Superconducting baryon crystal induced via the chiral anomaly

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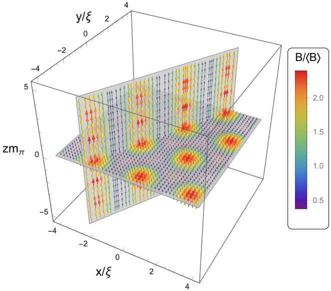
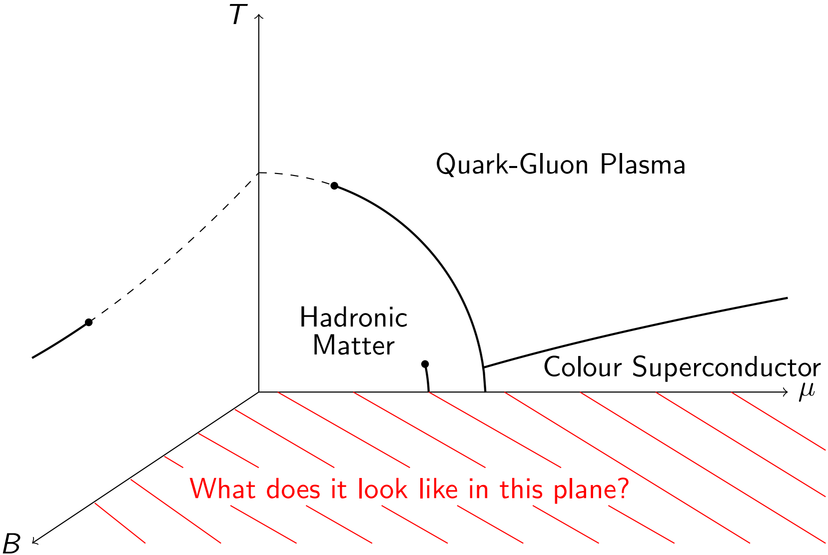


based on GWE and A. Schmitt, JHEP **09** (2022) + arXiv:2311.03880

QCD Phase Diagram extended along the B-axis



Chiral Perturbation Theory (ChPT) with chiral anomaly

Lagrangian with $N_f = 2$ ChPT, electromagnetic and chiral anomaly (WZW) terms [J. Wess and B. Zumino, PLB 37 (1971); E. Witten, NPB 223 (1983)]:

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\nabla_\mu \Sigma^\dagger \nabla^\mu \Sigma\right] + \frac{m_\pi^2 f_\pi^2}{4}\mathrm{Tr}\left[\Sigma + \Sigma^\dagger\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(A_\mu^B - \frac{e}{2}A_\mu\right) j_B^\mu \, ,$$

with $SU(2)$ chiral field Σ, covariant derivative ∇^μ, gauge fields A_μ (electromagnetic) and A_μ^B ("baryonic"), and anomalous baryon current [J. Goldstone and F. Wilczek, PRL **47** (1981)]

$$j_B^\mu = -\frac{\epsilon^{\mu\nu\rho\lambda}}{24\pi^2}\mathrm{Tr}\Big[(\Sigma\nabla_\nu\Sigma^\dagger)(\Sigma\nabla_\rho\Sigma^\dagger)(\Sigma\nabla_\lambda\Sigma^\dagger) + \frac{3ie}{4}F_{\nu\rho}\tau_3\left(\Sigma\nabla_\lambda\Sigma^\dagger + \nabla_\lambda\Sigma^\dagger\Sigma\right)\Big] \, .$$

Free energy density

- Parameterising $\Sigma(\pi^0, \pi^\pm) \to (\alpha, \varphi)$ and dropping time dependence, our thermodynamic potential (density) is

$$\Omega(\boldsymbol{r}) = \frac{\boldsymbol{B}^2}{2} + |\left[\nabla - i\left(e\boldsymbol{A} + \nabla\alpha\right)\right]\varphi|^2 + \frac{\left(\nabla|\varphi|^2\right)^2}{2\left(f_\pi^2 - 2|\varphi|^2\right)} + \frac{f_\pi^2 - 2|\varphi|^2}{2}\left(\nabla\alpha\right)^2 - m_\pi^2 f_\pi \sqrt{f_\pi^2 - 2|\varphi|^2}\cos\alpha - \mu n_B(\boldsymbol{r}),$$

  where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and

$$n_B(\boldsymbol{r}) = j_B^0 = \frac{e\nabla\alpha \cdot \boldsymbol{B}}{4\pi^2} + \frac{\nabla\alpha \cdot \nabla \times \boldsymbol{j}}{4\pi^2 e f_\pi^2}$$

  is the local baryon number density with electromagnetic current $\boldsymbol{j}$

- Obtain the free energy density from

$$F = \frac{1}{V}\int dV\, \Omega(\boldsymbol{r})$$

Chiral Soliton Lattice (CSL)

