Quantum mechanical description of heavy quark transport

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ExHIC-p workshop

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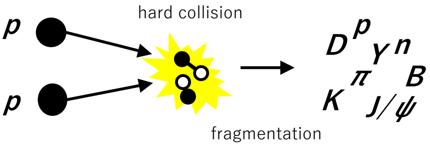
Introduction

Quarkonium production in experiments

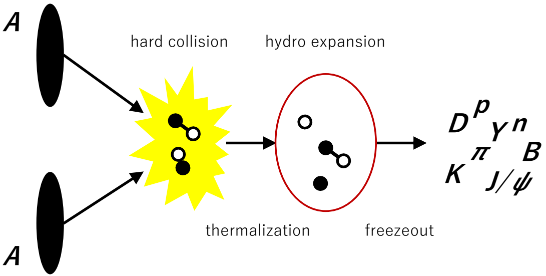
Proton collisions

- Pair production in parton collisions
- Some are bound



Nuclear collisions

- Pair production in parton collisions
- Bound states formed after traversing hot environment

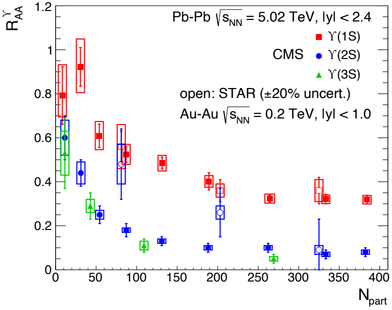


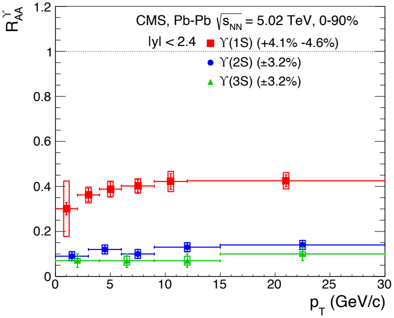
Experimental test of how color forces are modified in the QGP

Experimental data of Υ

Sequential melting is observed

- Almost no p_T dependence
- Υ (1S) suppression at STAR $\simeq$ CMS ?



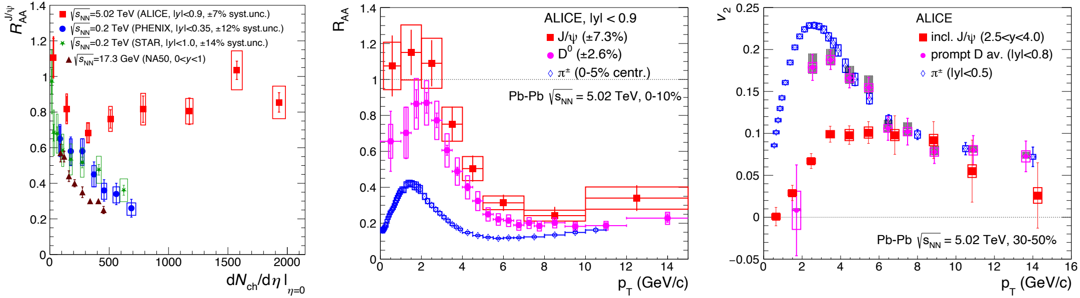


One-body problem of bottomonium

Experimental data of J/ψ

Recombination begins to dominate at LHC

- Recombination of (initially independent) charm quark pairs at low p_T
- Collective flow at low to intermediate p_T supports this picture

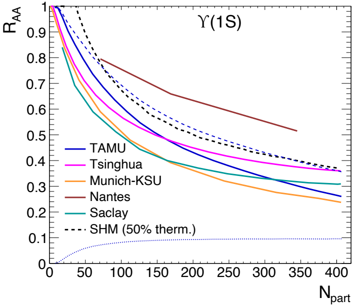


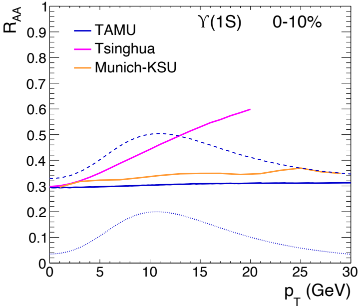
Many-body problem of charm and anticharm quarks

Simulations

Υ is simpler, but there exist many models, some of which are based on open quantum systems

- Not all the models use the same initial conditions or hydro backgrounds
- Detailed descriptions are given in [Andronic et al 2402.04366]





