Dispersive analysis of the pion distribution amplitude

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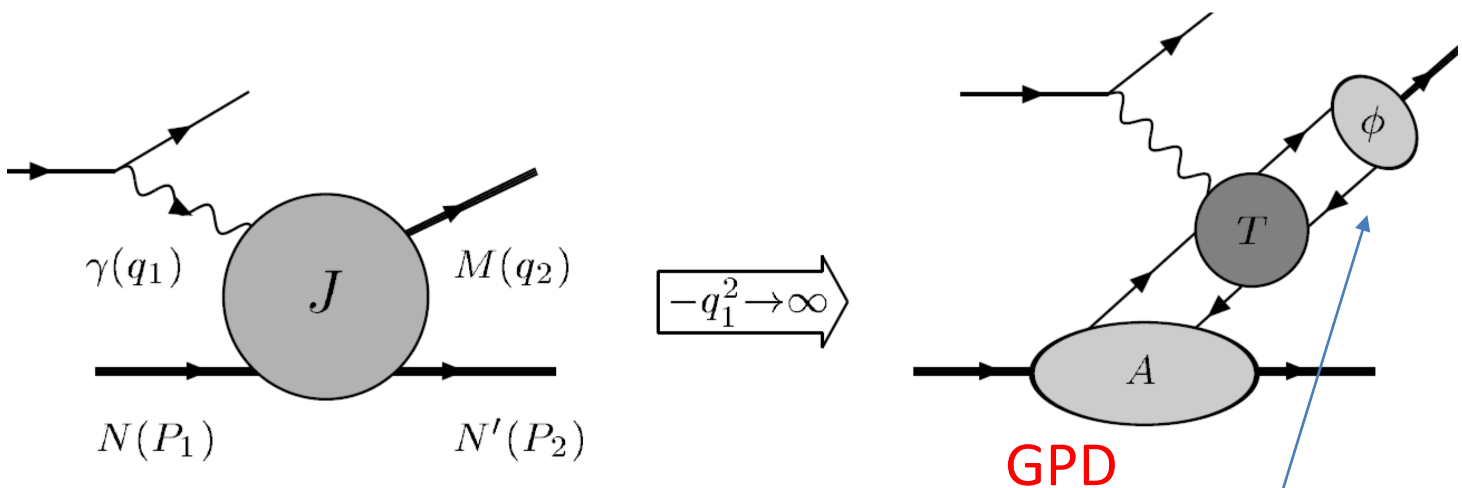
Hard exclusive electroproduction

- Hard exclusive meson electroproduction at EIC
- Collinear factorization at high energy
- To extract GPD, need to know meson DA



momentum fraction

$$\phi_\pi(x) = 6x(1-x) \sum_{n=1,2,\cdots} a^{\pi}_{2n-2} C^{(3/2)}_{2n-2}(2x-1)$$

Gegenbauer expansion

Introduction

- Distribution amplitude (DA) is nonpert fundamental input to collinear factorization for high-energy exclusive QCD processes
- Tremendous efforts devoted to hadron DAs:
- Lattice, sum rules limited to first few moments
- Quasi-correlation allows access to entire x range, but not reliable near endpoints of x
- Solutions for DAs from Dyson-Schwinger equations depend on kernels
- Global fits rely on theo and exp precisions

Challenge: x dependence

- Even all Mellin moments known, can reconstruct x dependence of DA?

$$\langle \xi^n \rangle \equiv \int_0^1 dx (2x-1)^n \phi_\pi(x)$$

- Gegenbauer coefficients vs moments

$$
\begin{aligned}
a_0^\pi &= \langle \xi^0 \rangle, \\
a_2^\pi &= \frac{7}{12} \left(5\langle \xi^2 \rangle - \langle \xi^0 \rangle \right), \\
a_4^\pi &= \frac{11}{24} \left(21\langle \xi^4 \rangle - 14\langle \xi^2 \rangle + \langle \xi^0 \rangle \right), \\
a_6^\pi &= \frac{5}{64} \left(429\langle \xi^6 \rangle - 495\langle \xi^4 \rangle + 135\langle \xi^2 \rangle - 5\langle \xi^0 \rangle \right), \\
a_8^\pi &= \frac{19}{384} \left(2431\langle \xi^8 \rangle - 4004\langle \xi^6 \rangle + 2002\langle \xi^4 \rangle - 308\langle \xi^2 \rangle + 7\langle \xi^0 \rangle \right), \\
a_{10}^\pi &= \frac{23}{1536} \left(29393\langle \xi^{10} \rangle - 62985\langle \xi^8 \rangle + 46410\langle \xi^6 \rangle - 13650\langle \xi^4 \rangle + 1365\langle \xi^2 \rangle - 21\langle \xi^0 \rangle \right)
\end{aligned}
$$

huge coefficients !
big number cancellation
theoretical or roundoff errors greatly amplified
highly nontrivial task

a30 ~ 10E+8 * moments

ill-posed problem

Zhong et al. 2102.03989

- Derived up to 10th moments in QSR

factorization scale

$$(\langle\xi^0\rangle, \langle\xi^2\rangle, \langle\xi^4\rangle, \langle\xi^6\rangle, \langle\xi^8\rangle, \langle\xi^{10}\rangle)|_{\mu=2\,\mathrm{GeV}}$$
$$= (1, 0.254, 0.125, 0.077, 0.054, 0.041)$$

