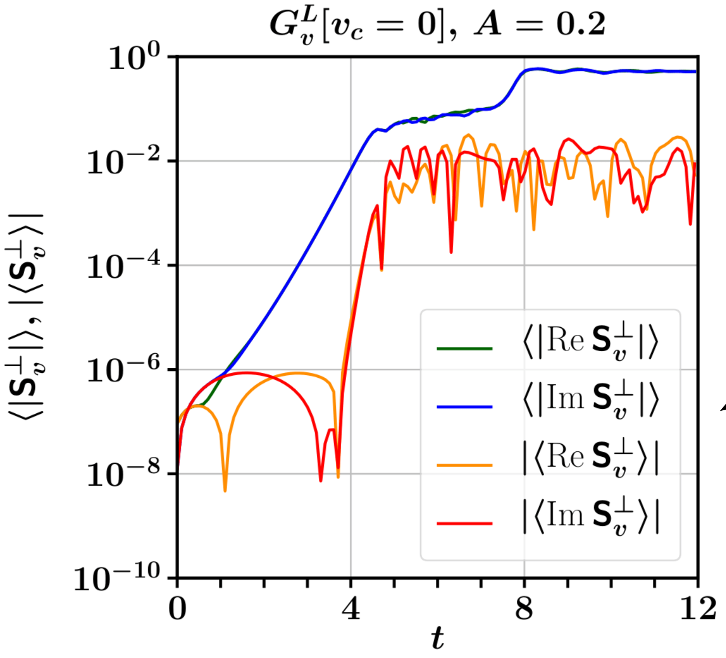
T2 relaxation in reality :



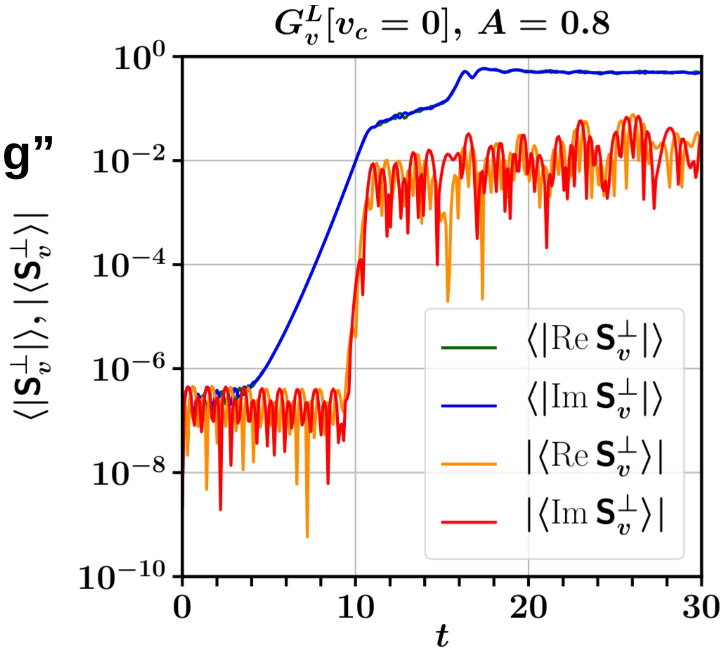


Nonlinear Analysis : Theory of extreme nonlinear behaviour

T2 relaxation in reality :

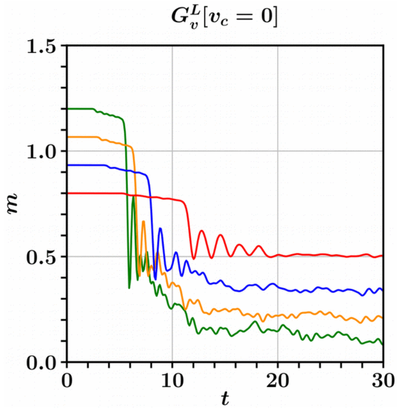


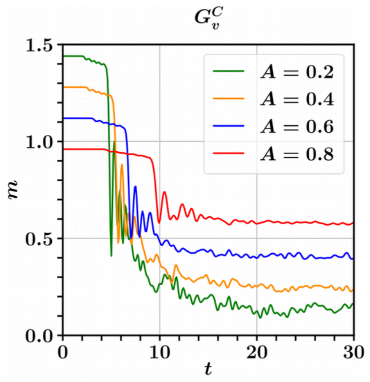
"Coarse-graining" Important

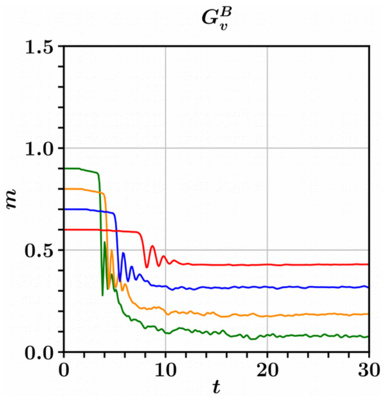


Nonlinear Analysis : Theory of extreme nonlinear behaviour

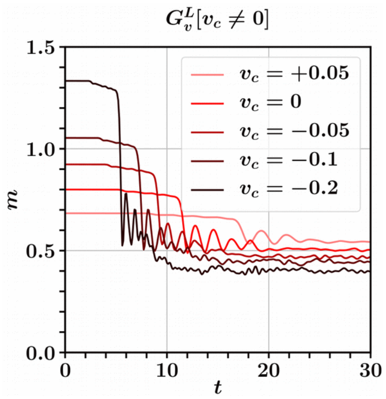
final state :







Final state depends on A and zero crossing



Nonlinear Analysis : Theory of extreme nonlinear behaviour

Flavor depolarized final state :

Pendulum dynamics + T2 relaxation + Coarse-graining

Conserved qty

$$E = \frac{\langle \mathsf{D} \rangle \cdot \langle \mathsf{D} \rangle}{2} + \langle \mathsf{M}_1 \rangle \cdot \langle \mathsf{B} \rangle = \text{const}$$

$$\sigma = \langle \mathsf{M}_1 \rangle \cdot \langle \mathsf{D} \rangle = \text{const}$$

$$\frac{\langle \mathsf{B} \rangle \cdot \langle \mathsf{B} \rangle}{2} + \langle \mathsf{K} \rangle \cdot \langle \mathsf{D} \rangle = \text{const}$$

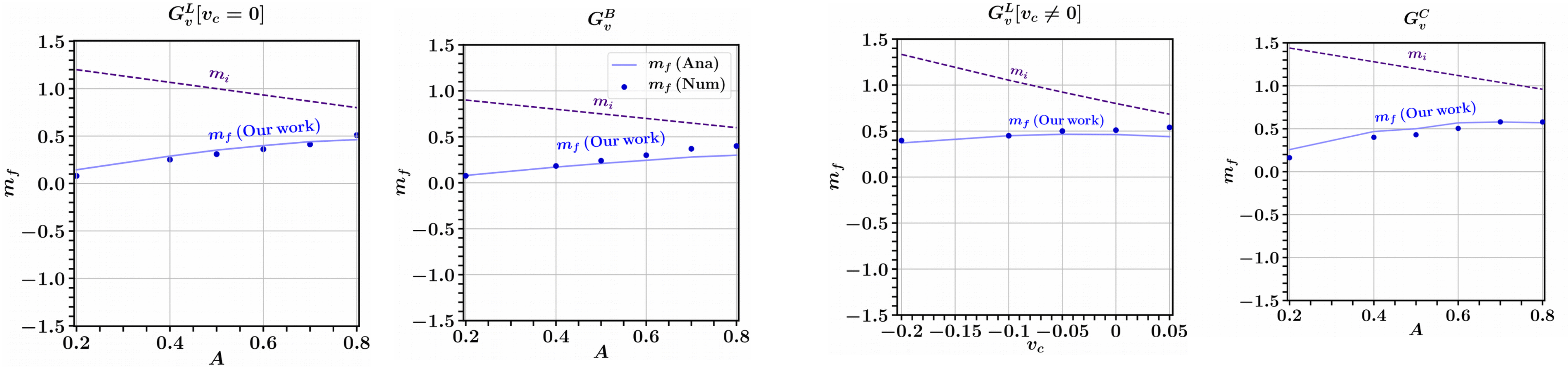
$$\langle \mathsf{B} \rangle \cdot \langle \mathsf{K} \rangle = \text{const}$$

$$m_f = \frac{-bu_i^2 \pm \sqrt{b^2 u_i^4 - 4\left(2b^2 - 4ku_i\right) bu_i^2 m_i}}{4b^2 - 8ku_i}$$

Nonlinear Analysis : Theory of extreme nonlinear behaviour

<u>Flavor depolarized final state :</u>

m_f **agress quite well with the numerics !!**



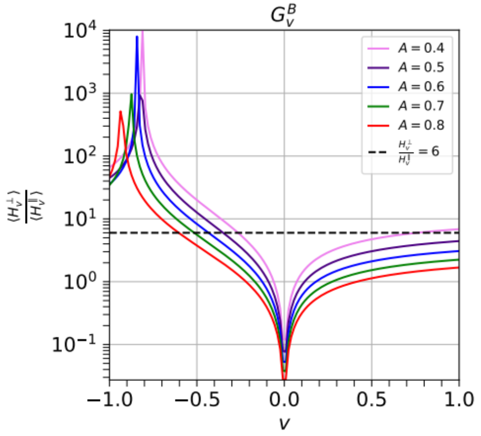
Nonlinear Analysis : Theory of extreme nonlinear behaviour

