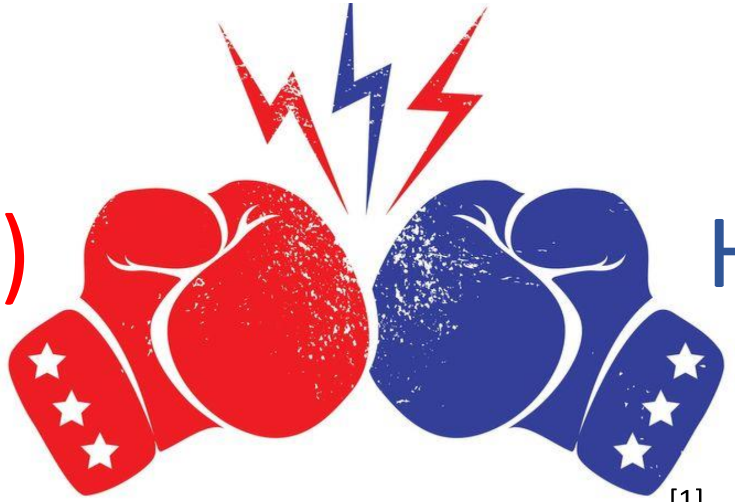
# $E=mc^2$ VS Symmetry for Lorentz Covariant Physics

$ISO(1,3)$



[1]

$H_R(1,3)$

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[1] https://www.jamf.com/blog/total-cost-of-ownership-mac-versus-pc-in-the-enterprise/

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# $E = mc^2$ versus symmetry for Lorentz covariant physics

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ABSTRACT

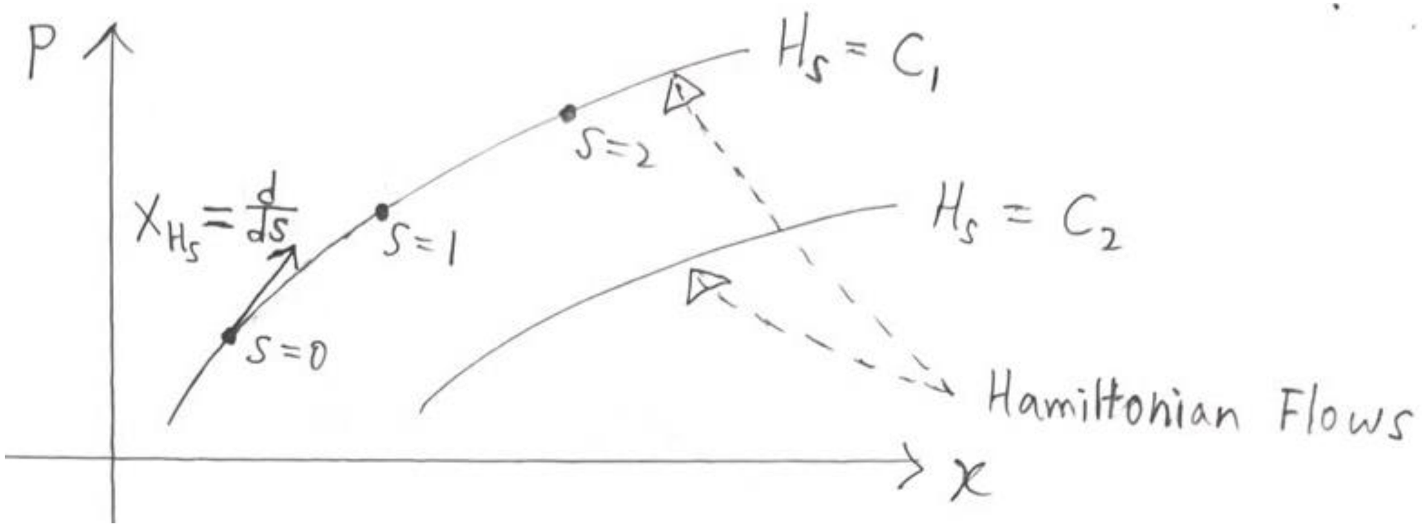
The famous equation $E = mc^2$ is a version of particle mass being essentially the magnitude of the (energy-)momentum four-vector in the setting of 'relativistic' dynamics, which can be seen as dictated by the Poincaré symmetry adopted as the relativity symmetry. However, as Einstein himself suggested, the naive notion of momentum as mass times velocity may not be right. The Hamiltonian formulation perspective gives exactly such a setting which in the case of motion of a charged particle under an electromagnetic field actually has the right, canonical, momentum four-vector with an evolving magnitude. The important simple result seems to have missed proper appreciation. In relation to that, we present clear arguments against taking the Poincaré symmetry as the fundamental symmetry behind 'relativistic' quantum dynamics, and discuss the proper symmetry theoretical formulation and the necessary picture of the covariant Hamiltonian dynamics with an evolution parameter that is, in general, not a particle proper time. In fact, it is obvious that the action of any position operator of a quantum state violates the on-shell mass condition. The phenomenologically quite successful quantum field theories are 'second quantized' versions of 'relativistic' quantum mechanics. We present a way for some reconciliation of that with our symmetry picture and discuss implications.