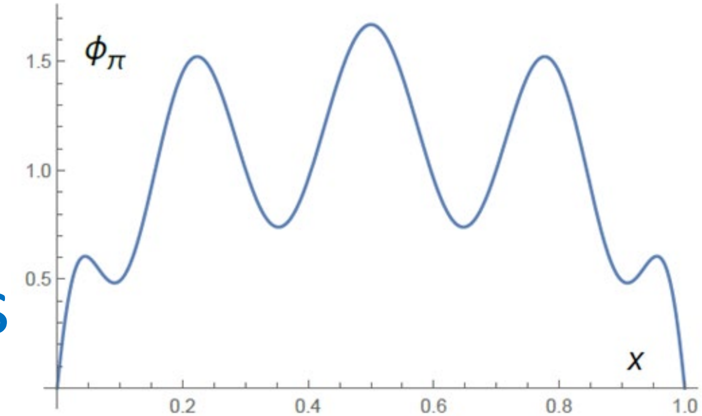
good convergence

- Inverted to Gegenbauer coefficients

$$(a_0^\pi, a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi)|_{\mu=2\,\mathrm{GeV}}$$
$$= (1, 0.157, 0.032, 0.035, 0.098, -0.046)$$

bad convergence

- Unrealistic fluctuating DA
- Eventually, fit DA parametrization to moments



Goals

- Develop analytical nonpert framework that gives all moments of DA --- dispersive approach
- Determine DA in entire x range unambiguously and reliably --- Tikhonov regularization
- Compatible with QCD evolution: DA solved at a scale and DA solved at another scale obey known evolution
- Precision can be improved systematically

Main ideas

consider correlator

$$\Pi_{2;\pi}^{(n,0)}(z,q) \;=\; i\int d^4x e^{iq\cdot x}\langle 0|T\{J_n(x)J_0^\dagger(0)\}|0\rangle$$

$$J_n(x) \;=\; \bar{d}(x)\not{z}\gamma_5(iz\cdot \overleftrightarrow{D})^n u(x) \qquad J_0^\dagger(0) \;=\; \bar{u}(0)\not{z}\gamma_5 d(0)$$

$$\langle 0|\bar{d}(0)\not{z}\gamma_5(iz\cdot \overleftrightarrow{D})^n u(0)|\pi(q)\rangle = i(z\cdot q)^{n+1}f_\pi\langle\xi^n\rangle$$

Wilson line direction

Dispersive integral (hadron side)

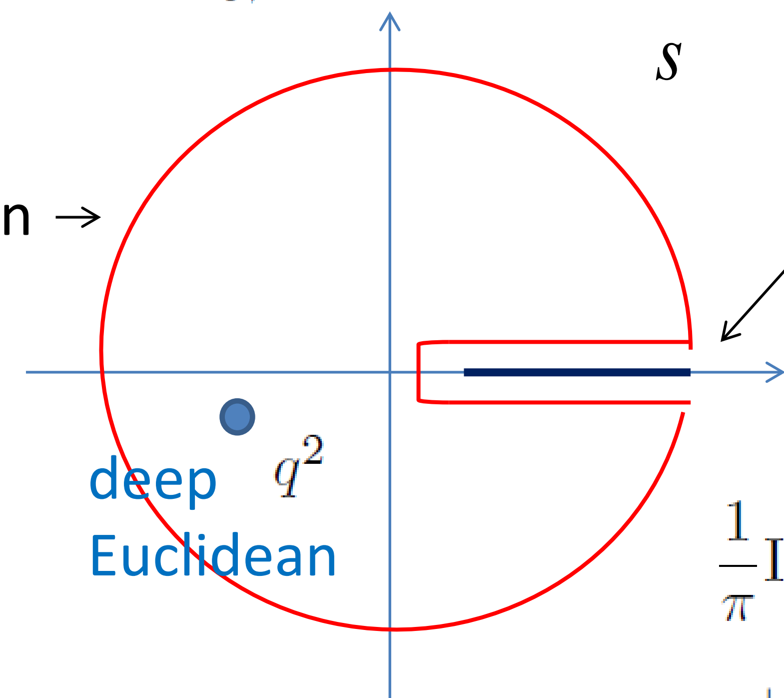
- For analytical function $\Pi(q^2)$

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}\Pi(s)}{s - q^2 - i\epsilon}$$

contain resonant nonpert contribution

branch cut caused by physical intermediate states due to time-like $s > 0$

contribution from large circle assumed negligible



threshold not greater than excited mass squared

$$\frac{1}{\pi}\mathrm{Im}I_n(s) = f_\pi^2\langle\xi^n\rangle\langle\xi^0\rangle\delta(s - m_\pi^2)$$

$$+ \ \pi\frac{3}{4\pi^2(n+1)(n+3)}\theta(s - s_\pi)$$

from excited states

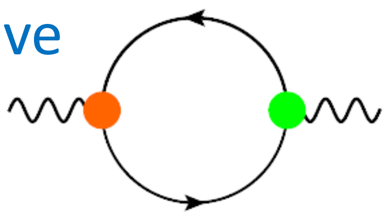
- Naive parametrization based on quark-hadron duality