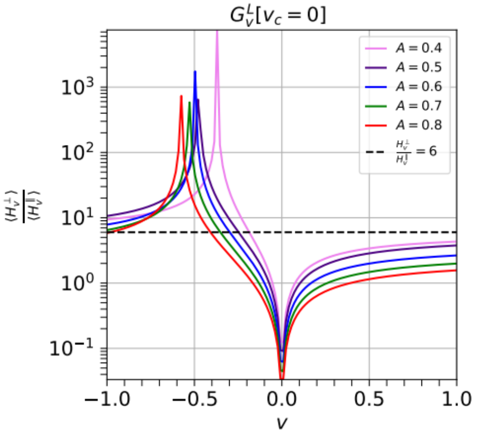
Flavor depolarized final state :

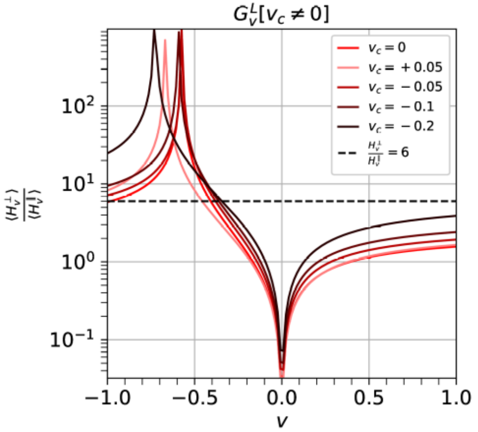
$$\frac{\langle H_v^{\perp}[t_f]\rangle}{\langle H_v^{\parallel}[t_f]\rangle} \approx \frac{v\sqrt{\dfrac{b(m_i-m_f)-\frac{1}{2}\left(\frac{u_i^2 m_i^2}{m_f^2}-u_i^2\right)}{2k\left(u_i-\frac{u_i m_i}{m_f}\right)}}}{\left|-\frac{A}{3}-v\,m_f\right|}$$

$\langle H_v^{\perp}[t_f]\rangle \gtrsim O(\langle H_v^{\parallel}[t_f]\rangle)$ → $[v_{min}, v_{max}]$

Range of fully flavor depolarized modes







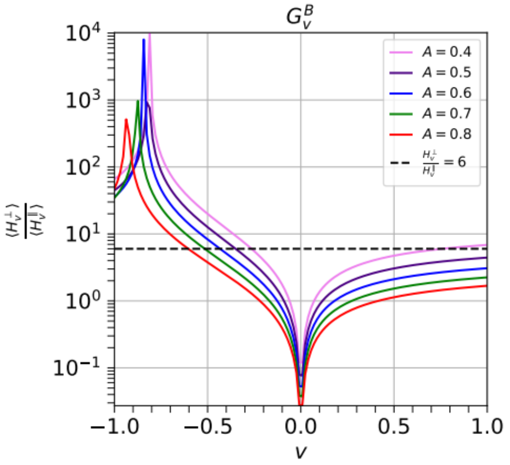
Nonlinear Analysis : Theory of extreme nonlinear behaviour

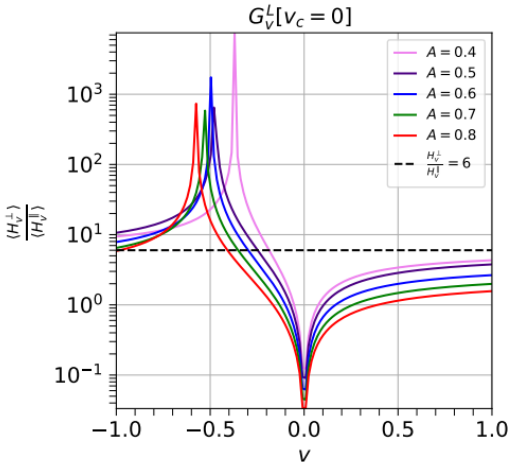
Flavor depolarized final state :

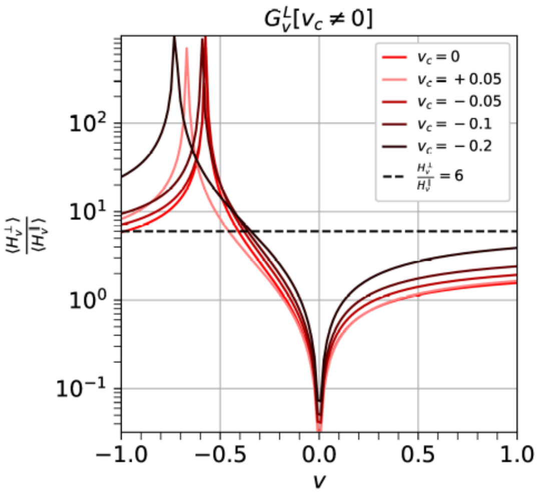
$$\frac{\langle H_v^{\perp}[t_f]\rangle}{\langle H_v^{\parallel}[t_f]\rangle} \gg 1 \xrightarrow{\;v<0\;} \left.\begin{array}{c} v_{min} \sim -1 \\ v_{max} \sim 0 \end{array}\right\}$$

$v \in [v_{min}, v_{max}]$ **: Full Depolarization**

$v \notin [v_{min}, v_{max}]$ **: Partial Depolarization** $\sim A\downarrow, v_c\downarrow$







Nonlinear Analysis : Theory of extreme nonlinear behaviour

Flavor depolarized final state :

Amount of flavor depolarization : $f_v^{\mathrm{D}} = \frac{1}{2}\left(1 - \langle \mathsf{S}_v^{\|} \rangle^{\mathrm{fin}} / \langle \mathsf{S}_v^{\|} \rangle^{\mathrm{ini}}\right)$

Full flavor depolarization : $f_v^{\mathrm{D}} = 0.5$ **Partial flavor depolarization :** $0 < f_v^{\mathrm{D}} < 0.5$ **No flavor depolarization :** $f_v^{\mathrm{D}} = 0$

$$\langle \mathsf{S}_v^{\|}[t_f] \rangle = 0 \quad \forall v < 0$$

$$\langle \mathsf{S}_v^{\|}[t_f] \rangle \approx \frac{\mathsf{S}_0}{2} + \frac{3\mathsf{S}_1}{2} v \quad \forall v > 0$$

$$\gamma_n = \int_{v_{max}}^{1} v^n G_v \, dv$$

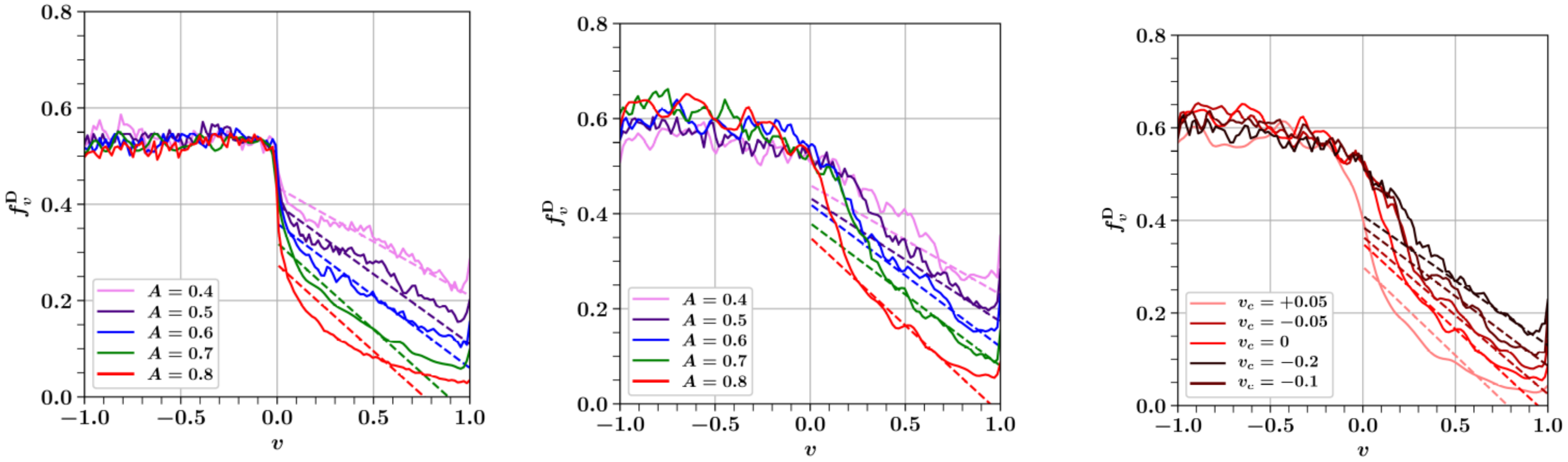
→ **Lepton number conservation**

$$f_v^{\mathrm{D}} \approx \begin{cases} \frac{1}{2} - \frac{A}{4\gamma_0} - -\frac{3A}{8\gamma_0}, & \text{if v>0}\,, \\ \frac{1}{2}, & \text{if } v < 0\,, \end{cases}$$