# Hamiltonian Dynamics

Generally, the **Hamiltonian function, $H_s$** :

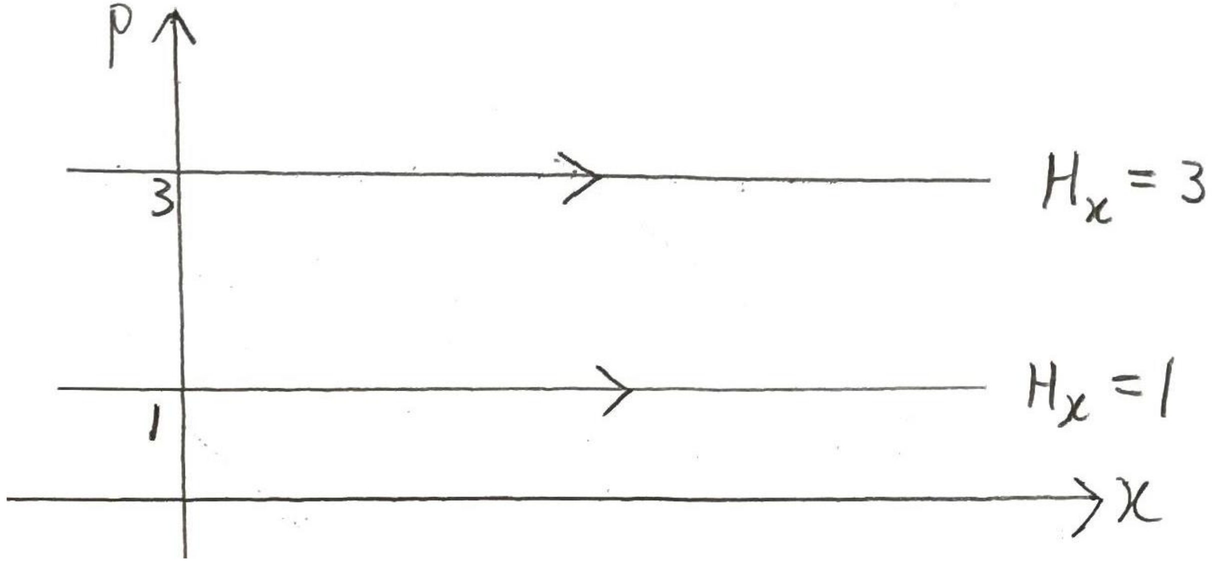
$$H_s = f(x, p)$$

- Smooth function on phase space
- Dictates an **evolution parameter, s**
- Generate **s-evolution of x & p on phase space (Hamiltonian flow)**