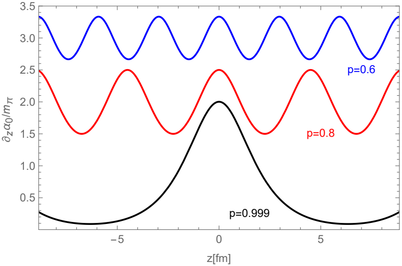
- Thermodynamic potential in the absence of $\pi^{\pm}$ $(\varphi = 0)$ [D. T. Son and M. A. Stephanov, PRD **77** (2008)]:

$$\Omega_0(\boldsymbol{r}) = \frac{\boldsymbol{B}^2}{2} + \frac{f_\pi^2}{2}(\nabla\alpha_0)^2 - m_\pi^2 f_\pi^2(\cos\alpha_0 - 1) - \frac{e\mu}{4\pi^2}\nabla\alpha_0 \cdot \boldsymbol{B}$$

- Solution of the α_0 equation of motion is [T. Brauner and N. Yamamoto, JHEP 4 (2017)]

$$\alpha_0(z, p) = 2 \arccos\left[-\mathrm{sn}(z, p^2)\right]$$

where $\mathrm{sn}(z, p^2)$ is the Jacobi elliptic sine function with elliptic modulus p



- Minimised free energy

$$F_0 = \frac{\boldsymbol{B}^2}{2} - 2m_\pi^2 f_\pi^2\left(\frac{1}{p^2} - 1\right)$$

- Preferred over vacuum above

$$eB_{\mathrm{CSL}} = \frac{16\pi m_\pi f_\pi^2}{\mu}$$

CSL $\pi^{\pm}$ instability

- Linearise EoMs in φ, find dispersion relation, and determine

$$eB_{c2} = \frac{m_\pi^2}{p^2}\left(2 - p^2 + 2\sqrt{p^4 - p^2 + 1}\right)$$

from the lowest energy excitation [T. Brauner and N. Yamamoto, JHEP 4 (2017)]

- p parameterises the instability curve B_{c2}

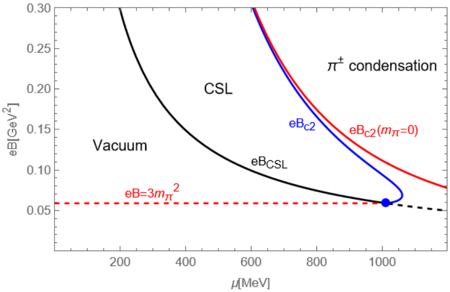
chiral limit ($p \to 0$):

$$eB_{c2} = \frac{16\pi^4 f_\pi^4}{\mu^2}$$

single domain wall ($p \to 1$):

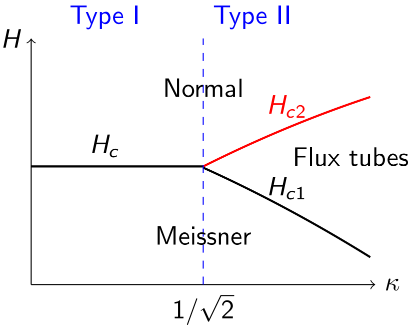
$$eB_{c2} = 3m_\pi^2$$

[D. T. Son and M. A. Stephanov, PRD **77** (2008)]



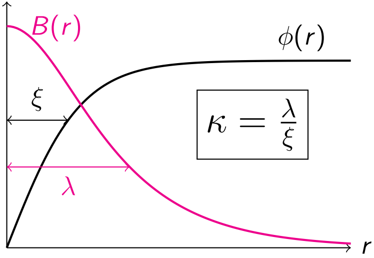
What phase is beyond B_{c2}?