I will talk about how open quantum system descriptions are developed

Minimum of open quantum system

[reviewed in Akamatsu (22), §2]

Basics of open quantum system

System + environment setup $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$

$$\rho_S(t) = \mathrm{Tr}_E\left[\rho(t)\right], \quad \dot{\rho}(t) = -i\left[H, \rho\right] \quad : \quad \text{von-Neumann equation}$$

Gorini-Kossakowski-Sudarshan-Lindblad (or Lindblad) equation for $\rho_S(t)$

- If the evolution is Markovian, trace-preserving, and completely positive

$$\dot{\rho}_S(t) = -i[H'_S, \rho_S] + \underbrace{\sum_\mu L_\mu \rho_S L_\mu^\dagger - \frac{1}{2}\left\{L_\mu^\dagger L_\mu, \rho_S\right\}}_{\text{dissipator } \mathcal{D}[\rho_S]}$$

- It can also be written by

$$\dot{\rho}_S(t) = \underbrace{-iH_{\text{eff}}\rho_S + i\rho_S H_{\text{eff}}^\dagger}_{\text{non-Hermitian evolution}} + \sum_\mu \underbrace{L_\mu \rho_S L_\mu^\dagger}_{\text{quantum jump}}, \quad H_{\text{eff}} = H'_S - \frac{i}{2}\sum_\mu L_\mu^\dagger L_\mu$$

Derivation of the Lindblad equation when system-environment coupling is weak

Born-Markov approximation

0. Notations and assumptions

$$H = H_S \otimes I_E + I_E \otimes H_E + \underbrace{\sum V_S^{(i)} \otimes V_E^{(i)}}_{\equiv V}, \quad \underbrace{\rho(0) = \rho_S(0) \otimes \rho_E(0)}_{\text{no initial correlation}}$$

1. Formally solve the von Neumann equation in the interaction picture

$$\rho(t) = \rho(0) - i \int_0^t ds [V(s), \rho(s)],$$

$$\frac{d}{dt}\rho(t) = -i[V(t), \rho(t)] = -i[V(t), \rho(0)] - \int_0^t ds [V(t), [V(s), \rho(s)]]$$

2. Trace out environment + Born approx. (weak coupling) + Markov approx. (see any textbook)

$$\frac{d}{dt}\rho_S(t) = \int_0^\infty ds \underbrace{\langle V_E(s) V_E(0) \rangle}_{\text{env. correlator}} \Big[V_S(t-s)\rho_S(t)V_S(t) - V_S(t)V_S(t-s)\rho_S(t) \Big] + h.c. + \mathcal{O}(V^3)$$

- If one point function $\mathrm{Tr}_E(\rho_E(0)V_E(t)) = c(t)$ exists, reshuffle $c(t)V_S(t)$ in H

Derivative expansion in quantum Brownian regime

3. Derivative expansion

$$\frac{d}{dt}\rho_S(t) = \int_0^\infty ds \underbrace{\langle V_E(s)V_E(0)\rangle}_{\text{fast decay}} \Big[\underbrace{V_S(t-s)}_{\text{slow}}\rho_S(t)V_S(t) - V_S(t)V_S(t-s)\rho_S(t)\Big] + h.c. + \mathcal{O}(V^3)$$

$$V_S(t-s) \simeq V_S(t) - s\dot{V}_S(t) + \cdots = V_S(t) - is[H_S, V_S(t)] + \cdots$$

4. Lindblad operator (c.f. Caldeira-Leggett model $L \propto x + \frac{i}{4T}\dot{x}$)

$$\gamma = \int_{-\infty}^{\infty} dt \, \langle V_E(t)V_E(0)\rangle > 0 \quad \to \quad \underbrace{L = \sqrt{\gamma}\left(V_S + \frac{i}{4T}\dot{V}_S + \cdots\right)}_{\text{approximate detailed balance}}$$

5. Hamiltonian part (omitted in this talk)

Heavy quark as an open quantum system

[reviewed in Akamatsu (22), §3.2]

Effective field theory for a heavy quark in medium

Heavy quark effective theory with $v^\mu = (1,0,0,0)$ in a thermal rest frame

- Lagrangian with $1/M$ expansion

$$\mathcal{L}_{\text{HQET}} = \psi^\dagger \left[iD_t + \frac{\vec{D}^2}{2M} - \frac{g\vec{\sigma}\cdot\vec{B}}{2M} + \mathcal{O}(M^{-2}) \right] \psi$$