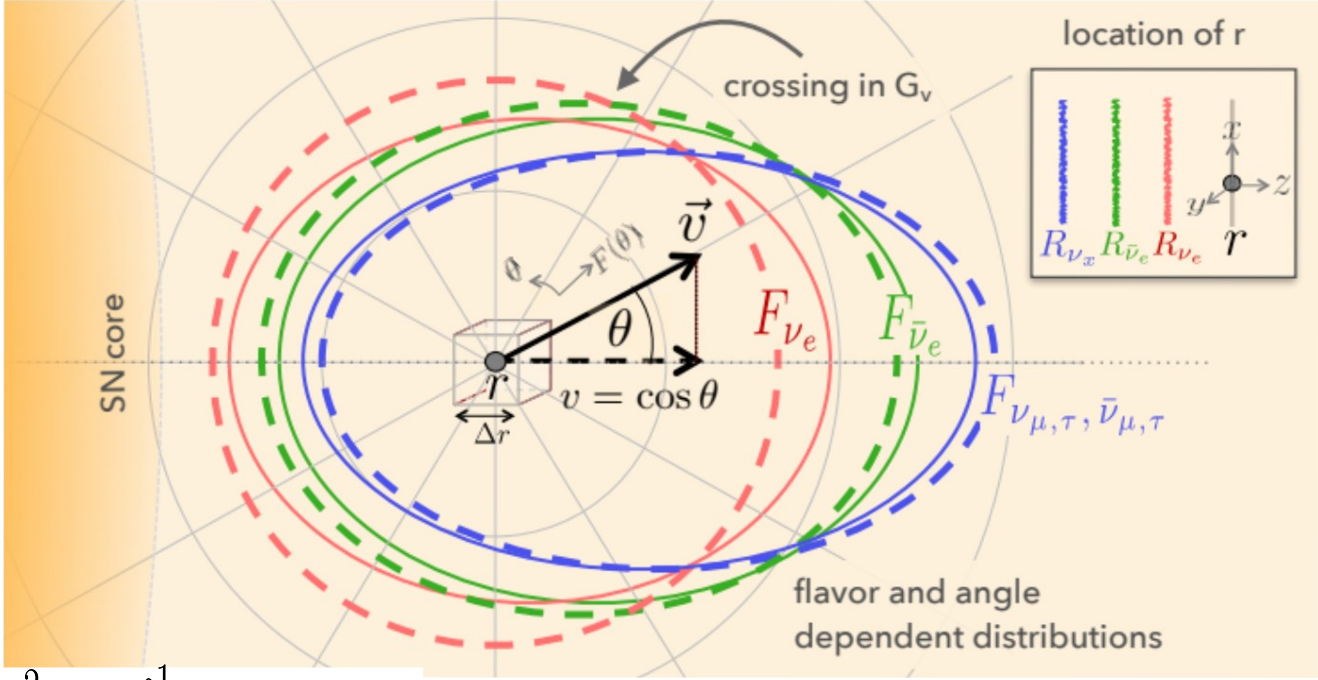
Nonlinear Analysis : Theory of extreme nonlinear behaviour

Flavor depolarized final state :

G_v^B $\qquad$ $G_v^L[v_c = 0]$ $\qquad$ $G_v^L[v_c \neq 0]$



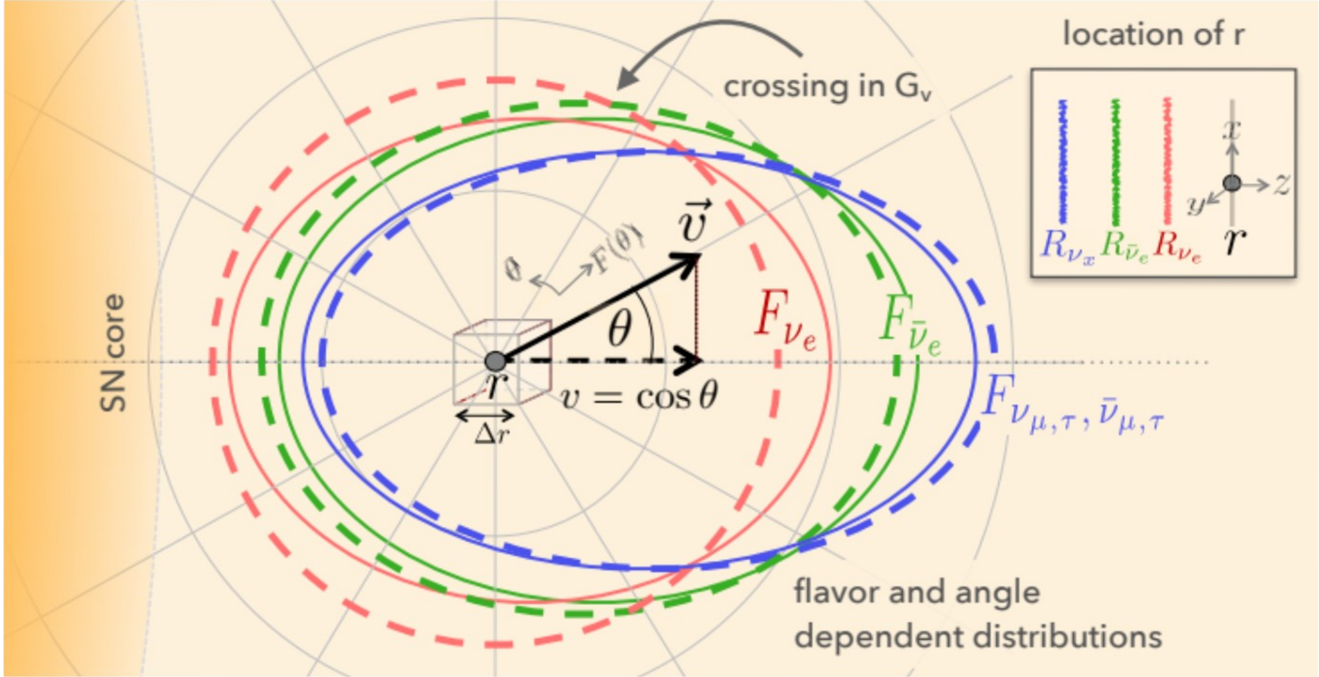
Nonlinear Analysis : Theory of extreme nonlinear behaviour



$$\Phi^{\nu_\alpha/\bar{\nu}_\alpha}[E,r] \propto \frac{r^2}{R^2}E^2 \int_0^1 dv\, F^{\nu_\alpha/\bar{\nu}_\alpha}_{v,E}[r]v$$

$$F^{\text{fin}}_{\nu_e,\,\nu_\mu}[\vec{p}] = (1 - f^{\text{D}}_{\vec{p}})F^{\text{ini}}_{\nu_e,\,\nu_\mu}[\vec{p}] + f^{\text{D}}_{\vec{p}}\,F^{\text{ini}}_{\nu_\mu,\,\nu_e}[\vec{p}]$$

Nonlinear Analysis : Theory of extreme nonlinear behaviour



$$F^{\rm fin}_{\nu_e,\nu_\mu}[\vec{p}] = (1 - f^{\rm D}_{\vec{p}})F^{\rm ini}_{\nu_e,\nu_\mu}[\vec{p}] + f^{\rm D}_{\vec{p}}F^{\rm ini}_{\nu_\mu,\nu_e}[\vec{p}]$$

Nonlinear Analysis :Take Away Message

- **Fast oscillations bring different neutrino flavors close to each other (Flavor depolarization) causing irreversibilty in the system**

- **We developed the first ever theory of fast oscillations in the nonlinear regime to show how, when, to what extent flavor depolarization happens and what are the various parameters controlling such behaviour.**

- **We gave a prediction for the final neutrino fluxes undergoing fast oscillations which can be predicted in future neutrino telescopes and can have important consequences in supernova neutrino phenomenology**

Results : Linear Analysis

Linear Analysis : Toy Model

Phase Space : 2 space + 1 vel + 1 time dimension $v^2 = v_x^2 + v_y^2 = 1$

$$\left(\partial_t + v_x\partial_x + v_y\partial_y\right)\mathsf{S}[v_x, v_y] = \mu \int_{-1}^{1}\int_{-1}^{1} dv_x^{'} dv_y^{'}\, \delta[v^{'} - 1] G[v_x^{'}, v_y^{'}] \left(1 - v_x v_x^{'} - v_y v_y^{'}\right) \mathsf{S}[v_x^{'}, v_y^{'}] \times \mathsf{S}[v_x, v_y]$$

Dispersion relation : Linear analysis

$$\mathsf{S}^{||}[v_x, v_y] \approx 1$$

$$det(\Pi^{\mu\nu}[k_x, k_y, \omega]) = 0 \qquad \Pi^{\mu\nu}[k_x, k_y, \omega] = \eta^{\mu\nu} + \int_{-1}^{1}\int_{-1}^{1} dv_x dv_y\; v^{\mu}v^{\nu}\delta[v-1]\frac{G[v_x, v_y]}{\omega - k_x v_x - k_y v_y}$$

$$\mathcal{D}[k_x, k_y, \omega] = -\left(\Pi^{ty}\right)^2 \Pi^{xx} + 2\Pi^{tx}\Pi^{ty}\Pi^{xy} - \Pi^{tt}\left(\Pi^{xy}\right)^2 - \left(\Pi^{tx}\right)^2 \Pi^{yy} + \Pi^{tt}\Pi^{xx}\Pi^{yy} = 0$$

Notation : $k_x = k\,\cos\,\beta$ $\quad k_y = k\,\sin\,\beta$ $\quad v_x = \cos\,\theta$ $\quad v_y = \sin\,\theta$ $\quad G[v_x, v_y] \equiv G[\theta]$ $\quad A = \int_0^{2\pi} G[\theta]\, d\theta$