Superconductivity refresher



- Instability to charged pions implies they condense $\rightarrow$ superconductivity
- Dispersion relation in chiral limit reminiscent of type-II Flux tube lattice/Normal transition

- (Above) H-κ phase diagram where κ is the Ginzburg-Landau (GL) parameter
- (Right) Flux tube profile: ϕ has coherence length ξ, B has penetration depth λ



Flux tube lattice

- Near second order phase transition $\rightarrow$ expand ϕ and $\boldsymbol{A}$ in small parameter $\epsilon \sim \sqrt{B_{c2} - B}$ [A. A. Abrikosov, Sov. Phys. JETP **5** (1957)]

$$\phi = \phi_0 + \delta\phi + \dots\,, \quad \boldsymbol{A} = \boldsymbol{A}_0 + \delta\boldsymbol{A} + \dots\,,$$

$$\Rightarrow \phi_0(x,y) = \sum_{n=-\infty}^{\infty} C_n e^{inqy} e^{-\frac{eB_{c2}}{2}\left(x - \frac{nq}{eB_{c2}}\right)^2}\,, \quad \boldsymbol{B} \simeq (\text{constant} - |\phi_0(x,y)|^2)\hat{\boldsymbol{z}}$$

- With unit cell lengths L_x, L_y, L_z, introduce

$$\langle f(\boldsymbol{r})\rangle_{x,y,z} \equiv \frac{1}{L_x}\frac{1}{L_y}\frac{1}{L_z}\int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\, f(\boldsymbol{r})$$

and parameter

$$\beta \equiv \frac{\langle|\phi_0|^4\rangle_{x,y,z}}{\left(\langle|\phi_0|^2\rangle_{x,y,z}\right)^2}$$

- Minimised free energy up to and including ϵ^4 terms is

$$F \simeq \frac{\langle B\rangle^2}{2} - \frac{1}{2}\frac{\left(B_{c2} - \langle B\rangle\right)^2}{(2\kappa^2 - 1)\,\beta + 1}$$

Expansion near B_{c2}

- Adopt Abrikosov's expansion with $\epsilon \equiv \sqrt{|\langle B\rangle - B_{c2}|/B_{c2}}$,

$$\varphi = \varphi_0 + \delta\varphi + \dots\,, \quad \boldsymbol{A} = \boldsymbol{A}_0 + \delta\boldsymbol{A} + \dots\,, \quad \alpha = \alpha_0 + \delta\alpha + \dots$$

- Lowest order equations solved by CSL solution $\alpha_0(z,p)$, gauge-like condition $e\boldsymbol{A}_0 + \nabla\alpha_0 = eB_{c2}x\hat{\boldsymbol{y}}$, and

$$\varphi_0(x,y,z) = f_0(z)\phi_0(x,y)$$

where $f_0(z)$ is the lowest eigenfunction of the $m=2$ Lamé equation

- To solve remaining equations in Fourier space, employ Fourier series

$$|\phi_0(x,y)|^2 = \sum_{\boldsymbol{k}_\perp} e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r}}\hat{\omega}(\boldsymbol{k}_\perp)\,, \quad f_0(z)^2 = \sum_{k_z} e^{ik_z z}\hat{s}(k_z)\,,$$

where $\boldsymbol{k}_\perp = (k_x, k_y, 0)$ and

$$\hat{\omega}(\boldsymbol{k}_\perp) = \langle e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{r}}|\phi_0(x,y)|^2\rangle_{x,y}\,, \quad \hat{s}(k_z) = \langle e^{-ik_z z}f_0(z)^2\rangle_z$$

δB

- Use Coulomb gauge $\nabla \cdot \delta \boldsymbol{A}$ and Fourier series ansatz

$$\delta \boldsymbol{A} = cx\hat{\boldsymbol{y}} + \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \delta \hat{\boldsymbol{A}}(\boldsymbol{k}) \quad \Rightarrow \quad \delta \boldsymbol{B} = c\hat{\boldsymbol{z}} + \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \delta \hat{\boldsymbol{B}}(\boldsymbol{k})$$

  where $\boldsymbol{k} = (k_x, k_y, k_z)$ and c is a constant

- Solutions in Fourier space are

$$\delta \hat{B}_x(\boldsymbol{k}) = \frac{k_x k_z}{k^2} e\hat{s}(k_z)\hat{\omega}(\boldsymbol{k}_\perp)\,,$$
$$\delta \hat{B}_y(\boldsymbol{k}) = \frac{k_y k_z}{k^2} e\hat{s}(k_z)\hat{\omega}(\boldsymbol{k}_\perp)\,,$$
$$\delta \hat{B}_z(\boldsymbol{k}) = -\frac{k_\perp^2}{k^2} e\hat{s}(k_z)\hat{\omega}(\boldsymbol{k}_\perp)$$