- Power counting near equilibrium $A \sim T$ and $\partial_t \sim T, \nabla \sim \sqrt{MT}$ for ψ

$$\mathcal{L}_{\text{HQET}} \simeq \psi^\dagger \left[i\partial_t + \frac{\vec{\nabla}^2}{2M} - gA_0 + \cdots) \right] \psi$$

Born-Markov approximation requires weak coupling $g \ll 1$

Lindblad equation for a heavy quark [Akamatsu (15, 20)]

Non-relativistic quantum mechanics

$$H(x,p) = \frac{p^2}{2M} + gA_0(x) = \frac{p^2}{2M} + \underbrace{\int_k e^{ikx} t^a \otimes g\tilde{A}_0^a(k)}_{= \sum_a \int_k V_S^a(k) \otimes V_E^a(k)}$$

Environmental correlator

$$\int_{-\infty}^{\infty} dt \underbrace{\left\langle g\tilde{A}_0^a(t,k) g\tilde{A}_0^b(0,k') \right\rangle}_{\text{perturbatively with HTL}} = \tilde{\gamma}(k)(2\pi)^3 \delta^3(k-k')\delta_{ab}$$

Lindblad operator

$$L_k^a = \underbrace{\sqrt{\frac{\pi g^2 T m_D^2}{k(k^2+m_D^2)^2}}}_{=\sqrt{\tilde{\gamma}(k)}} \underbrace{e^{ikx/2}\left(1 - \frac{kp}{4MT}\right) e^{ikx/2} t^a}_{\text{scattering with color rotation}} + \cdots$$

Evolution in color space

Simplified in the $M \to \infty$ limit

$$\rho_S^{ij}(t,x,y) \equiv \langle x, i|\rho_S(t)|y, j\rangle \quad : \quad N_c \times N_c \text{ matrix}$$
$$\partial_t \rho_S(t,x,y) = -\gamma(x-y) t^a \rho(t,x,y) t^a + C_F \gamma(0) \rho(t,x,y)$$

Equivalent to random color rotation

$$\rho_S(t,x,y) \equiv \langle \psi(t,x)\psi^*(t,y)\rangle_\theta \quad : \quad \psi = \text{wave function with } N_c \text{ components}$$
$$\psi(t+dt,x) = e^{-i\theta^a(t,x)t^a dt}\psi(t,x), \quad \langle \theta^a(t,x)\theta^b(t',x')\rangle = \gamma(x-y)\delta(t-t')\delta^{ab}$$

Time scale of color relaxation

$$\gamma(0) = \frac{g^2 T}{4\pi} \quad \to \quad \underbrace{\tau_{\rm color} \sim \frac{1}{g^2 T} \gg \tau_{\rm electric} \sim \frac{1}{gT}}_{\partial\text{-expansion justified when } g \ll 1}$$

Evolution in phase space

Master equation in full space is complicated, so trace out the color space

$$\bar{\rho}_S(t,x,y) \equiv \mathrm{Tr}_c \rho_S(t,x,y)$$

Turns out to be equivalent to

$$L_k^a \quad \to \quad L_k = \sqrt{C_F \tilde{\gamma}(k)}\, \underbrace{e^{ikx/2}\left(1 - \frac{kp}{4MT}\right) e^{ikx/2}}_{\text{(color averaged) scattering}}$$

Master equation for quantum Brownian motion (truncating from the full Lindblad equation)

$$\frac{\partial}{\partial t}\bar{\rho}_S(t,x,y) = i\frac{\nabla_x^2 - \nabla_y^2}{2M}\bar{\rho}_S(t,x,y) - C_F\Big[\underbrace{\gamma(0) - \gamma(x-y)}_{\text{decoherence}} - \underbrace{\frac{\nabla\gamma(x-y)}{4MT}\cdot(\nabla_x - \nabla_y)}_{\text{dissipation}}\Big]\bar{\rho}_S(t,x,y)$$

- Wave packet limit $\gamma(x-y) \simeq \gamma(0) + \frac{1}{2}(x-y)^2\gamma''(0)$ reproduces Caldeira-Leggett master equation

Quarkonium as an open quantum system

[reviewed in Akamatsu (22), §4.2]