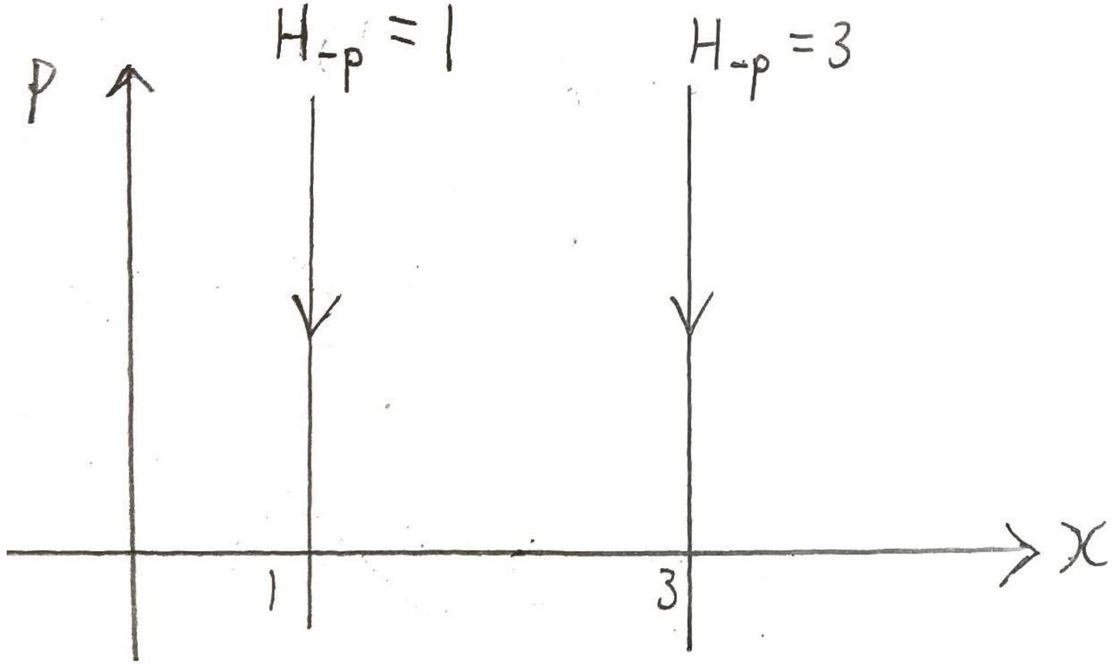


# Examples

(i) When $H_s = p$, then $s = x$




(ii) When $H_s = x$, then $s = -p$



# When $H_s$ is the physical Hamiltonian :

$$H_{s=t} = \frac{1}{2m}(p_i - \frac{e}{c}A_i)(p^i - \frac{e}{c}A^i) + V(x^i)$$

# s is the Newtonian time, t !

**Hamilton's Equations of Motion :**

$$\frac{dx^i}{dt} = \frac{\partial H_t}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H_t}{\partial x^i}$$

# Lorentz Covariant Hamiltonian

$$H_s = \frac{1}{2m}(p_\mu - \frac{e}{c}A_\mu)(p^\mu - \frac{e}{c}A^\mu) + V(x^\mu)$$