- Determine c from boundary condition $\langle B \rangle \equiv \langle B_z \rangle_{x,y}$

$$\Rightarrow c = \langle B \rangle - B_{c2} + e\hat{\omega}_0\,, \quad \text{where} \quad \hat{\omega}_0 \equiv \hat{\omega}(\mathbf{0})$$

$\delta\alpha$

- Extend CSL solution from p at B_{c2}, to $p + \delta p$ at $\langle B \rangle$ → Topological contribution + Fourier series ansatz:

$$\delta\alpha = \alpha_1 \delta p + \frac{\omega_0}{f_\pi^2} \delta\alpha_1 \,, \quad \text{with} \quad \delta\alpha_1 = \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \delta\hat{\alpha}(\boldsymbol{k})$$

and

$$\alpha_1 = \frac{\partial \alpha_0}{\partial p} = -\frac{\mathcal{E}(\bar{z}, p^2)\partial_{\bar{z}}\alpha_0 + \partial_{\bar{z}}^2 \alpha_0}{p(1-p^2)} \,, \quad \delta p = -\frac{pE(p^2)}{K(p^2)} \frac{\langle B \rangle - B_{c2}}{B_{c2}} + \mathcal{O}(\epsilon^4)$$

where $\bar{z}$ is dimensionless z, $\mathcal{E}$ is the Jacobi epsilon function, and K and E are the complete elliptic integrals of the first and second kind respectively

- Inhomogeneous differential equation reduces to a coupled set of linear equations that must be solved to obtain $\delta\hat{\alpha}(\boldsymbol{k})$

Free energy

- Do not solve $\delta\varphi$ equation, use instead to show

$$\langle|\varphi_0(x,y,z)|^2\rangle_{x,y,z} = e\hat{\omega}_0 = \mathcal{G}(p)\frac{\langle B\rangle - B_{c2}}{(2\kappa^2-1)\beta + 1 + 2(\mathcal{H}_1 - \mathcal{H}_2)}\,,$$

where $\mathcal{H}_{1,2}$ are infinite sums over $\boldsymbol{k}$ and κ is an *effective* GL parameter

- $\mathcal{G}(p)$ related to $eB_{c2}(\mu)$ "turning point"
- Up to and including ϵ^4 terms,

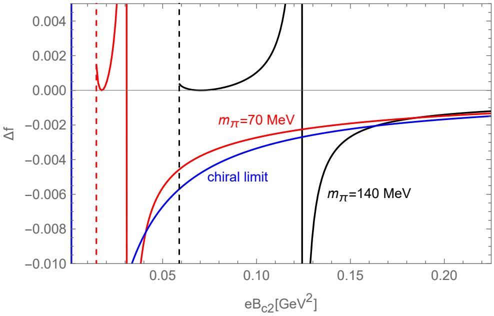
$$F \simeq F_0 + \Delta f\left(\langle B\rangle - B_{c2}\right)^2\,,$$

where

$$\Delta f = -\frac{\mathcal{G}^2}{2}\frac{1}{(2\kappa^2-1)\beta + 1 + 2(\mathcal{H}_1 - \mathcal{H}_2)}$$

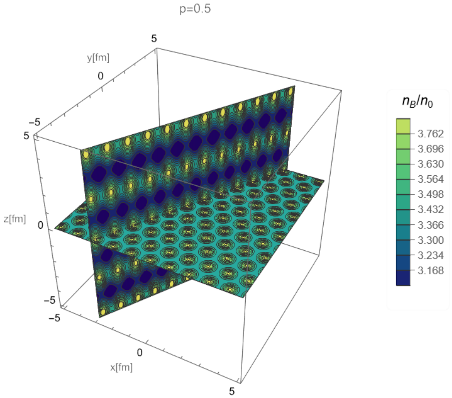
Δf - Is it preferred over pure CSL?