OPE (quark side)

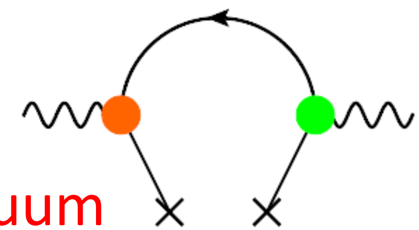
- Calculate correlator at q^2 via OPE directly

$$I_n^{\mathrm{OPE}}(q^2) = \frac{3}{4\pi^2(n+1)(n+3)} \ln \frac{\mu^2}{-q^2} + \frac{m_u \langle \bar{u} u \rangle + m_d \langle \bar{d} d \rangle}{(q^2)^2} + \ldots$$

perturbative piece



quark condensate



nontrivial vacuum

- Express perturbative piece into dispersive integral

$$I_n^{\mathrm{OPE}}(q^2) = \frac{1}{\pi} \int_0^\infty ds \frac{\mathrm{Im} I_n^{\mathrm{pert}}(s)}{s - q^2} + I_n^{\mathrm{cond}}(q^2) \longleftarrow$$

condensates, higher powers

imaginary part for s > 0

Conventional sum rules

- Equate two calculations

Borel transform $q^2 \to M^2$

$$\frac{f_\pi^2 \langle \xi^n \rangle \langle \xi^0 \rangle}{M^2 e^{m_\pi^2/M^2}} = \frac{3}{4\pi^2 (n+1)(n+3)} \left(1 - e^{-s_\pi/M^2}\right)$$

$$+ \frac{m_u \langle \bar{u} u \rangle + m_d \langle \bar{d} d \rangle}{M^4} + \frac{1}{12\pi} \frac{\langle \alpha_s G^2 \rangle}{M^4} - \frac{n\theta(n-2)}{48\pi^2} \frac{\langle g_s^3 f G^3 \rangle}{M^6} + \ldots$$

- Perturbative (condensate) piece decreases (increases) with n; OPE deteriorates with n
- Enlarge Borel mass M to suppress latter
- $1 - e^{-s_\pi/M^2}$ diminishes with M for threshold

s_π < excited states, otherwise more resonances

Quark-hadron duality

- Reason why QSR limited to few moments
- Weakness of conventional QSR originates from assumption of quark-hadron duality
- Our spectral density along branching cut

$$\frac{1}{\pi}\mathrm{Im}I_n(s) = f_\pi^2\langle\xi^n\rangle\langle\xi^0\rangle\delta(s-m_\pi^2) + \rho_n(s)$$

resonance excited state contribution

- Last term unknown, smooth function, may not be equal to perturbative piece in OPE
- Solve it directly, can go for all moments

Integral equation

- How to solve such an ill-posed inverse problem (integral equation) is technical

spectral density, unknown

notoriously difficult to solve

$$\int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x) \longleftarrow \text{OPE input}$$

- We can get exact solutions and all moments
- Refer details to 2205.06746

Polynomial decomposition

- Suppose $\rho(y)$ decreases quickly enough
- Expansion into powers of 1/x justified

$$\frac{1}{x-y} = \sum_{m=1}^{N} \frac{y^{m-1}}{x^m} \qquad \omega(x) = \sum_{n=1}^{N} \frac{b_n}{x^n}$$

true for OPE

- Suppose $\omega(x)$ can be expanded
- Decompose

$$\rho(y) = \sum_{n=1}^{N} a_n y^{\alpha} e^{-y} L_{n-1}^{(\alpha)}(y)$$

orthogonal generalized Laguerre polynomials

depend on $\rho(y)$ at $y \to 0$

$$\rho_n(s) \sim s \to \alpha = 1$$

Azizi et al, 2010

Inverse matrix method

- Equate coefficients of $1/x^n$ on two sides

$$Ma = b \qquad M_{mn} = \int_0^\infty dy y^{m-1+\alpha} e^{-y} L_{n-1}^{(\alpha)}(y)$$

matrix

input $b = (b_1, b_2, \cdots, b_N)$

unknown $a = (a_1, a_2, \cdots, a_N)$

- Solution $a = M^{-1}b$, easy by using Math
- True solution can be approached by increasing N, before M^{-1} diverges, stability in N