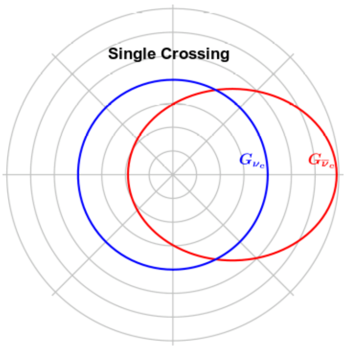
Initial Condition : $\mathsf{S}_v^{||}[x, y, 0] = 1 \forall v$

$\mathsf{S}_v^{\perp}[x, y, 0] = O(10^{-6})$

Linear Analysis : Toy Model

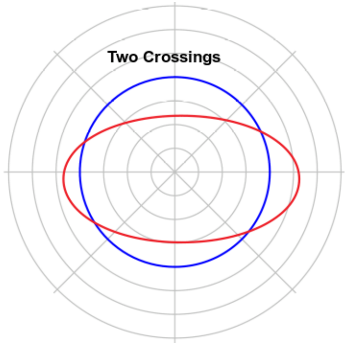
$$G[v_x, v_y] = G^{\nu_e}[v_x, v_y] - G^{\bar{\nu}_e}[v_x, v_y]$$

Forward excess



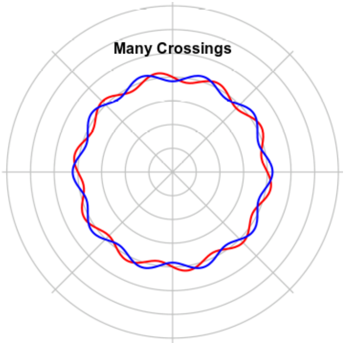
$$G[\theta] = \begin{cases} \frac{A-1}{2\pi}, & \text{if } v_x = \cos\theta > 0 \\ \frac{1}{2\pi}, & \text{if } v_x = \cos\theta < 0\,. \end{cases}$$

Forward and Backward excess



$$G[\theta] = \begin{cases} \frac{A-1}{2\pi}, & \text{if } v_x = \cos\theta > 0 \;\&\; v_y = \sin\theta > 0 \\ \frac{1}{2\pi}, & \text{if } v_x = \cos\theta < 0 \;\&\; v_y = \sin\theta > 0 \\ \frac{A-1}{2\pi}, & \text{if } v_x = \cos\theta < 0 \;\&\; v_y = \sin\theta < 0 \\ \frac{1}{2\pi}, & \text{if } v_x = \cos\theta > 0 \;\&\; v_y = \sin\theta < 0\,. \end{cases}$$

Turbulent medium



$$G[\theta] = \frac{A}{2\pi} + c_1\,\cos m\theta + c_2\,\sin m\theta$$

Linear Analysis : Results for Single Crossing

Symmetry :

$$\Pi^{\mu\nu}[k_x, k_y, \omega] \xrightarrow{k_y \to -k_y} \eta^{\mu\nu} + \int_{-1}^{1}\int_{-1}^{1} dv_x dv_y \ \delta[v-1] \frac{G[v_x, v_y]}{\omega - k_x v_x + k_y v_y} v^\mu v^\nu \xrightarrow{v_y \to -v_y} \pm\Pi^{\mu\nu}[k_x, k_y, \omega]$$

$$\Pi^{xy} \to -\Pi^{xy} \qquad \Pi^{ty} \to -\Pi^{ty} \qquad \Pi^{tx} \to \Pi^{tx} \qquad \Pi^{xx} \to \Pi^{xx} \qquad \Pi^{yy} \to \Pi^{yy} \qquad \Pi^{tt} \to \Pi^{tt}$$

↓

$$\mathcal{D}[k_x, k_y, \omega] = -\left(\Pi^{ty}\right)^2 \Pi^{xx} + 2\Pi^{tx}\Pi^{ty}\Pi^{xy} - \Pi^{tt}\left(\Pi^{xy}\right)^2 - \left(\Pi^{tx}\right)^2 \Pi^{yy} + \Pi^{tt}\Pi^{xx}\Pi^{yy} = 0$$

↓

$$\mathcal{D}[k_x, -k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$$

Linear Analysis : Results for Single Crossing

K = 0 mode :

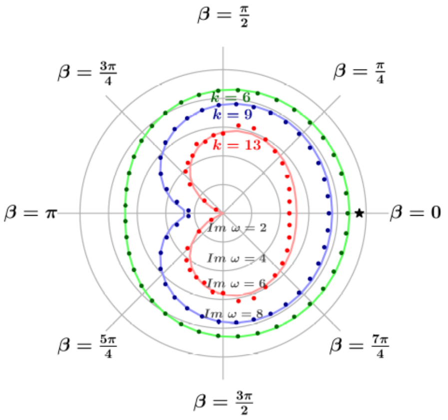
$$\left.\begin{array}{l}\Pi^{xy} \to -\Pi^{xy} \\ \Pi^{ty} \to -\Pi^{ty}\end{array}\right\} \xrightarrow[\omega \equiv \omega_0]{k_x = k_y = 0} \left.\begin{array}{l}\Pi^{ty}[0,0,\omega_0] = 0 \\ \Pi^{xy}[0,0,\omega_0] = 0\end{array}\right\} \to \mathcal{D}[0,0,\omega_0]$$

$\mathcal{D}[0,0,\omega_0]$ →

- $\Pi_{yy}[0,0,\omega_0] = 0$, ω_0 **real**
- $$\Pi_{tx}[0,0,\omega_0]\Pi_{tx}[0,0,\omega_0] - \Pi_{tt}[0,0,\omega_0]\Pi_{xx}[0,0,\omega_0] = 0$$
  ↓
  $$\omega_0^2 + \omega_0\Big(\phi_{tt} - \phi_{xx}\Big) - \Big(\phi_{tt}\phi_{xx} - (\phi_{tx})^2\Big) = 0$$
  ω_0 **complex if** $\left(\phi_{tt} + \phi_{xx}\right)^2 - 4\left(\phi_{tx}\right)^2 < 0$

$$\phi^{\mu\nu} = \int_{-1}^{1}\int_{-1}^{1} dv_x dv_y \; v^\mu v^\nu \delta[v-1] G[v_x, v_y]$$

Linear Analysis : Results for Single Crossing



$$A = 0$$

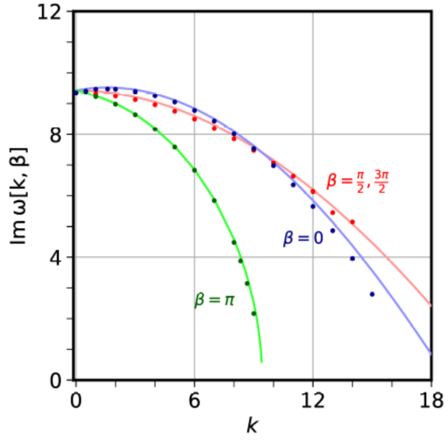
1) $\mathcal{D}[k_x, -k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$

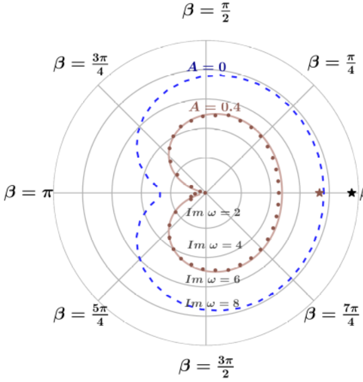
2) $k_x = k_y = 0$ **unstable**

3) **Maximum growing fourier mode along** k_x

4) $\text{Im}\,\omega$ **monotonically decreases w.r.t** k

5) **Rate of decrease more along** k_y



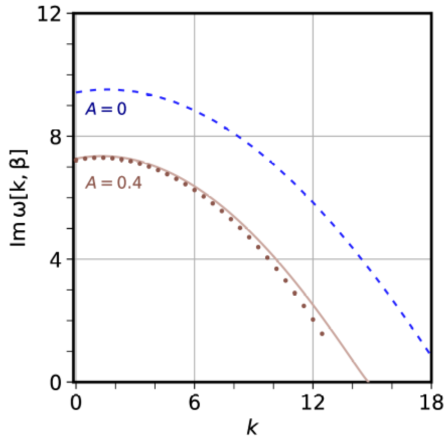


$$A \neq 0$$

1) **Above (1)-(5) points remain unchanged**

2) **Decrease as a function of** k **is much faster**

3) **Overall growth rate is suppressed.**



Linear Analysis : Results for Two Crossings

Symmetry :

$$\Pi^{\mu\nu}[k_x, k_y, \omega] \xrightarrow[k_x \to -k_x]{k_y \to -k_y} \eta^{\mu\nu} + \int_{-1}^{1}\int_{-1}^{1} dv_x dv_y \, \frac{G[v_x, v_y]}{\omega + k_x v_x + k_y v_y} \, \delta[v-1] \, v^\mu v^\nu \xrightarrow[v_x \to -v_x]{v_y \to -v_y} \pm\Pi^{\mu\nu}[k_x, k_y, \omega]$$