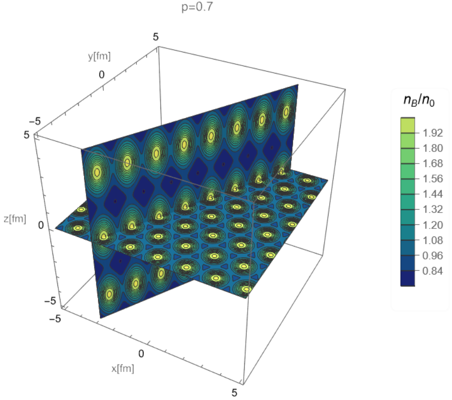
- Minimum of Δf at $R = \sqrt{3}$ for all p $\rightarrow$ hexagonal lattice
- $\Delta f < 0$ for $eB_{c2} \gtrsim 0.12\,\mathrm{GeV}^2$, $\mu \lesssim 910\,\mathrm{MeV}$
- Preferred for all $e\langle B\rangle > eB_{c2}$, $\mu \lesssim 10\,\mathrm{GeV}$ in chiral limit



We have constructed a phase which is preferred over the CSL for $e\langle B\rangle \gtrsim 0.12\,\mathrm{GeV}^2$ and $\mu \lesssim 910\mathrm{MeV}$!

Baryon number density - What does it look like?



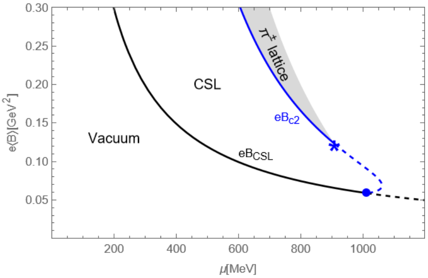


- Transverse plane - hexagonal flux tube lattice reflected in n_B
- At fixed distance from B_{c2}, varying p changes periodicity in x, y, and z
- (Chiral limit) $p \to 0$: Baryon tubes (no z-dependence) [GWE and A. Schmitt, JHEP **09** (2022)], (Domain wall) $p \to 1$: lattice confined to a single sheet

Summary

- CSL phase instability to $\pi^{\pm}$ fluctuations implies they condense to a superconducting phase
- Adapting Abrikosov's original calculation, we constructed a superconducting flux tube lattice which lowers the free energy of the CSL phase for $e\langle B\rangle \gtrsim 0.12\,\mathrm{GeV}^2$, $\mu \lesssim 910\,\mathrm{MeV}$
- Baryon number density is non-zero and inhomogeneous with periodicity in $(x, y, z) \to$ 3D Baryon crystal

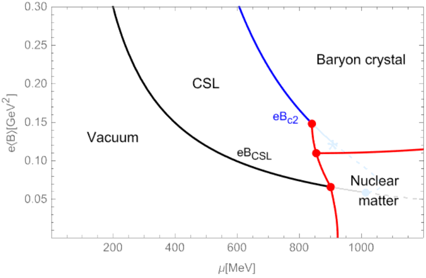
Updated phase diagram



Outlook

- Domain wall skyrmion phase found in this region of the μ-B plane - competition between phases? [M. Eto et al., arXiv:2304.02940,2311.01112 [hep-ph]]
- $\pi^{\pm}$ lattice preferred up to near baryon onset - include baryons for a more realistic calculation
- CSL and charged pion superconductivity emerge in the μ_I-B plane - can we extend our results to this plane? [T. Brauner et al., JHEP **12** (2019); P. Adhikari et al., PRC **91** (2015); M. S. Grønli and T. Brauner, Eur. Phys. J. C **82** (2022)]

Conjectured phase diagram



Back-up slides

Equations of motion

From the Lagrangian/free energy we obtain the equations of motion for φ, $\boldsymbol{A}$ and α

$$0 = \left[\mathcal{D} + \frac{\nabla^2|\varphi|^2}{f_\pi^2 - 2|\varphi|^2} + \frac{\left(\nabla|\varphi|^2\right)^2}{(f_\pi^2 - 2|\varphi|^2)^2} + m_\pi^2 \cos\alpha \left(1 - \frac{f_\pi}{\sqrt{f_\pi^2 - 2|\varphi|^2}}\right)\right]\varphi\,,$$

$$\nabla \times \boldsymbol{B} = -ie\left(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*\right) - 2e\left(e\boldsymbol{A} + \nabla\alpha\right)|\varphi|^2\,,$$

$$\nabla \cdot \left[\left(1 - \frac{2|\varphi|^2}{f_\pi^2}\right)\nabla\alpha\right] = m_\pi^2\sqrt{1 - \frac{2|\varphi|^2}{f_\pi^2}}\sin\alpha\,,$$

respectively, where

$$\mathcal{D} \equiv \nabla^2 - i\nabla\cdot(e\boldsymbol{A} + \nabla\alpha) - 2i\left(e\boldsymbol{A} + \nabla\alpha\right)\cdot\nabla - (e\boldsymbol{A} + \nabla\alpha)^2 + (\nabla\alpha)^2 - m_\pi^2\cos\alpha\,.$$