Effective field theory for quarkonium

Potential non-relativistic QCD

- Lagrangian with $1/M$ and dipole expansion $(p_i \sim Mv,\ \omega \sim P_i \sim Mv^2)$

$$\mathcal{L}_{\rm pNRQCD} = \int_r {\rm Tr_c}\left[{\rm S}^\dagger\left(i\partial_t + \frac{\nabla_r^2}{M} + \frac{C_F\alpha_s}{r} + \cdots\right){\rm S} + {\rm O}^\dagger\left(iD_t + \frac{\nabla_r^2}{M} - \frac{\alpha_s}{2N_c r} + \cdots\right){\rm O}\right]$$
$$+ {\rm Tr_c}\left[{\rm O}^\dagger \vec{r}\cdot g\vec{E}{\rm S} + {\rm S}^\dagger \vec{r}\cdot g\vec{E}{\rm O}\right] + \frac{1}{2}{\rm Tr_c}\left[{\rm O}^\dagger \vec{r}\cdot g\vec{E}{\rm O} + {\rm O}^\dagger{\rm O}\vec{r}\cdot g\vec{E}\right] + \cdots,$$
$${\rm S}(t,R,r) \equiv \frac{S(t,R,r)}{\sqrt{N_c}}\mathbf{1}, \quad {\rm O}(t,R,r) \equiv \sqrt{2}O^a(t,R,r)t_F^a$$

- Gauge interaction of octet quarkonium is absorbed in field redefinition $(D_t \to \partial_t)$

$${\rm O}(t,R,r) = \Omega_{\rm F}(t,-\infty){\rm O}'(t,R,r)\Omega_{\rm F}^\dagger(t,-\infty), \quad \vec{E}(t,R) = \Omega_{\rm F}(t,-\infty)\vec{E}'(t,R)\Omega_{\rm F}^\dagger(t,-\infty)$$
$$\Omega_{\rm F}(t,-\infty) \equiv {\rm P}\exp\left[-ig\int_{-\infty}^t dt' A_0^a(t',R)t_{\rm F}^a\right],$$

Born-Markov approximation requires r to be short, but not necessarily $g \ll 1$

Lindblad equation for quarkonium [Brambilla et al (17,18)]

Non-relativistic quantum mechanics (s = singlet, $a, b, c, \cdots$ = octet)

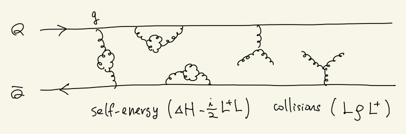
$$H = \frac{p^2}{M} - \frac{C_F\alpha_s}{r}|s\rangle\langle s| + \frac{\alpha_s}{2N_c r}|a\rangle\langle a| - r_i\Big[\underbrace{\sqrt{\frac{1}{2N_c}}\left(|a\rangle\langle s| + |s\rangle\langle a|\right)}_{\text{singlet} \leftrightarrow \text{octet}} + \underbrace{\frac{1}{2}d^{abc}|b\rangle\langle c|}_{\text{octet} \leftrightarrow \text{octet}}\Big] gE_i^a(\vec{R})$$

Environmental correlator

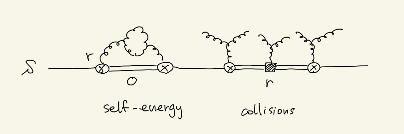
$$\int_{-\infty}^{\infty} dt\, \underbrace{\left\langle gE_i'^a(t,R)gE_j'^b(0,R)\right\rangle}_{\text{gauge invariant?}} = \gamma\delta_{ij}\delta_{ab}$$

Lindblad operator

$$L_{ai} = \sqrt{\gamma}r_i\left[\sqrt{\frac{1}{2N_c}}\left(|a\rangle\langle s| + |s\rangle\langle a|\right) + \frac{1}{2}d_{abc}|b\rangle\langle c|\right] + \cdots$$



small dipole →



Careful treatment of field redefinition

Gauge invariant correlator for singlet-octet rate [Yao (22)]

$$\gamma_{\rm so} = \frac{1}{3(N_c^2-1)} \int_{-\infty}^{\infty} dt \left\langle g E_i^a(t,R) [\Omega_{\rm A}(t,0)]_{ab} g E_i^b(0,R) \right\rangle$$