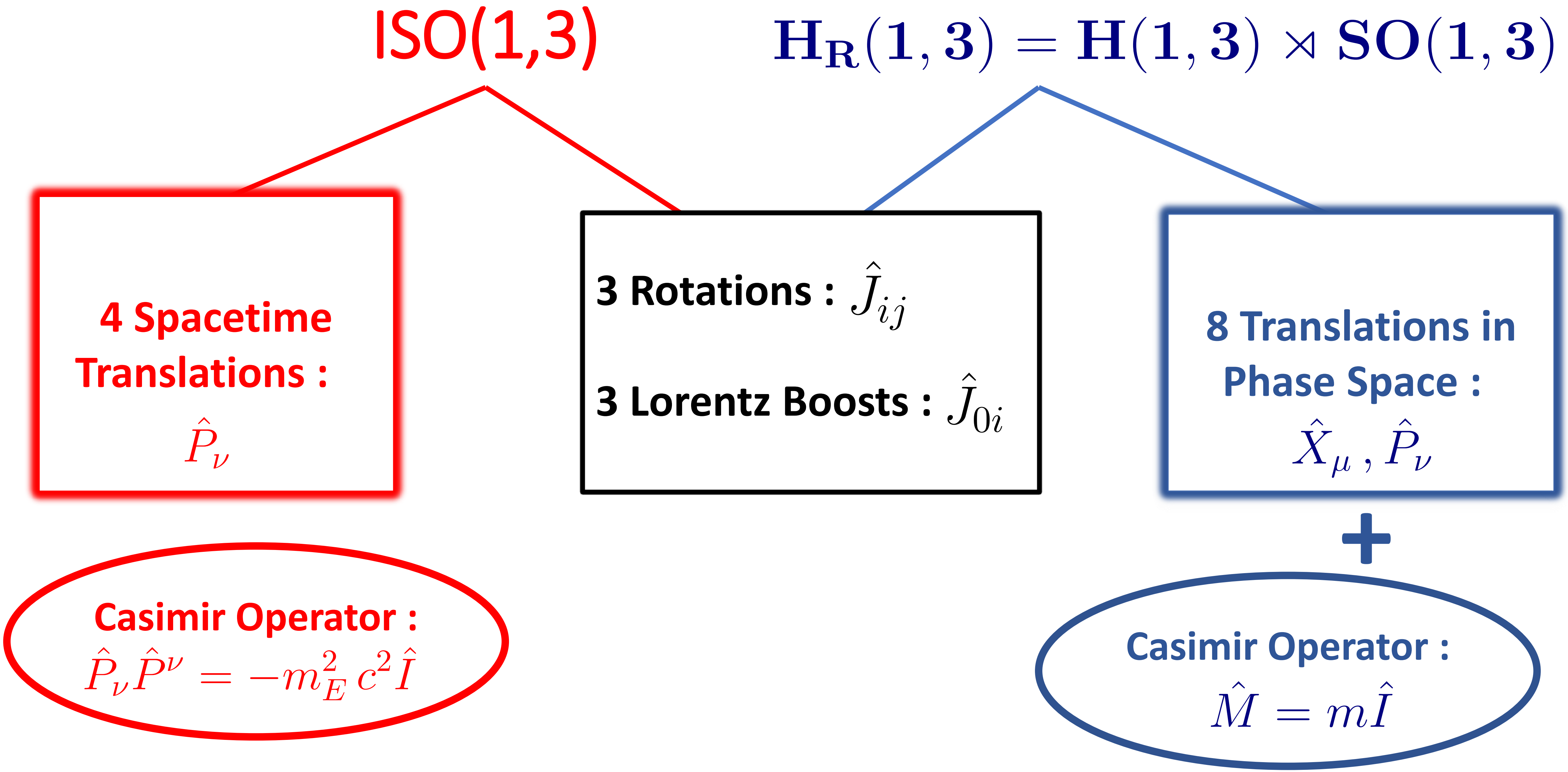
- **t is the spacetime coordinate, NOT evolution parameter !**
- **$H_s$** dictates an evolution parameter, **s**

**s is LORENTZ-FRAME INDEPENDENT !**

# Charged particle under Lorentz force

$$\mathbf{H_s} = \frac{\mathbf{1}}{\mathbf{2m}}(\mathbf{p}_\mu - \frac{\mathbf{e}}{\mathbf{c}}\mathbf{A}_\mu)(\mathbf{p}^\mu - \frac{\mathbf{e}}{\mathbf{c}}\mathbf{A}^\mu)$$

- **We've** $\mathbf{s = k\tau + C}$ **!**
- C is arbitrary constant (Can choose C=0)
- **s = (rescaled) proper time,** $\tau$
- For convenience, use simple notation: $H_s = H_{k\tau} \to H_\tau$