CSL $\pi^{\pm}$ instability

- Linearise EoMs in φ and use product ansatz $\varphi = e^{-iwt} g(x,y) f(z)$ to find the (z-dependent) dispersion relation [T. Brauner and N. Yamamoto, JHEP 4 (2017)]

$$w^2 = (2l+1)\, eB - \frac{m_\pi^2}{p^2}\left[4 + p^2 - 6p^2 \mathrm{sn}^2(\bar{z}, p^2)\right] - f^{-1}\partial_z^2 f\ ,$$

  where $g(x,y)$ is the solution to Schrödinger equation for the quantum harmonic oscillator

- Above can be cast into a Lamé equation with lowest eigenvalue

$$\varepsilon_0 = 2(1 + p^2 - \sqrt{p^4 - p^2 + 1})$$

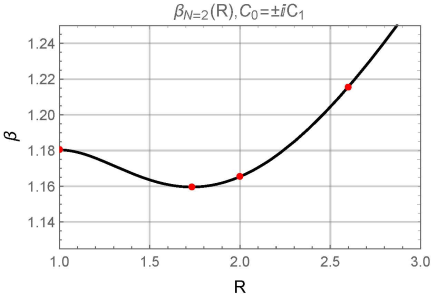
  and corresponding eigenfunction

$$f_0(z) = \frac{1}{N(p)}\left(\frac{\sqrt{p^4 - p^2 + 1} + 1 - 2p^2}{3p^2} + \sin^2\frac{\alpha_0}{2}\right)\ ,$$

  where $N(p)$ is a normalisation factor

β parameter and lattice configurations

- Minimise $\beta \rightarrow$ minimise F
- Depends on periodicity condition $C_n = C_{n+N}$
- Explore a continuum of geometries with $N = 2$ and $C_0 = \pm iC_1$ [W. H. Kleiner et al., PR 133 **5A** (1964)]



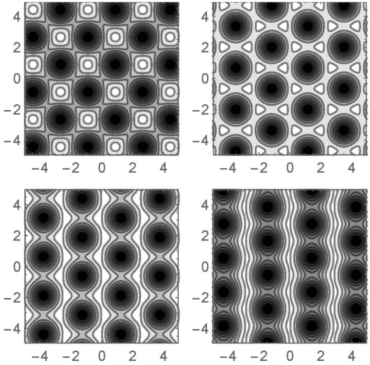


Figure: $R = L_x/L_y$. *Left*: Red dots correspond to contour plots on the right. *Right*: $|\phi_0(x,y)|^2$ in the x-y plane. Dark regions correspond to flux tubes.

Chiral Limit

- Adopt Abrikosov's expansion with $\epsilon \sim \sqrt{B - B_{c2}}$,

$$\varphi = \varphi_0 + \delta\varphi + \ldots\,, \quad \boldsymbol{A} = \boldsymbol{A}_0 + \delta\boldsymbol{A} + \ldots\,, \quad \alpha = \alpha_0 + \delta\alpha + \ldots$$

- With $m_\pi = 0$ for simplicity,

$$\boldsymbol{B}_0 = B_{c2}\hat{\boldsymbol{e}}_z\,, \quad \alpha_0(z) = \frac{e\mu}{4\pi^2 f_\pi^2} B_{c2} z\,, \quad \varphi_0(x,y) = \phi_0(x,y)$$

- Next to leading order correction to $\boldsymbol{B}$ and α become

$$\delta\boldsymbol{B}(x,y) = \left[\langle B\rangle - B_{c2} + e\left(\langle|\varphi_0(x,y)|^2\rangle - |\varphi_0(x,y)|^2\right)\right]\hat{\boldsymbol{e}}_z\,,$$

$$\delta\alpha(z) = \frac{e\mu}{4\pi^2 f_\pi^2}\left(\langle B\rangle - B_{c2}\right) z$$

Free energy result in chiral limit

Do not solve $\delta\varphi$ equation, use it instead to show

$$e\langle|\varphi_0|^2\rangle = \frac{\langle B\rangle - B_{c2}}{(2\kappa^2 - 1)\,\beta + 1}\,, \quad \text{where} \quad \beta = \frac{\langle|\varphi_0|^4\rangle}{\langle|\varphi_0|^2\rangle^2}\,,$$

and $\kappa = \sqrt{eB_{c2}}/\sqrt{2}ef_\pi$ is an effective Ginzburg-Landau parameter.