N=15~20 usually

Tikhonov regularization

- To get x dependence, work on dispersion relations for Gegenbauer coefficients directly
- Linearly combine OPE inputs for moments into those for Gegenbauer coefficients BV^{-1}

$$V_{kn} = 6\int_0^1 dx x(1-x)(2x-1)^{2n-2}C_{2k-2}^{(3/2)}(2x-1),$$

V more singular than U(=M)

search for solutions insensitive to parameter

- Solutions to $UAV = B$ diverge
- Employ Tikhonov regularization $UA(V+\lambda H) = B,$
- Freedom to choose H, set $H = I$

unknown

OPE inputs

- Condensate inputs in OPE

Zhong et al.
2102.03989

$$m_u\langle \bar{u}u\rangle + m_d\langle \bar{d}d\rangle = -(1.651\pm 0.003)\times 10^{-4}\ \text{GeV}^4,$$

$$\beta_0 = 11 - 2n_f/3$$

$$\langle g_s\bar{q}q\rangle^2 = (2.082^{+0.734}_{-0.697})\times 10^{-3}\left[\frac{\alpha_s(\mu)}{\alpha_s(2\ \text{GeV})}\right]^{-4/\beta_0}\ \text{GeV}^6,$$

$$n_f = 4$$

$$\sum_{u,d,s}\langle g_s^2\bar{\psi}\psi\rangle^2 = (2+r_c^2)\langle g_s^2\bar{q}q\rangle^2,\quad \langle g_s^2\bar{q}q\rangle^2 = (7.420^{+2.614}_{-2.483})\times 10^{-3}\ \text{GeV}^6,$$

$$\langle \alpha_s G^2\rangle = 0.038\pm 0.011,\ \text{GeV}^4,$$

$$r_c \equiv \langle \bar{s}s\rangle/\langle \bar{q}q\rangle \qquad r_c = 0.74\pm 0.03$$

$$m_u\langle g_s\bar{u}\sigma TGu\rangle + m_d\langle g_s\bar{d}\sigma TGd\rangle = -(1.321\pm 0.033)\times 10^{-4}\left[\frac{\alpha_s(\mu)}{\alpha_s(2\ \text{GeV})}\right]^{14/(3\beta_0)}\ \text{GeV}^4$$

$$\Lambda_{\text{QCD}} = 0.22\ \text{GeV} \qquad \mu = 2\ \text{GeV}$$

evolution

$$\langle g_s^3 fG^3\rangle = (8.2\pm 1.0)\ \text{GeV}^2\times\langle \alpha_s G^2\rangle$$

Narison 2010

Results

$0.210 \pm 0.013\ (\mathrm{stat.}) \pm 0.034\ (\mathrm{sys.})$

from HOPE (lattice) 2022

- Moments

$$(\langle \xi^2 \rangle, \langle \xi^4 \rangle, \langle \xi^6 \rangle, \langle \xi^8 \rangle, \langle \xi^{10} \rangle, \langle \xi^{12} \rangle, \cdots)|_{\mu=2\,\mathrm{GeV}}$$
$$= (0.2672, 0.1333, 0.0871, 0.0658, 0.0546, 0.0480, \cdots)$$
$$\boxed{(0.2609, 0.1362, 0.0890, 0.0652, 0.0511, 0.0420, \cdots)}$$

- Can get all moments in principle
- Corresponding Gegenbauer coefficients

$$(a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi, a_{12}^\pi, \cdots)|_{\mu=2\,\mathrm{GeV}}$$
$$= (0.1960, 0.0268, 0.1918, 0.1376, 0.4034, -0.1319, \cdots)$$

bad convergence

Tikhonov regularization works!

- Solution with Tikhonov regularization

overcome ill-posedness

$$(a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi, a_{12}^\pi, \cdots, a_{32}^\pi, a_{34}^\pi)|_{\mu=2\,\mathrm{GeV}}$$
$$= (0.1775^{+0.0036}_{-0.0040}, 0.0957^{+0.0011}_{-0.0012}, 0.0762^{+0.0006}_{-0.0003}, 0.0688^{+0.0016}_{-0.0012}, 0.0643^{+0.0021}_{-0.0017}, 0.0603^{+0.0024}_{-0.0019}, \cdots, 0.0089^{+0.0004}_{-0.0006}, 0.0028^{+0.0001}_{-0.0003}),$$