ISO(1,3)
$\mathbf{H_R(1,3) = H(1,3) \rtimes SO(1,3)}$
4 Spacetime Translations : $\hat{P}_\nu$
3 Rotations : $\hat{J}_{ij}$
3 Lorentz Boosts : $\hat{J}_{0i}$
8 Translations in Phase Space : $\hat{X}_\mu\,,\hat{P}_\nu$
+
Casimir Operator : $\hat{P}_\nu \hat{P}^\nu = -m_E^2\, c^2\, \hat{I}$
Casimir Operator : $\hat{M} = m\hat{I}$

Poincare Symmetry says :

$$p_\mu p^\mu = -m_E^2\, c^2$$

$$(\, \hat{P}_\mu \hat{P}^\mu = -m_E^2\, c^2\, \hat{I}\, )$$

- Momentum[2] is a Casimir Operator !
- Fixed the Wavefunction as $\phi(p^i)$ , NOT $\phi(p^\mu)$ !

# Imposing On-shell Mass (Charged Particle) :

$$H_\tau = \frac{1}{2m}(p_\mu - \frac{e}{c}A_\mu)(p^\mu - \frac{e}{c}A^\mu) \equiv \frac{1}{2m}\pi_\mu \pi^\mu$$

True Momentum !

Kinematic Momentum ?

- $p^\mu$ ($\neq mv^\mu$) is the true (canonical) momentum, not $\pi^\mu$ !
- $p_\mu p^\mu$ evolves with $\tau$
- **Using $\pi_\mu \pi^\mu = -\mathrm{m_E^2\, c^2}$ ? ( No Dynamical Content ! )**

# Quantum Version

$$\hat{H}_\tau = \frac{1}{2m}(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu)(\hat{P}^\mu - \frac{e}{c}\hat{A}^\mu) \equiv \frac{1}{2m}\hat{\pi}_\mu\hat{\pi}^\mu$$

- Hamilton's Equations: $\frac{d\hat{X}^\mu}{d\tau} = \frac{\partial \hat{H}_\tau}{\partial \hat{P}_\mu} \quad ; \quad \frac{d\hat{P}_\mu}{d\tau} = -\frac{\partial \hat{H}_\tau}{\partial \hat{X}^\mu}$

- **Problem 1 : NO $\hat{\mathrm{X}}^\mu$ from ISO(1,3) !**

- **Problem 2 : $\hat{\pi}^\mu$ is NOT the true (canonical) Momentum Operator !**

- Also, QM tells us that :

$$[\,\hat{\mathbf{X}}_\nu\,,\hat{\mathbf{P}}_\mu\hat{\mathbf{P}}^\mu\,] = \mathbf{2i}\hbar\,\hat{\mathbf{P}}_\nu \Rightarrow \hat{\mathbf{P}}_\mu\hat{\mathbf{P}}^\mu$$ **is NOT a Casimir Operator !**

- So, **ISO(1,3)** is **the WRONG symmetry** !

- **Then, why $H_R(1,3)$ is the right symmetry ?**

# Finding the Irr. Representations from a Symmetry

# Irr. Representations of Angular Momentum from SO(3) Lie Algebra

$$[\hat{J}_i, \hat{J}_j] = i\hbar\, \delta_{ij} \hat{J}_k$$

- Casimir Operator : $\hat{J}^2$
- Casimir Invariants : $j(j+1)\hbar^2, \text{ where } j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

- **Each j** value gives **1 irr. representation**

**e.g. j=1/2 :**

$$\hat{J}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{J}_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_z = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{J}_y = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

# *Then… What Does $H_R(1,3)$ Tell Us?*

- Casimir Operator : $\hat{M} = m\hat{I} \quad (\text{ from } [\hat{Y}_\mu , \hat{P}_\nu] = i\hbar\, \eta_{\mu\nu}\hat{M}\ )$

- Casimir Invariants : m (Newtonian Mass)

- **Each m** value gives **1 irr. representation**

- **On-shell condition** is just the **initial condition** applied when solving Hamilton's equations of motion for a **free particle only** !