$\Pi^{xy} \to \Pi^{xy}$ $\Pi^{ty} \to -\Pi^{ty}$ $\Pi^{tx} \to -\Pi^{tx}$ $\Pi^{xx} \to \Pi^{xx}$ $\Pi^{yy} \to \Pi^{yy}$ $\Pi^{tt} \to \Pi^{tt}$

$$\downarrow$$

$$\mathcal{D}[k_x, k_y, \omega] = -\left(\Pi^{ty}\right)^2 \Pi^{xx} + 2\Pi^{tx}\Pi^{ty}\Pi^{xy} - \Pi^{tt}\left(\Pi^{xy}\right)^2 - \left(\Pi^{tx}\right)^2 \Pi^{yy} + \Pi^{tt}\Pi^{xx}\Pi^{yy} = 0$$

$$\downarrow$$

$$\mathcal{D}[-k_x, -k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$$

Linear Analysis : Results for Two Crossings

Symmetry :

$$\Pi^{\mu\nu}[k_x,k_y,\omega] \xrightarrow[k_x\,\to\,-k_x]{k_y\,\to\,-k_y} \eta^{\mu\nu} + \int_{-1}^{1}\int_{-1}^{1} dv_x dv_y\, \frac{G[v_x,v_y]}{\omega + k_x v_x + k_y v_y}\, \delta[v-1]\, v^\mu v^\nu \xrightarrow[v_x\,\to\,-v_x]{v_y\,\to\,-v_y} \pm\Pi^{\mu\nu}[k_x,k_y,\omega]$$

$\Pi^{xy} \to \Pi^{xy}$ $\quad \Pi^{ty} \to -\Pi^{ty}$ $\quad \Pi^{tx} \to -\Pi^{tx}$ $\quad \Pi^{xx} \to \Pi^{xx}$ $\quad \Pi^{yy} \to \Pi^{yy}$ $\quad \Pi^{tt} \to \Pi^{tt}$

$$\mathcal{D}[k_x,k_y,\omega] = -\left(\Pi^{ty}\right)^2 \Pi^{xx} + 2\Pi^{tx}\Pi^{ty}\Pi^{xy} - \Pi^{tt}\left(\Pi^{xy}\right)^2 - \left(\Pi^{tx}\right)^2 \Pi^{yy} + \Pi^{tt}\Pi^{xx}\Pi^{yy} = 0$$

$$\mathcal{D}[-k_x,-k_y,\omega] = \mathcal{D}[k_x,k_y,\omega] \quad + $$

$$\mathcal{D}[-k_x,-k_y,\omega] = \mathcal{D}[k_x,k_y,\omega]$$

$$\mathcal{D}[-k_x,-k_y,\omega] = \mathcal{D}[k_x,k_y,\omega]$$

Linear Analysis : Results for Two Crossings

K = 0 mode :

$$\left.\begin{array}{l}\Pi^{tx} \to -\Pi^{tx} \\ \Pi^{ty} \to -\Pi^{ty}\end{array}\right\} \xrightarrow[\omega \equiv \omega_0]{k_x = k_y = 0} \left.\begin{array}{l}\Pi^{tx}[0,0,\omega_0] = 0 \\ \Pi^{ty}[0,0,\omega_0] = 0\end{array}\right\} \to \mathcal{D}[0,0,\omega_0]$$

$\mathcal{D}[0,0,\omega_0]$ → $\Pi_{tt}[0,0,\omega_0] = 0$

ω_0 **real**

$\mathcal{D}[0,0,\omega_0]$ →

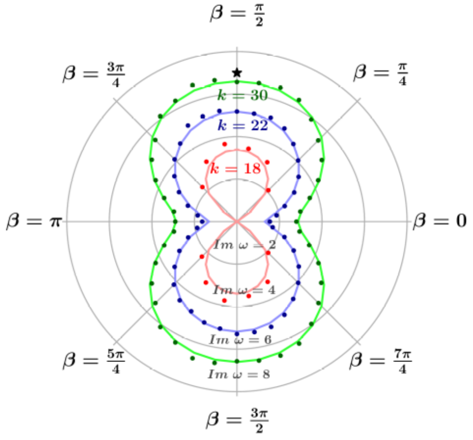
$$\Pi_{xy}[0,0,\omega_0]\Pi_{xy}[0,0,\omega_0] - \Pi_{xx}[0,0,\omega_0]\Pi_{yy}[0,0,\omega_0] = 0$$

↓

$$\omega_0^2 - \omega_0\Big(\phi_{xx} + \phi_{yy}\Big) + \Big(\phi_{xx}\phi_{yy} - (\phi_{xy})^2\Big) = 0$$

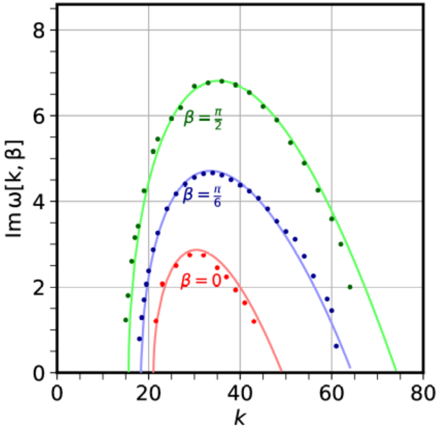
ω_0 **complex if** $\big(\phi_{xx} - \phi_{yy}\big)^2 + 4\big(\phi_{xy}\big)^2 < 0$

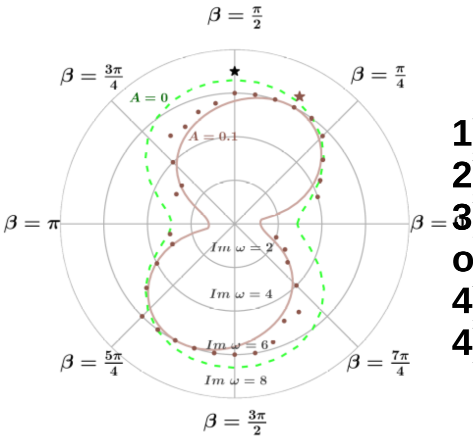
Linear Analysis : Results for Two Crossings



$$A = 0$$

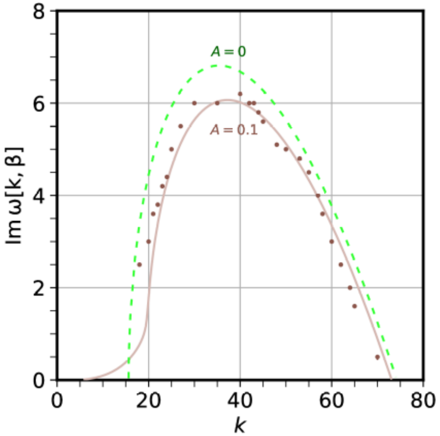
1) $\mathcal{D}[k_x, -k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$
$\mathcal{D}[-k_x, k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$
$\mathcal{D}[-k_x, -k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$
2) $k_x = k_y = 0$ **stable**
3) **Maximum growing fourier mode along** k_y
4) $\mathrm{Im}\,\omega$ **shows lorentzian nature w.r.t** k
5) **More wide along** k_y





$$A \neq 0$$

1) $\mathcal{D}[-k_x, -k_y, \omega] = \mathcal{D}[k_x, k_y, \omega]$
2) Points (2), (4), (5) remain same
3) Maximum growing fourier mode along one of the diagonal
4) Wideness of the lorentzian decreases
4) Overall growth rate is suppressed.



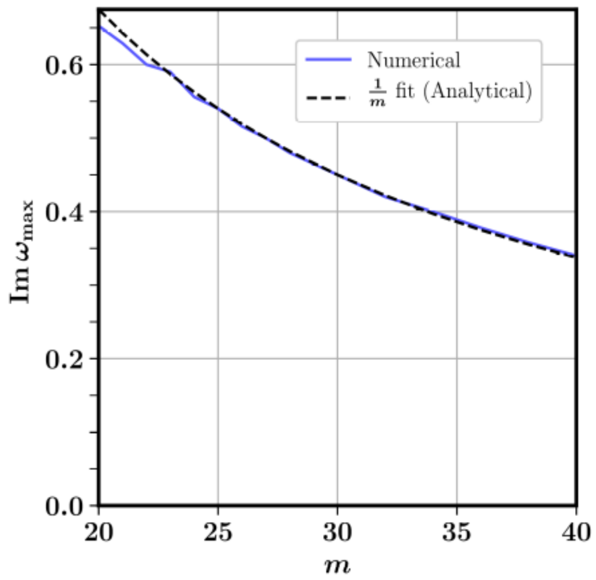
Linear Analysis : Results for Many Crossings

$$\Pi^{\mu\nu} = \eta^{\mu\nu} + \int_0^{2\pi} d\theta \, \frac{1}{\omega - k\cos(\theta - \beta)} G[\theta] W^{\mu\nu}[\theta] = \eta^{\mu\nu} + \int_0^{2\pi} d\theta \, \exp[F^{\mu\nu}[\theta]]$$

$$W[\theta] = \begin{pmatrix} 1 & \cos\theta & \sin\theta \\ \cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta & \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

Saddle point approximation

$$\Pi^{\mu\nu}[k_{max}, \beta_{max}, \omega] \xrightarrow[G[\theta] = \frac{A}{2\pi} + c_1 \cos m\theta + c_2 \sin m\theta]{F^{\mu\nu}[\theta_0] + (\theta - \theta_0)^2 \left. \frac{d^2 F^{\mu\nu}[\theta]}{d\theta^2} \right|_{\theta=\theta_0}} \mathrm{Im}\,\omega_{\max} \propto \frac{1}{m}$$