- $\gamma_{\rm so}$ vanishes in strong coupling limit by AdS/CFT [Nijs-ScheihingHitschfeld-Yao (23)]
- $\gamma_{\rm so}$ is similar to but different from heavy quark momentum diffusion constant [CasalderreySolana-Teaney (06), CaronHuot-Moore (08)]

$$\kappa = \frac{1}{3N_c} \int_{-\infty}^{\infty} dt \left\langle {\rm Tr}_c \Omega_{\rm F}(-\infty,t) g E_i(t,R) \Omega_{\rm F}(t,0) g E_i(0,R) \Omega_{\rm F}(0,-\infty) \right\rangle$$

Octet-octet rate is also different from $\gamma_{\rm so}$ [Akamatsu, in progress]

$$\gamma_{\rm oo} = \frac{1}{3(N_c^2-1)} \int_{-\infty}^{\infty} dt \left\langle {\rm Tr}_c \Omega_{\rm A}(-\infty,t) g(d\circ E_i)(t,R) \Omega_{\rm A}(t,0) g(d\circ E_i)(0,R) \Omega_{\rm A}(0,-\infty) \right\rangle$$

$$d^{abc} E_i^a =: (d\circ E_i)_{bc}$$

Transitions of quarkonium

Lindblad operators

- Color dynamics inside octet is fast in a fixed basis $\rightarrow$ trace out the octet sector

$$\rho_S^{\rm s} = \langle s|\rho_S|s\rangle, \quad \rho_S^{\rm o} = \langle a|\rho_S|a\rangle, \quad \bar{\rho}_S = \begin{pmatrix} \rho_S^{\rm s} & 0 \\ 0 & \rho_S^{\rm o} \end{pmatrix}$$

- Lindblad operators split into three

$$L_{i,{\rm s}\to{\rm o}} = \sqrt{\gamma_{\rm so}}\left[\sqrt{C_F} r_i + \cdots\right]\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_{i,{\rm o}\to{\rm s}} = \sqrt{\gamma_{\rm so}}\left[\frac{r_i}{\sqrt{2N_c}} + \cdots\right]\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$L_{i,{\rm o}\to{\rm o}} = \sqrt{\gamma_{\rm oo}}\left[\frac{r_i}{2} + \cdots\right]\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Terms "$\cdots$" are responsible for satisfying the approximate detailed balance relation

Discussion on heavy quark spin from the open system perspective

[Akamatsu, in progress]

Heavy quark spin as an open quantum system

Heavy quark effective theory with $v^\mu = (1,0,0,0)$ in a thermal rest frame

- Lagrangian with $1/M$ expansion

$$\mathcal{L}_{\mathrm{HQET}} = \psi^\dagger \left[iD_t + \frac{\vec{D}^2}{2M} - \frac{g\vec{\sigma}\cdot\vec{B}}{2M} + \mathcal{O}(M^{-2}) \right] \psi$$

- Focus on spin dynamics of a static heavy quark

$$\mathcal{L}_{\mathrm{HQET}} \simeq \psi^\dagger \left[iD_t - \frac{g\vec{\sigma}\cdot\vec{B}}{2M} \right] \psi$$

Gauge interaction of the heavy quark is absorbed in field redefinition

$$\psi(t,x) = \Omega_{\mathrm{F}}(t,-\infty)\psi'(t,x), \quad \vec{B}(t,x) = \Omega_{\mathrm{F}}(t,-\infty)\vec{B}'(t,x)\Omega_{\mathrm{F}}^\dagger(t,-\infty)$$

$$\Omega_{\mathrm{F}}(t,-\infty) \equiv \mathrm{P}\exp\left[-ig\int_{-\infty}^{t} dt' A_0^a(t',R)t_{\mathrm{F}}^a \right],$$

Born-Markov approximation requires $M \gg T$, but not necessarily $g \ll 1$

Lindblad equation for heavy quark spin

Only non-relativistic spin remains

$$H = \frac{1}{2M}\sigma_i t^a \otimes gB_i'^a(x)$$

Environmental correlator

$$\int_{-\infty}^{\infty} dt \underbrace{\left\langle gB_i'^a(t,x) gB_j'^b(0,x)\right\rangle}_{\text{gauge invariant?}} = \gamma_{\rm s}\delta_{ij}\delta_{ab}$$

Lindblad operators

$$L_{ai} = \frac{\sqrt{\gamma_{\rm s}}}{2M}\underbrace{(\sigma_i t^a + \cdots)}_{\text{spin \& color rot.}} = \frac{\sqrt{\gamma_{\rm s}}}{2M}\sigma_i t^a \ (\because H_S = 0)$$