- Lorentz Covariant Wavefunction : $\phi(p^\mu)$

## NEXT, WE CAN GO FURTHER & ASK :

# HOW DO WE LOOK AT QFT FROM $H_R(1,3)$ SYMMETRY ?

# QFT as an Irr. Representation of $H_R(1,3)$

- **m=0**
- H(1,3): $[\hat{Y}_\mu , \hat{P}_\nu] = i\hbar\, \eta_{\mu\nu}\, m\hat{I} = 0$
- 1-dim, Irr. Representation:

$$\hat{Y}_\mu \to y_\mu$$
$$\hat{P}_\mu \to p_\mu$$
$$\hat{M} \to 0$$

- 1-dim Hilbert space spanned by single $|p'_\mu\rangle$
- Unitary Transformation:

$$e^{\frac{i}{\hbar}(\tilde{v}^\mu y_\mu - \tilde{x}^\mu p_\mu)}\, |p'_\mu\rangle$$

# QFT as an Irr. Representation of $H_R(1,3)$

- Irr. Representation :

$$H(1,3) \xrightarrow[\text{Extend}]{} \text{spin-0 } H_R(1,3)$$

$$1 \text{ dim} \xrightarrow[\text{Extend}]{} \text{infinite dim}$$

- Representation space :

$$|p'_\mu\rangle \xrightarrow[\text{Extend}]{} \{\, |p'_\mu\rangle, |p''_\mu\rangle, \ldots \,\}$$

$$p'_\mu p'^\mu = p''_\mu p''^\mu = \ldots = -m_E^2 c^2$$

# QFT as an Irr. Representation of $H_R(1,3)$

- $-m_E^2 c^2$ are Casimir Invariants of $\hat{P}_\mu \hat{P}^\mu$
- Implementing 2nd Quantization :

$$\hat{a}_{\vec{p}} \left|0\right\rangle = 0$$

$$\hat{a}^\dagger_{\vec{p}} \left|0\right\rangle \sim \left|1_{\vec{p}}\right\rangle$$

$$\hat{a}^\dagger_{\vec{p}_1} \, \hat{a}^\dagger_{\vec{p}_2} \ldots \hat{a}^\dagger_{\vec{p}_n} \left|0\right\rangle \sim \left|1_{\vec{p}_1} \, 1_{\vec{p}_2} \ldots 1_{\vec{p}_n}\right\rangle$$

- **Define :**

$$\hat{\phi}(x) \equiv \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} (e^{-ipx} \, \hat{a}_{\vec{p}} + e^{ipx} \, \hat{a}^\dagger_{\vec{p}})$$

$$E_{\vec{p}} \equiv c\sqrt{|\vec{p}|^2 + m_E^2 c^2}$$

# CONCLUSION

QFT from *$H_R(1,3)$ Symmetry* :

(i) m=0 Irr. Representation of $H_R(1,3)$

(ii) Heisenberg-Weyl Commutator = 0

(iii) Indefinite-number Particle Theory

| | ISO(1,3) | $H_R(1,3)$ |
|---|---|---|
| $\hat{X}_\mu$ | × | ✓ |
| $[\,\hat{X}_\mu\,,\,\hat{P}_\nu\,] = i\hbar\,\eta_{\mu\nu}\,\hat{I}$ | × | ✓ |
| **On-shell Mass Condition** | **Fixed the Irr. Representations** | **Initial Condition for Free Particle** |
| **Charged Particle Under Lorentz Force** | $\pi_\mu \pi^\mu \neq -m_E^2\, c^2$ | **NO On-shell Mass Constraint** |
| **Wavefunction** | $\phi(p^i)$ | $\phi(p^\mu)$ |

- **ISO(1,3) is the WRONG identification of relativity symmetry**
- **$H_R(1,3)$ should be identified as the RIGHT symmetry**