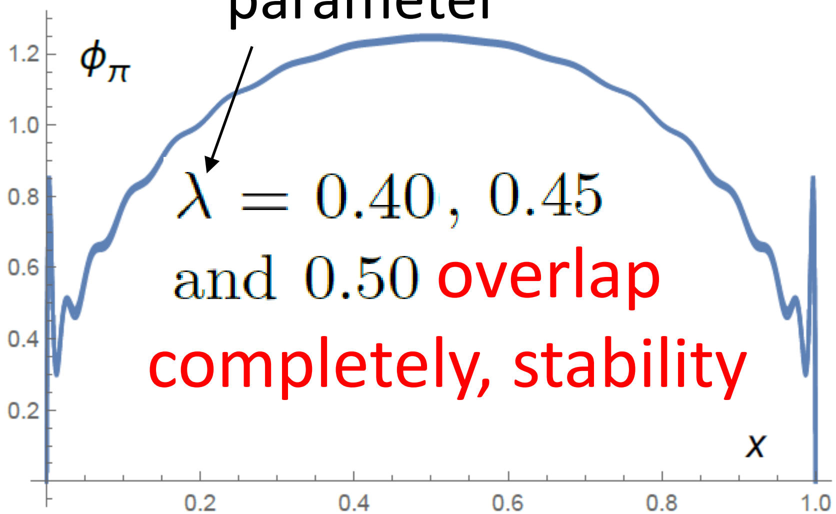
good convergence

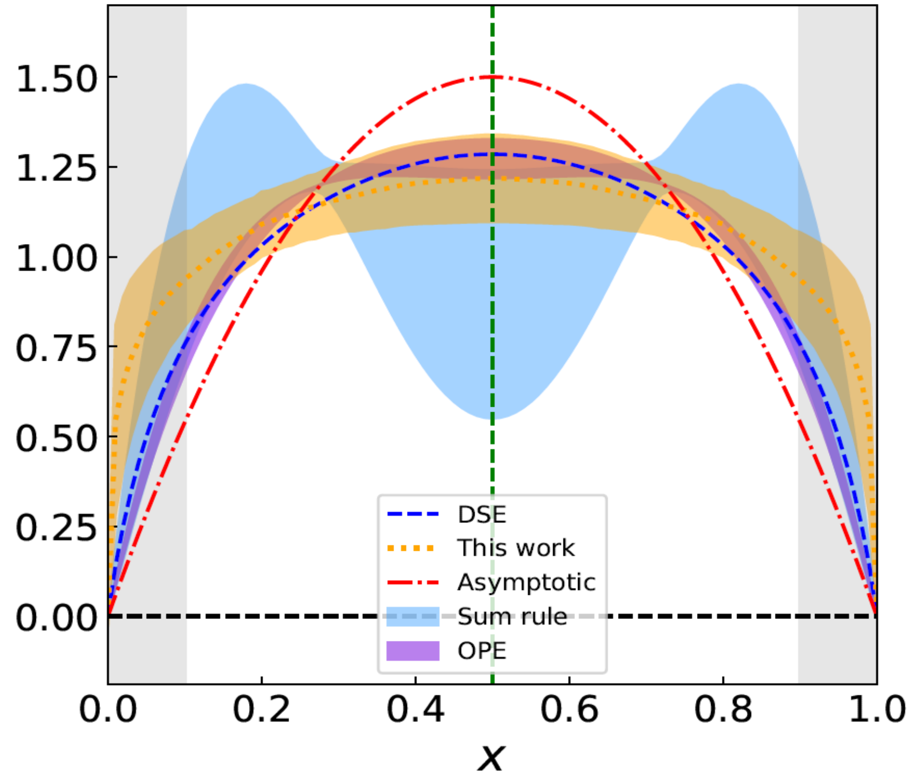
x dependence

- Sum over 18 Gegenbauer coefficients

Tikhonov regularization parameter

$\lambda = 0.40,\ 0.45$ and 0.50 overlap completely, stability





Hua et al 2021
from quasi-correlator

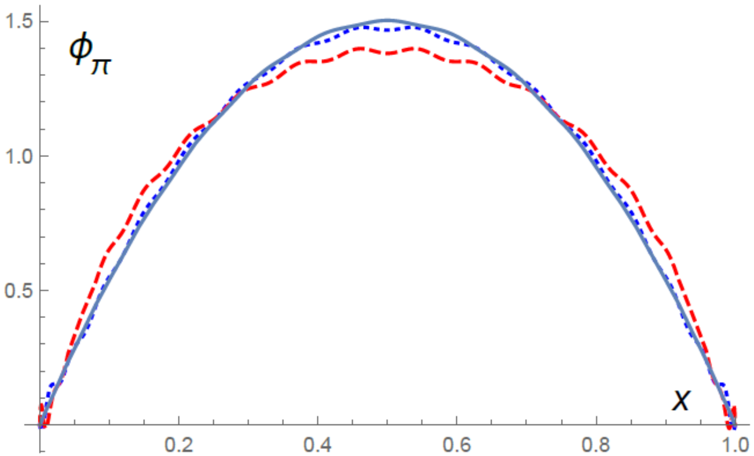
- Fit to parametrization

$$\frac{\Gamma(2p+2)}{\Gamma(p+1)^2}x^p(1-x)^p, \quad p = 0.45 \pm 0.02,$$

from variation of λ

Asymptotic limit

- Drop higher-power (condensate) contribution, keep only perturbative contribution



solid and dotted lines almost overlap stability

$\lambda = 0$ (solide line), 0.01 (dotted line) and 0.10 (dashed line)

$$(a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi, a_{12}^\pi, \cdots, a_{32}^\pi, a_{34}^\pi) \qquad \lambda = 0$$
$$= (\sim 0, \sim 0, \sim 0, \sim 0, \sim 0, 4.4 \times 10^{-9}, \cdots, 0.0001, -0.0001)$$

- This is asymptotic pion DA!