Careful treatment of field redefinition

Gauge invariant correlator for spin relaxation rate

$$\gamma_{\rm s} = \frac{2}{3(N_c^2-1)} \int_{-\infty}^{\infty} dt \, \langle {\rm Tr_c} \Omega_{\rm F}(-\infty, t) g B_i(t,0) \Omega_{\rm F}(t,0) g B_i(0,0) \Omega_{\rm F}(0,-\infty) \rangle$$

Lindblad operators

- Color dynamics is fast in a fixed basis $\rightarrow$ trace out color space $\bar{\rho}_S \equiv {\rm Tr_c} \rho_S$

$$L_{ai} \rightarrow L_i = \frac{\sqrt{C_F \gamma_{\rm s}}}{2M} \underbrace{\sigma_i}_{\text{spin rot.}}$$

- Convenient to introduce $\kappa_{\rm s} \equiv C_F \gamma_{\rm s}$

$$\kappa_{\rm s} = \frac{1}{3N_c} \int_{-\infty}^{\infty} dt \, \langle {\rm Tr_c} \Omega_{\rm F}(-\infty, t) g B_i(t,0) \Omega_{\rm F}(t,0) g B_i(0,0) \Omega_{\rm F}(0,-\infty) \rangle$$

$$\text{c.f.} \quad \kappa = \frac{1}{3N_c} \int_{-\infty}^{\infty} dt \, \langle {\rm Tr_c} \Omega_{\rm F}(-\infty, t) g E_i(t,R) \Omega_{\rm F}(t,0) g E_i(0,R) \Omega_{\rm F}(0,-\infty) \rangle$$

Can anyone measure $\kappa_{\rm s}$ on the lattice? $\rightarrow$ Yes, measured as $1/M$-correction for κ

Heavy quark spin relaxation rate

Lindblad equation

$$\frac{\partial}{\partial t}\bar{\rho}_S = \frac{\kappa_{\rm s}}{4M^2}\left(\sigma_j \bar{\rho}_S \sigma_j - 3\bar{\rho}_S\right)$$

Relaxation of the averaged spin

$$\langle S_i \rangle \equiv \mathrm{Tr}_{\rm s}\left(\bar{\rho}_S \frac{1}{2}\sigma_i\right), \quad \frac{d}{dt}\langle S_i \rangle = -\frac{\kappa_s}{M^2}\langle S_i \rangle$$

Agrees with hydrodynamic derivation [Hongo-Huang-Kaminski-Stephanov-Yee (22)]

$$\Gamma_{\rm s} = \frac{1}{6T\chi_s} G_{12}^{\Theta_i \Theta_i}(\omega \to 0, k = 0), \quad \vec{\Theta} \equiv -\frac{g}{2M}\psi^\dagger (\vec{B} \times \vec{\sigma})\psi$$

Heavy quark spin polarization

Equilibrium environment

- It is tempting to think that we chose

$$\mathcal{H} = [\mathcal{H}_{\text{spin}} \otimes \mathcal{H}_{\text{color}}]_S \otimes [\mathcal{H}_{\text{QGP}}]_E \quad \to \quad \langle \vec{B}^a(t,x) \rangle = \vec{0}$$

- However, we traced out heavy quark color, which is equivalent to

$$\mathcal{H} = [\mathcal{H}_{\text{spin}}]_S \otimes [\mathcal{H}_{\text{color}} \otimes \mathcal{H}_{\text{QGP}}]_E \quad \to \quad \langle \vec{B}(t,x) \rangle = \vec{0}$$

  It is natural because color rotation is determined by temporal Wilson line

Vortical environment $\vec{\omega} \neq \vec{0}$

- Heavy quark color state and QGP with vorticity are correlated

$$\underbrace{\vec{B}_{\text{eff}}(\vec{\omega})}_{\text{U(1)}} \equiv \langle \vec{B}(t,x) \rangle_\omega = \lambda \vec{\omega} + \mathcal{O}(\omega^3),$$

Recall: The effect of one point function $\mathrm{Tr}_E(\rho_E(0) V_E(t)) = c(t)$ in the Born-Markov approximation

Lindblad operator with approximate detail balance

1. Reshuffle the one-point function

$$H = \frac{1}{2M}\vec{\sigma}\otimes g\vec{B}(x) = \underbrace{\frac{1}{2M}\vec{\sigma}\cdot g\vec{B}_{\rm eff}(\vec{\omega})}_{=H_S(\vec{\omega})}\otimes I_E + \frac{1}{2M}\vec{\sigma}\otimes g\underbrace{\left(\vec{B}(x)-\vec{B}_{\rm eff}(\vec{\omega})\right)}_{\equiv\Delta\vec{B}(x)}$$