Linear Analysis :Take Away Message

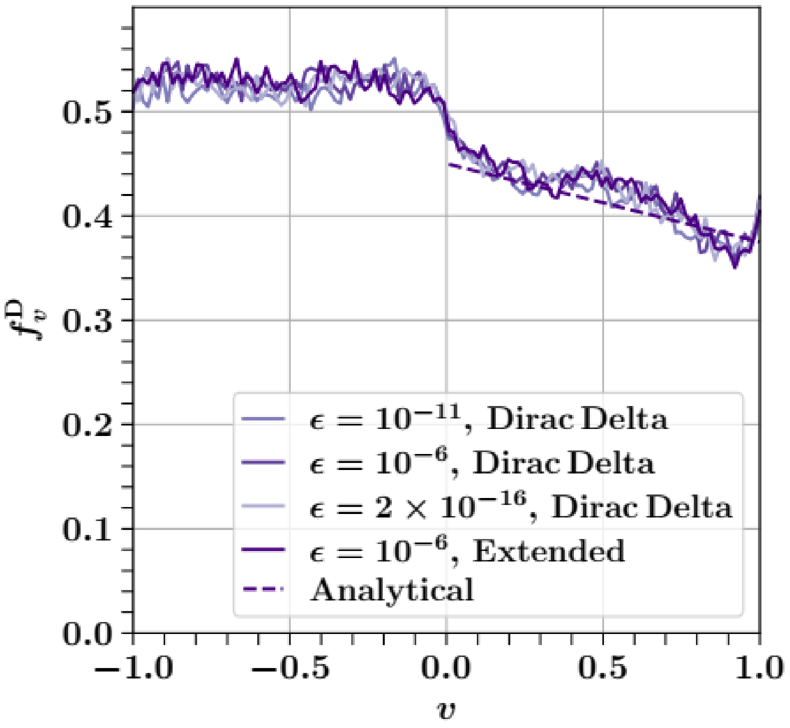
- We did a linear study (both numerically and analytically) in higher dimensional phase space

- Our study suggests the linear behaviour of fast oscillations is connected to the number of zero crossings, shapes, symmetry and various other nature (lepton asymmetry) of neutrino angular distributions in momentum space

- ELN’s with large number of zero crossings lead to a relatively smaller growth rate and thus can have interesting consequences in realistic turbulent SN environment

on-linear Analysis
An Aside

Nonlinear Analysis : An Aside



Universality of Depolarization Factor

Dirac delta seed

$$\mathsf{S}_v^{(1),(2)}[z, t=0] \sim \epsilon\delta[z]$$

Extended seed

$$\mathsf{S}_v^{(1)}[z, t=0] = 10^{-6} \cos\, \phi[z]$$

$$\mathsf{S}_v^{(2)}[z, t=0] = 10^{-6} \sin\, \phi[z]$$

$$\phi[z] = \frac{1}{N_z} \sum_{j=0}^{N_z-1} \cos[\frac{2\pi j z}{L}]$$

Nonlinear Analysis : An Aside

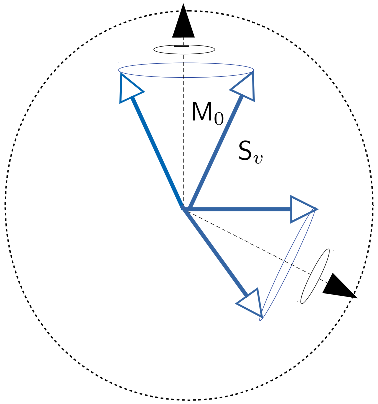
- **Approximate steady state behaviour in time**
- **Bloch vector** S_v **: Spatial precession with frequency** $\frac{1}{v}$ **around a common axis** M_0

$$d_x \mathsf{S}_v[x] = \frac{\mathsf{M}_0[x]}{v} \times \mathsf{S}_v[x] \qquad \mathsf{M}_n = \int_{-1}^{1} G_v L_n[v]\, \mathsf{S}_v dv$$

- M_0 **: Gyroscopic pendulum in space under the action of spatially varying magnetic field** B

$$\mathsf{M}_0 \times d_x^2 \mathsf{M}_0 + (\mathsf{D}.\mathsf{M}_0)\, d_x \mathsf{M}_0 = M_0^2 \mathsf{B} \times \mathsf{M}_0$$

$$d_x \mathsf{B}[x] = \sum_{p,r,n=0}^{\infty} \ell_{0p}\ell_{pr}\ell_{rn} \left(\mathsf{M}_0[x] \times \mathsf{M}_n[x] \right)$$



$$\mathsf{B} = \sum_{r,n=0}^{\infty} \ell_{0r}\ell_{rn} \mathsf{M}_n$$

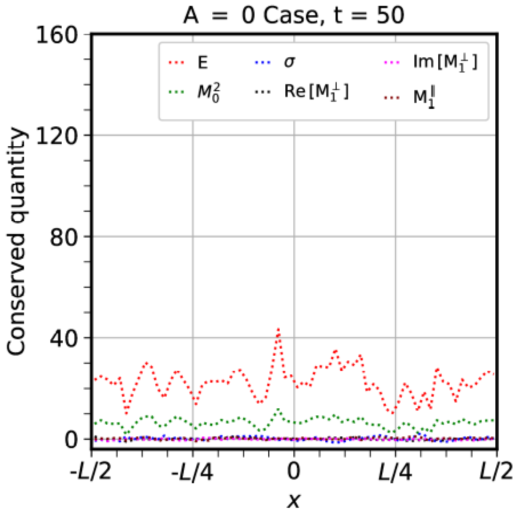
$$\ell_{rn} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} \frac{L_n(v) L_r(v)}{v} dv$$

Nonlinear Analysis : An Aside

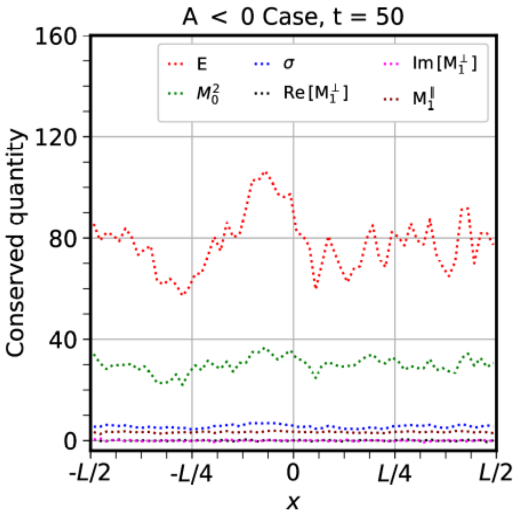
- **Spatially conserved quantities : Pendulum's energy** E**, spin** σ**, length** M_0 **and** M_1

$$E = \frac{\mathsf{D}^2}{2} + \mathsf{M}_0 \cdot \mathsf{B} \qquad \sigma = \mathsf{M}_0 \cdot \mathsf{D} \qquad M_0 = |\mathsf{M}_0|$$

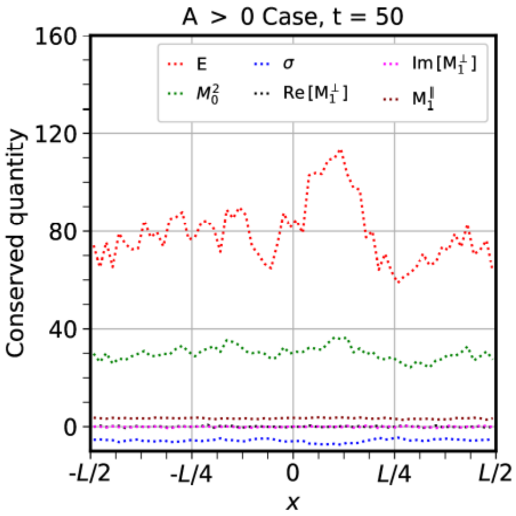
$G_v^{'}$



$G_v^{''}$



$G_v^{'''}$



Nonlinear Analysis : An Aside

- **Non-separable solution in x & v. Non-collective nature for any A**

$$d_x \mathsf{S}_v[x] = \frac{\mathsf{M}_0[x]}{v} \times \mathsf{S}_v[x] \longrightarrow \mathsf{S}_v = \begin{pmatrix} f_1[x]h_1[v] \\ f_2[x]h_2[v] \\ f_3[x]h_3[v] \end{pmatrix} \longrightarrow \left.\begin{array}{l} v h_i[v] = H_j h_k[v] - h_k[v] H_j \\ \dfrac{d}{dx} f_i[x] = f_j[x] f_k[x] \end{array}\right\} \begin{array}{l} \mathsf{S}_v = 0 \\ v \neq 0 \end{array}$$

$$R^i_{v_1, v_2}[x,t] = \frac{s^i_{v_1}[x,t]}{s^i_{v_1}[x,t]}$$

For fixed (v_1, v_2)

For fixed t

Independent of x → **Separable in x & v**

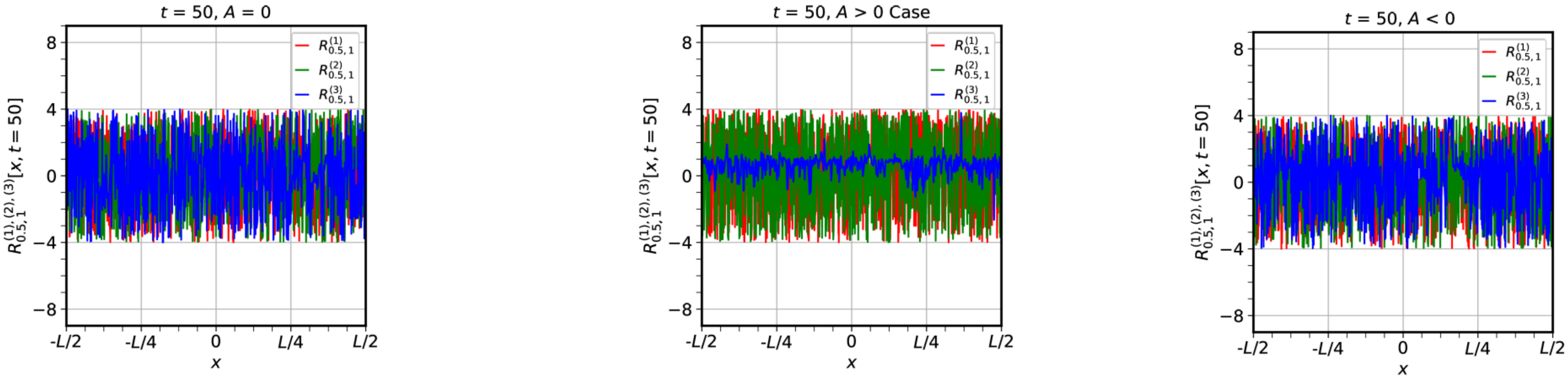
Dependent of x → **Non-separable in x & v**