Results in various approaches

Methods	a_2^π	a_4^π	
This work	$0.1775^{+0.0036}_{-0.0040}$	$0.0957^{+0.0011}_{-0.0012}$	
Lattice QCD [13]	0.101 ± 0.023		RQCD, 2020
Lattice QCD [23]	0.258 ± 0.087	0.122 ± 0.055	Hua et al, 2022
Lattice QCD [63]	0.233 ± 0.065		Arthur et al, 2011
Lattice QCD [64]	0.136 ± 0.021		Braun et al, 2015
QCD sum rules [2]	$0.057^{+0.024}_{-0.019}$	$-0.013^{+0.022}_{-0.019}$	Stefanis, 2014
QCD sum rules [30]	$0.149^{+0.052}_{-0.043}$	$-0.096^{+0.063}_{-0.058}$	Bukulev et al, 2004
QCD sum rules [32]	0.157 ± 0.029	0.032 ± 0.007	Zhong et al, 2021
LFQM [65]	0.092 (0.038)	-0.002 (-0.020)	Choi, Ji, 2007
LCSR fit [68]	0.085	-0.020	Mikhailov et al, 2021
LCSR fit [70]	0.205 ± 0.036	0.125 ± 0.042	Cheng et al, 2020
Global fit [37]	0.491 ± 0.058	0.084 ± 0.029	Hua et al, 2021

Summary

- Have developed analytical nonpert framework that gives all moments of DA
- Have determined DA in entire x range unambiguously and reliably
- Precision can be improved systematically by including subleading contributions to OPE
- Can be extended to other meson DAs, like rho
- Results help extraction of GPD in hard exclusive electroproduction

Back-up slides

How?

- Typical Fredholm integral equation

notoriously difficult to solve

spectral density, unknown

$$\int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x) \longleftarrow \text{OPE input}$$

- Discretize integral equation usually

$$\sum_i M_{ij}\rho_j = \omega_i \qquad M_{ij} = \begin{cases} 1/(i-j), & i \neq j \\ 0, & i = j \end{cases}$$

unknowns input

- Rows Mij and M(i+1)j become almost identical and matrix M becomes singular quickly for fine meshes, solution diverges

Test with Mock data

- Consider sample DA and continuum functions

$$(a_0^\pi, a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi, \cdots) = (1, 0.20, -0.15, 0.10, 0, 0, \cdots)$$

$$\Delta\rho_{2n-2}(y) = ye^{-ny}, \quad n = 1, 2, \cdots$$

- Mock data for input

$$B_i^{(n)} = r_m^{i-1} \int_0^1 dy(2y-1)^{2n-2}\phi_\pi(y) + \int_0^\infty dy y^i e^{-ny}$$

pion mass

- Comparison with true solution

$$(\langle\xi^0\rangle, \langle\xi^2\rangle, \langle\xi^4\rangle, \langle\xi^6\rangle, \langle\xi^8\rangle, \langle\xi^{10}\rangle, \langle\xi^{12}\rangle)$$

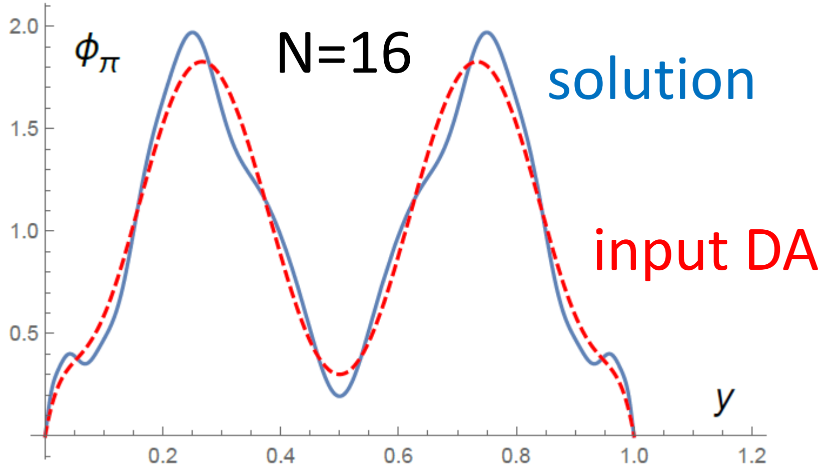
$$= (1, 0.2686, 0.1158, 0.0638, 0.0408, 0.0288, 0.0217)$$