Up to to fourth order, the free energy density in the chiral limit F_{m0} is

$$F_{m0} \simeq F_{0,m0} - \frac{1}{2}\frac{(\langle B\rangle - B_{c2})^2}{(2\kappa^2 - 1)\,\beta + 1}\,,$$

where $F_{0,m0}$ is the free energy in the "homogeneous CSL phase".

We have constructed a phase which is preferred above B_{c2} in the chiral limit!

Charged pion condensate and baryon number density

Oscillation in baryon number density comes primarily from the vorticity term $\nabla \times \boldsymbol{j} \simeq e\nabla^2|\varphi_0|^2\hat{\boldsymbol{e}}_z$.

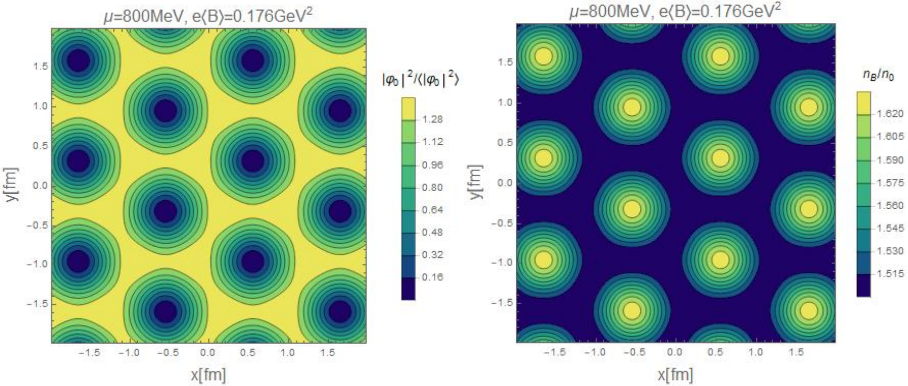
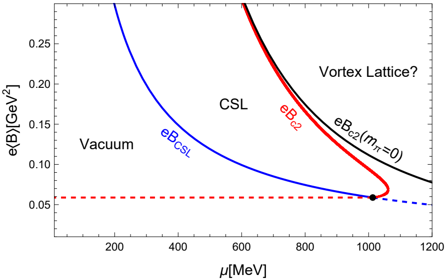


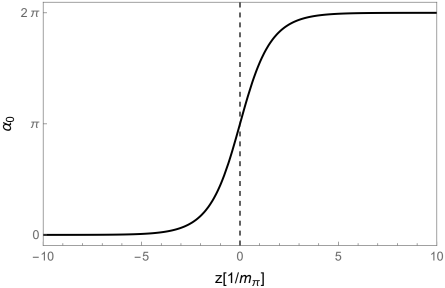
Figure: Charged pion vortex lattice (left) and local baryon number density (right).

Single domain wall

- Instability now occurs at

$$B \leq B_{c2} = \frac{3m_\pi^2}{e}$$





- Single domain wall CSL

$$\alpha_0(z) = 4\arctan\left(e^{m_\pi z}\right)$$

Free energy in single domain wall case

- First order solution becomes

$$\varphi_0(x, y, z) = \frac{\phi_0(x, y)}{\cosh^2(m_\pi z)}$$

- Derive semi-analytical results in Fourier space for $\delta\alpha$ and $\delta\boldsymbol{B}$ to obtain

$$\mathcal{F} \simeq \mathcal{F}_{\text{DW}} - \frac{2}{3m_\pi}\frac{(B_{c2} - \langle B\rangle)^2}{D(\beta)},$$

where $\mathcal{F}_{\text{DW}}$ is the domain wall free energy and $D(\beta)$ must be evaluated numerically

- Find $D < 0$ for physical values of m_π, e, and f_π

Single domain wall CSL preferred over superconducting baryon crystal below B_{c2}