Nonlinear Analysis : An Aside

- **Non-separable solution in x & v. Non-collective nature for any A**

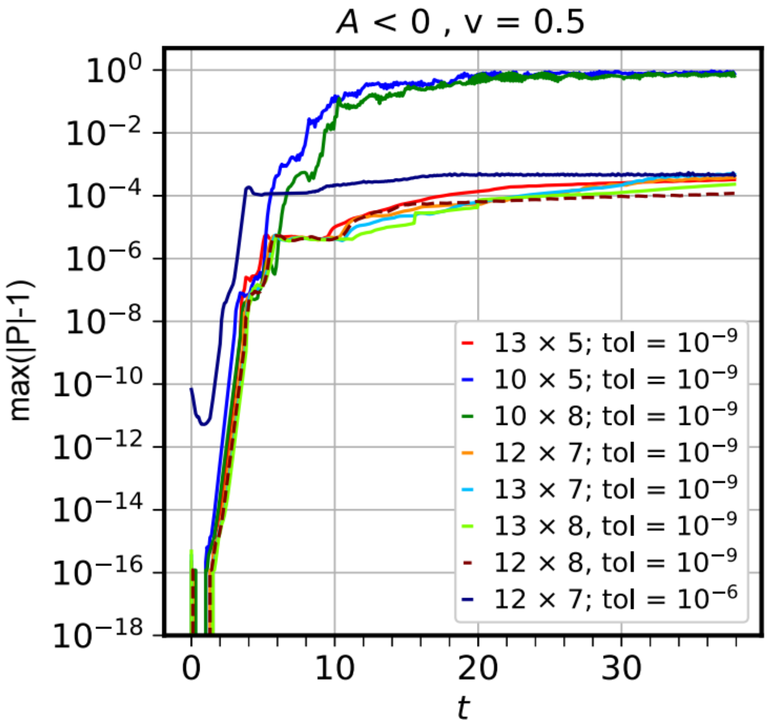
$$d_x \mathsf{S}_v[x] = \frac{\mathsf{M}_0[x]}{v} \times \mathsf{S}_v[x] \longrightarrow \mathsf{S}_v = \begin{pmatrix} f_1[x]h_1[v] \\ f_2[x]h_2[v] \\ f_3[x]h_3[v] \end{pmatrix} \longrightarrow \left. \begin{array}{l} v h_i[v] = H_j h_k[v] - h_k[v] H_j \\ \frac{d}{dx} f_i[x] = f_j[x] f_k[x] \end{array} \right\} \begin{array}{l} \mathsf{S}_v = 0 \\ v \neq 0 \end{array}$$

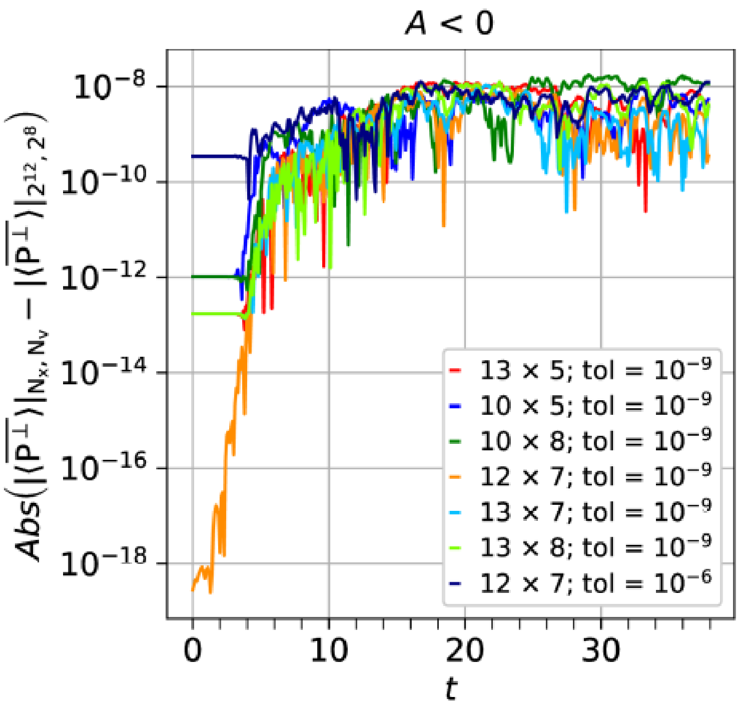
$G_v^{'}$ $G_v^{''}$ $G_v^{'''}$



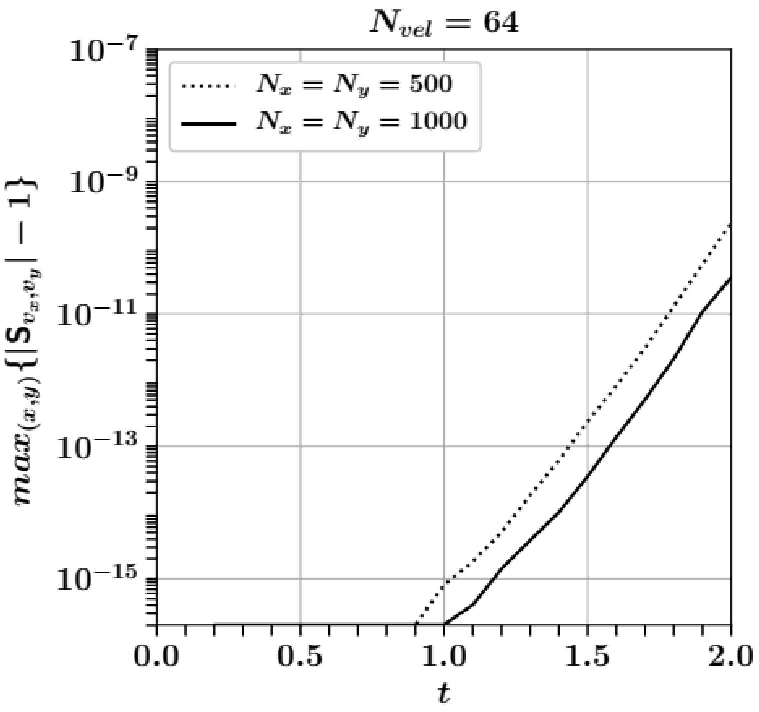
Backup Slides

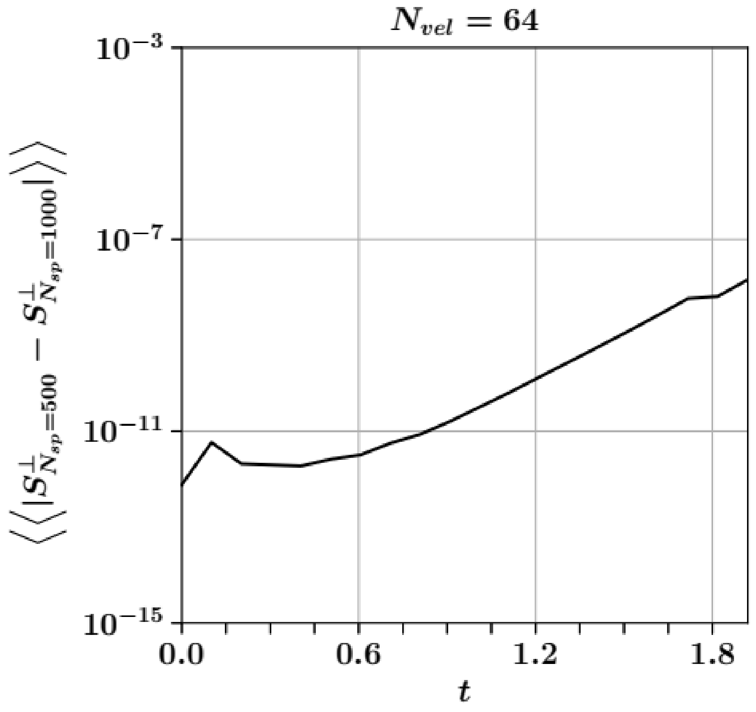
Nonlinear Analysis





Linear Analysis





Derivation of irreversible pendulum equation

$$d_t\langle \mathsf{M}_n\rangle - \langle \mathsf{M}_0\rangle \times \langle \mathsf{M}_n\rangle = -\langle \mathsf{M}_1\rangle \times \left(\frac{2n}{2n+1}\langle \mathsf{M}_{n-1}\rangle + \frac{2n+2}{2n+1}\langle \mathsf{M}_{n+1}\rangle\right)$$

$$d_t\langle \mathsf{M}_0\rangle = 0$$

$$\left.\begin{aligned} d_t\langle \mathsf{M}_1\rangle &= \langle \mathsf{D}\rangle \times \langle \mathsf{M}_1\rangle \\ d_t\mathsf{D} &= \mathsf{B} \times \mathsf{M}_1 \end{aligned}\right\} \longrightarrow$$

$$d_t^2\mathsf{M}_1 = (\mathsf{B} \times \mathsf{M}_1) \times \mathsf{M}_1 + \mathsf{D} \times d_t\mathsf{M}_1$$

$$\langle \mathsf{M}_1\rangle \times d_t^2\langle \mathsf{M}_1\rangle + \big(\langle \mathsf{D}\rangle.\langle \mathsf{M}_1\rangle\big) d_t\langle \mathsf{M}_1\rangle = |\langle \mathsf{M}_1\rangle|^2\langle \mathsf{B}\rangle \times \langle \mathsf{M}_1\rangle$$

$$d_t\mathsf{B} = \mathsf{K} \times \mathsf{M}_1$$

$$\mathsf{A} \times (\mathsf{B} \times \mathsf{C}) = (\mathsf{A}\cdot\mathsf{C})\,\mathsf{B} - (\mathsf{A}\cdot\mathsf{B})\,\mathsf{C}$$