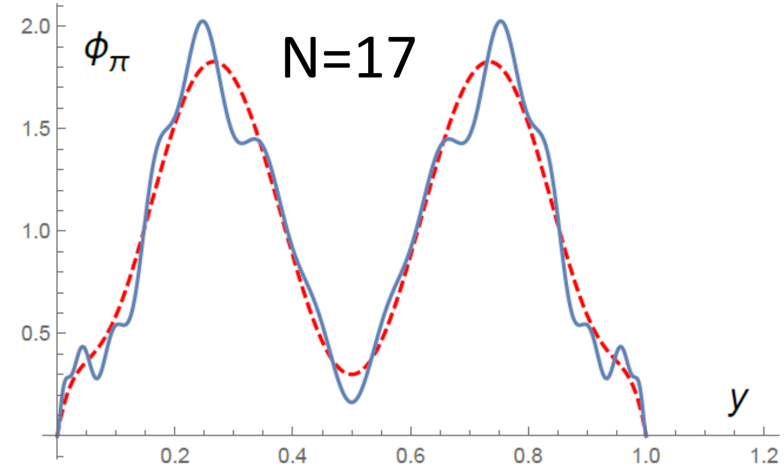
our solution 1, 0.2686, 0.1159, 0.0642, 0.0417, 0.0300, 0.0232

but Gegenbauer 1, 0.2001, -0.1496, 0.1119, 0.0306, -0.0233, 0.2339

Solutions for Gegenbauer without regularization

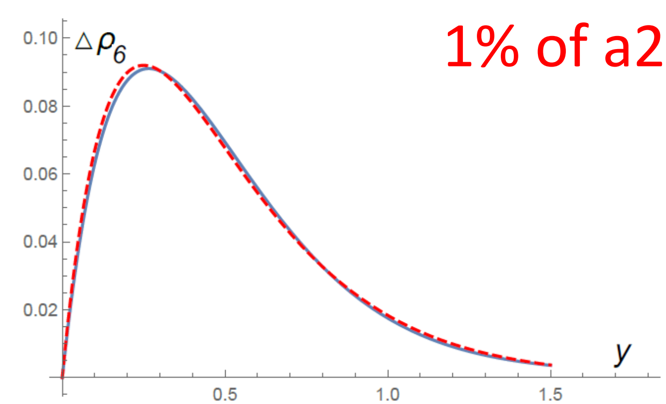
- Solutions stable as N>13, oscillate as N>17





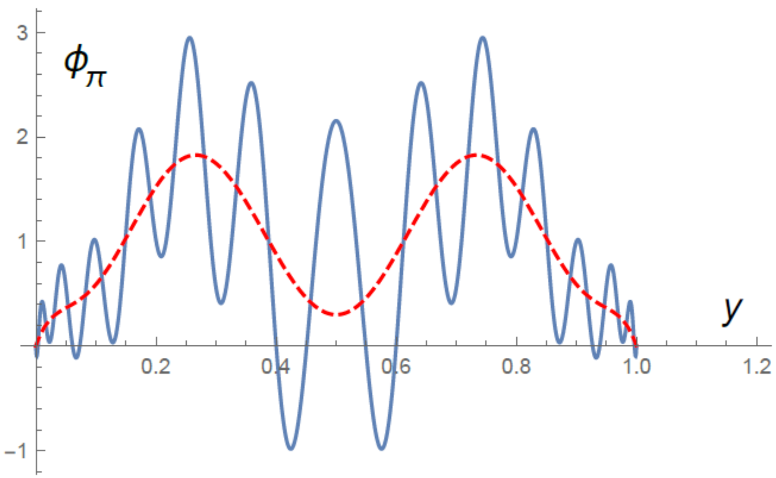
$$(a_0^\pi, a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi, a_{12}^\pi, \cdots, a_{28}^\pi, a_{30}^\pi)$$
$$= (1, 0.2000, -0.1472, 0.1212, 0.0335, -0.0059, -0.0098, \cdots, -0.0029, 0.0013)$$

- Continuum functions
- First two functions reproduced exactly



Add noise

- Enhance an element in input B by 0.05%
- Solution for x dependence of DA without Tikhonov regularization goes out of control completely