2. Rate in vortical environment

$$\kappa_{\rm s}(\vec{\omega}) = \frac{1}{3N_c}\int_{-\infty}^{\infty}dt\,\langle {\rm Tr_c}\Omega_{\rm F}(-\infty,t)g\Delta B_i(t,0)\Omega_{\rm F}(t,0)g\Delta B_i(0,0)\Omega_{\rm F}(0,-\infty)\rangle_\omega = \kappa_{\rm s}+\mathcal{O}(\omega^2)$$

3. System-environment coupling

$$L_k(\vec{\omega}) = \frac{\sqrt{\kappa_{\rm s}(\vec{\omega})}}{2M}\left(\vec{\sigma}+\underbrace{\frac{i}{4MT}g\vec{B}_{\rm eff}(\vec{\omega})\times\vec{\sigma}}_{=i\dot{\vec{\sigma}}/4T}\right)_k \simeq \frac{\sqrt{\kappa_{\rm s}}}{2M}\left(\vec{\sigma}+\frac{ig\lambda}{4MT}\vec{\omega}\times\vec{\sigma}\right)_k+\mathcal{O}(\omega^2)$$

Conclusion and outlook on heavy quark spins

Theoretical description

- In the static limit, the Lindblad equation can be derived nonperturbatively
- In the weak rotation, κ_{s} and λ characterizes the dynamics
- Renormalization of the Hamiltonian by coupling to environment
- Need to examine possible interplays with and complications from other $1/M$ effects
- Matching between QCD and static HQET should be considered
- Collaborations welcome!

Experimental issues

- Initial spin polarization of heavy quarks?
- Hadronic phase?

Back up

Approximate detailed balance in quantum Brownian regime

Ratio of the rates for $E_1 \to E_2$ and $E_2 \to E_1$ by Lindblad operator $L = V_S + \frac{i}{4T}\dot{V}_S$

$$\langle E_2|L|E_1\rangle \propto \langle E_2|V_S|E_1\rangle \left(1 - \frac{E_2 - E_1}{4T}\right),$$

$$\frac{\Gamma_{1\to 2}}{\Gamma_{2\to 1}} = \frac{|\langle E_2|L|E_1\rangle|^2}{|\langle E_1|L|E_2\rangle|^2} = \left[\frac{1 - \frac{E_2 - E_1}{4T}}{1 - \frac{E_1 - E_2}{4T}}\right]^2 \simeq \exp\left[-\frac{E_2 - E_1}{T}\right],$$

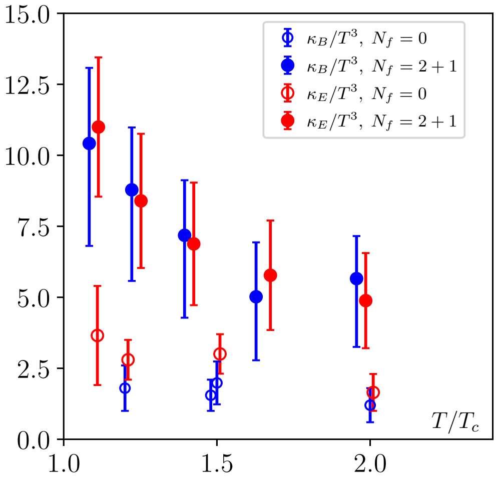
$$\because \quad \left(\frac{1 + x/4}{1 - x/4}\right)^2 \simeq 1 + x + \frac{1}{2}x^2 + \underbrace{\frac{3}{16}}_{\simeq 1/6} x^3 + \cdots \simeq e^x$$

Numerically, the detailed balance is satisfied with 3% level when $\Delta E \lesssim T$

Lattice simulation for κ_s

$\kappa_E (= \kappa)$ and $\kappa_B (= \kappa_s)$ [HotQCD Collaboration (24) (PRL's supplemental material)]

$$\kappa_{\rm tot} = \kappa_E + \underbrace{\frac{2}{3}\langle v^2\rangle \kappa_B}_{\simeq \frac{2T}{M}\kappa_B}$$



See also [Banerjee-Datta-Laine (20), TUMQCD Collaboration (23), Altenkort-delaCruz-Kaczmarek-Moore-Shu (24)]