Vector Identity

$$\mathsf{D} = \frac{\mathsf{M}_0}{3} + \frac{2\mathsf{M}_2}{3}$$

$$\mathsf{B} = \frac{2\mathsf{M}_3}{5} - \frac{9\mathsf{M}_1}{35}$$

Derivation of Conserved quantities

$$\left.\begin{aligned} d_t\langle \mathsf{M}_1\rangle &= \langle \mathsf{D}\rangle \times \langle \mathsf{M}_1\rangle \\ d_t\langle \mathsf{D}\rangle &= \langle \mathsf{B}\rangle \times \langle \mathsf{M}_1\rangle \\ d_t\langle \mathsf{B}\rangle &= \langle \mathsf{K}\rangle \times \langle \mathsf{M}_1\rangle \end{aligned}\right\} \longrightarrow \begin{aligned} E &= \frac{\langle \mathsf{D}\rangle \cdot \langle \mathsf{D}\rangle}{2} + \langle \mathsf{M}_1\rangle \cdot \langle \mathsf{B}\rangle = \text{const} \\ \sigma &= \langle \mathsf{M}_1\rangle \cdot \langle \mathsf{D}\rangle = \text{const} \end{aligned}$$

$$d_t E = \langle \mathsf{D}\rangle \cdot (\langle \mathsf{B}\rangle \times \langle \mathsf{M}_1\rangle) + (\langle \mathsf{D}\rangle \times \langle \mathsf{M}_1\rangle) \cdot \langle \mathsf{B}\rangle = 0$$

$$d_t \sigma = (\langle \mathsf{D}\rangle \times \langle \mathsf{M}_1\rangle) \cdot \langle \mathsf{D}\rangle + \langle \mathsf{M}_1\rangle \cdot (\langle \mathsf{B}\rangle \times \langle \mathsf{M}_1\rangle) = 0$$

$$\mathsf{A}.\,(\mathsf{B} \times \mathsf{C}) = \mathsf{B}.\,(\mathsf{C} \times \mathsf{A})$$

Derivation of pendulum's final position

EOM :

$$d_t\langle \mathsf{M}_1\rangle = \langle \mathsf{D}\rangle \times \langle \mathsf{M}_1\rangle$$

$$d_t\langle \mathsf{D}\rangle = \langle \mathsf{B}\rangle \times \langle \mathsf{M}_1\rangle$$

$$d_t\langle \mathsf{B}\rangle = \langle \mathsf{K}\rangle \times \langle \mathsf{M}_1\rangle$$

Conserved qty:

$$E = \frac{\langle \mathsf{D}\rangle \cdot \langle \mathsf{D}\rangle}{2} + \langle \mathsf{M}_1\rangle \cdot \langle \mathsf{B}\rangle = \text{const}$$

$$\sigma = \langle \mathsf{M}_1\rangle \cdot \langle \mathsf{D}\rangle = \text{const}$$

$$\frac{\langle \mathsf{B}\rangle \cdot \langle \mathsf{B}\rangle}{2} + \langle \mathsf{K}\rangle \cdot \langle \mathsf{D}\rangle = \text{const}$$

$$\langle \mathsf{B}\rangle \cdot \langle \mathsf{K}\rangle = \text{const}$$

Notations : $\mathsf{M}_1^{\parallel} \equiv m \quad \mathsf{M}_1^{\perp} \equiv \mathsf{M} \quad \mathsf{D}^{\parallel} \equiv u \quad \mathsf{B}^{\parallel} \equiv b \quad \mathsf{K}^{\parallel} \equiv k$

$i \equiv \text{initial} \quad f \equiv \text{final}$

At pendulum's southernmost position :

$$d_t m_f = 0$$

Derivation of pendulum's final position

EOM :

$$d_t\langle \mathsf{M}_1\rangle = \langle \mathsf{D}\rangle \times \langle \mathsf{M}_1\rangle$$

$$d_t\langle \mathsf{D}\rangle = \langle \mathsf{B}\rangle \times \langle \mathsf{M}_1\rangle$$

$$d_t\langle \mathsf{B}\rangle = \langle \mathsf{K}\rangle \times \langle \mathsf{M}_1\rangle$$

Conserved qty:

$$E = \frac{\langle \mathsf{D}\rangle \cdot \langle \mathsf{D}\rangle}{2} + \langle \mathsf{M}_1\rangle \cdot \langle \mathsf{B}\rangle = \text{const} \quad \text{(1)}$$

$$\sigma = \langle \mathsf{M}_1\rangle \cdot \langle \mathsf{D}\rangle = \text{const} \quad \text{(2)}$$

$$\frac{\langle \mathsf{B}\rangle \cdot \langle \mathsf{B}\rangle}{2} + \langle \mathsf{K}\rangle \cdot \langle \mathsf{D}\rangle = \text{const} \quad \text{(3)}$$

$$\langle \mathsf{B}\rangle \cdot \langle \mathsf{K}\rangle = \text{const} \quad \text{(4)}$$

Notations : $\mathsf{M}_1^{\parallel} \equiv m \quad \mathsf{M}_1^{\perp} \equiv \mathsf{M} \quad \mathsf{D}^{\parallel} \equiv u \quad \mathsf{B}^{\parallel} \equiv b \quad \mathsf{K}^{\parallel} \equiv k$

$i \equiv \text{initial} \quad f \equiv \text{final}$

At pendulum's southernmost position :

$$d_t m_f = 0$$

$\rightarrow \mathsf{D}_f = (0, 0, u_f)$ $\xrightarrow{\text{Eq.(1)}}$ $m_i u_i = m_f u_f$ (5)

Eq.(4) $\rightarrow$ $b_f = b_i = b$ (6)

Eq.(1), (3) $\rightarrow$ $\dfrac{u_f^2 - u_i^2}{2} = b\,(m_i - m_f) - M_f B_f^{\perp} \cos\theta_{MG}$ (7)

$$\left(B_f^{\perp}\right)^2 = 2k\,(u_i - u_f) \quad \text{(8)}$$

$t = t_f \rightarrow \mathsf{B}||\mathsf{M}_1$ **approximation**

$$\left(\frac{b}{B_f^{\perp}}\right)^2 = \left(\frac{m_f}{M_f}\right)^2 \quad \text{(9)}$$

Derivation of pendulum's final position

$$\frac{u_f^2 - u_i^2}{2} = b\,(m_i - m_f) - M_f B_f^{\perp} \cos\theta_{MG} \quad (7)$$

$$\left(B_f^{\perp}\right)^2 = 2k\,(u_i - u_f) \quad (8)$$

$$\left(\frac{b}{B_f^{\perp}}\right)^2 = \left(\frac{m_f}{M_f}\right)^2 \quad (9)$$

$$\left(2b^2 m_f^2 + (m_f + m_i) b u_i^2\right)^2 - 16k^2 u_i^2 m_f^4$$

$$2b^2 m_f^2 + (m_f + m_i) b u_i^2 = \pm 4k u_i m_f^2$$

$$m_f^{--}, m_f^{++}, m_f^{-+}, m_f^{+-}$$

Derivation of diffusion equation

$$\partial_t \mathsf{M}_n - \mathsf{M}_0 \times \mathsf{M}_n = \partial_z \left(\frac{n+1}{2n+1}\mathsf{M}_{n+1} + \frac{n}{2n+1}\mathsf{M}_{n-1} \right) - \mathsf{M}_1 \times \left(\frac{n+1}{2n+1}\mathsf{M}_{n+1} + \frac{n}{2n+1}\mathsf{M}_{n-1} \right)$$

Dot with M_n

$$\partial_t M_n - \mathsf{u}_n \cdot \partial_z \mathsf{T}_n = (\mathsf{M}_1 \times \mathsf{u}_n) \cdot \mathsf{T}_n \qquad \mathsf{T}_n = \frac{n+1}{2n+1}\mathsf{M}_{n+1} + \frac{n}{2n+1}\mathsf{M}_{n-1}$$

$$\mathsf{T}_n = \frac{n+1}{2n+1}\mathsf{M}_{n+1} + \frac{n}{2n+1}\mathsf{M}_{n-1} \approx \mathsf{M}_n + \frac{\partial_n \mathsf{M}_n}{2n} + \frac{\partial_n^2 \mathsf{M}_n}{2} + \ldots$$

Averaging procedure distributes in a phase averaged sense

Diffusion equation