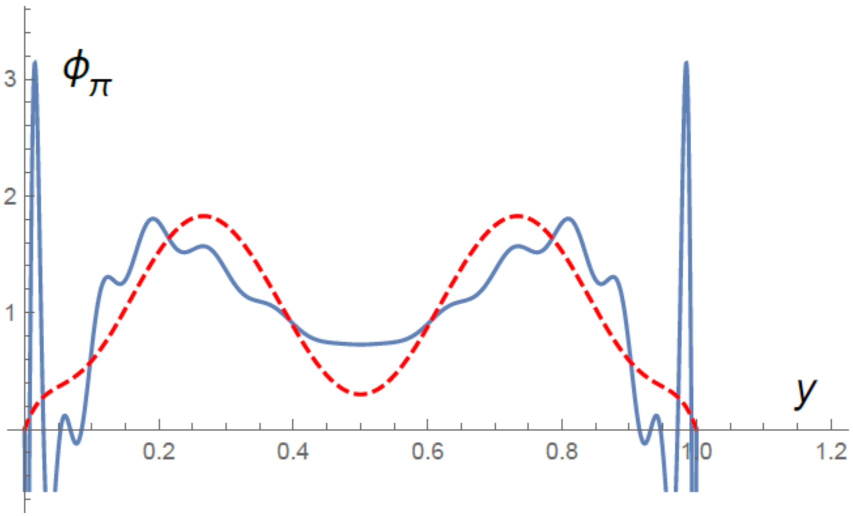


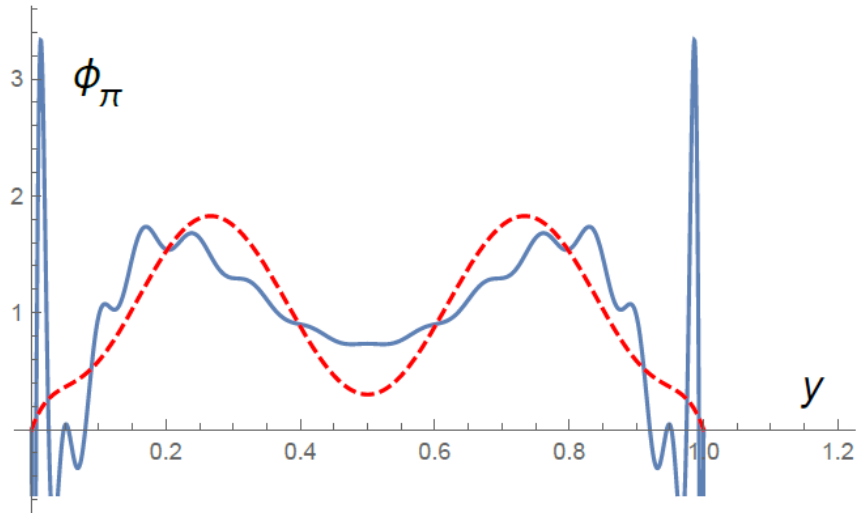
$\lambda = 0$ with $N = 16$ (solid line)
input one (dashed line)

- ill-pose nature

Solution under noise

- Implement Tikhonov regularization, shape of DA reproduced reliably





$\lambda = 0.0054$
$N = 15$

stability in λ

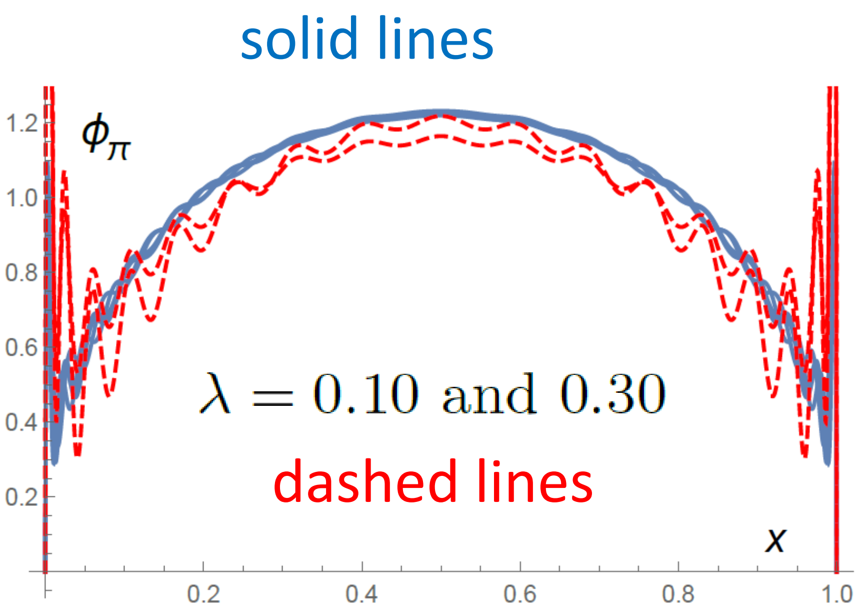
$\lambda = 0.0058$
$N = 16$

$$a_2^\pi = 0.1980,\ a_4^\pi = -0.1289,\ a_6^\pi = 0.0597, \ldots.$$

Compatible with QCD evolution

- Compare DA evolved from 2 GeV to 1.5 GeV with DA solved at 1.5 GeV
- Latter derived with OPE at 1.5 GeV

$$a_n^\pi(\mu) = a_n^\pi(\mu_0) E_n(\mu, \mu_0), \quad n = 2, 4, 6, \cdots \qquad E_n(\mu, \mu_0) = \left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{\gamma_n^{(0)}/(2\beta_0)}$$



$$\gamma_n^{(0)} = 8C_F\left[\psi(n+2) + \gamma_E - \frac{3}{4} - \frac{1}{2(n+1)(n+2)}\right]$$

curves at low scale bumpier due to larger condensates, stronger fluctuation