The lattice computation of the TMD soft function using the auxiliary field representation of the Wilson line

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with

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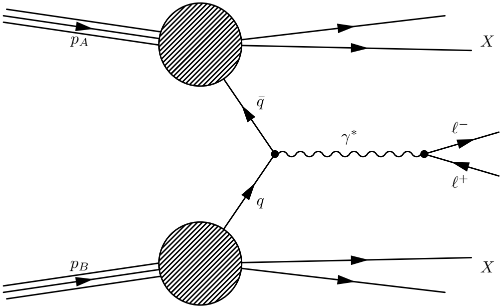
TMD factorization



Drell-Yan scattering

- invariant mass: Q
- rapidity of lepton pair: Y
- transverse momentum: $\vec{q}_\perp$
- momentum fraction: x_a, x_b
- rapidity scale: ν
- Collins-Soper (CS) scale: ζ_a, ζ_b

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q,\mu) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$
$$\times S_i(b_\perp, \mu, \nu)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right)\right]$$

Beam and soft function renormalization

$$B_i\left(x,\vec{b}_\perp,\mu,\zeta/\nu^2\right)=\lim_{\epsilon\to0}\lim_{\tau\to0}Z_B^i\left(b_\perp,\mu,\nu,\epsilon,\tau,xP^+\right)\frac{B_i^{0(\mathrm{u})}\left(x,\vec{b}_\perp,\epsilon,\tau,xP^+\right)}{S_i^{0(\mathrm{subt})}\left(b_\perp,\epsilon,\tau\right)}$$

$$S_i\left(b_\perp,\mu,\nu\right)=\lim_{\epsilon\to0}\lim_{\tau\to0}Z_S^i\left(b_\perp,\mu,\nu,\epsilon,\tau\right)S_i^0\left(b_\perp,\epsilon,\tau\right),$$

$$f_i\left(x,\vec{b}_\perp,\mu,\zeta\right)=\lim_{\substack{\epsilon\to0\\ \tau\to0}}Z_{UV}^i\left(\mu,\epsilon,\zeta\right)B_i^{0(\mathrm{u})}\left(x,\vec{b}_\perp,\epsilon,\tau,xP^+\right)\frac{\sqrt{S_i^0\left(b_\perp,\epsilon,\tau\right)}}{S_i^{0(\mathrm{subt})}\left(b_\perp,\epsilon,\tau\right)},$$

- Subtract possible double counting with $S_i^{0(\mathrm{subt})}$
- TMDPDF independent of ν
- TMDPDF depends on CS scale, ζ

Operator definitions

$$B_i^{0(u)}\left(x,\vec{b}_\perp,\epsilon,\tau,xP^+\right)=\int\frac{\mathrm{d}b^-}{2\pi}e^{-ib^-(xP^+)}\langle P|\left[\bar{\psi}_i^0\left(b^-,\vec{b}_\perp\right)W_{\bar{n}}\left(b^-,\vec{b}_\perp;-\infty,0\right)\right.$$
$$\left.\times\ W_\perp\left(-\infty\bar{n};0,b_\perp\right)W_{\bar{n}}\left(0;0,-\infty\right)\frac{\gamma^+}{2}\psi_i^0\left(0\right)\right]_\tau|P\rangle$$

$$S^0\left(b_\perp,\epsilon,\tau\right)=\frac{1}{N_c}\langle 0|\left[W_n\left(b_\perp;0,-\infty\right)W_{\bar{n}}\left(b_\perp;-\infty,0\right)W_\perp\left(-\infty\bar{n};0,b_\perp\right)\right.$$
$$\left.\times\ W_{\bar{n}}\left(0;0,-\infty\right)W_n\left(0;-\infty,0\right)W_\perp\left(-\infty n;b_\perp,0\right)\right]_\tau|0\rangle$$

$$W_n\left(x;a,b\right)=P\exp\left\{-ig_0\int_a^b\mathrm{d}s\,n^\mu A_\mu^{c0}\left(x+sn\right)t^c\right\}$$

$$n=(1,0,0,1),\qquad\bar{n}=(1,0,0,-1)$$

Soft function

Naive soft function:

$$S\left(b_\perp,\epsilon\right) = \frac{1}{N_c}\left\langle 0\right| \mathrm{Tr}\, S_n^\dagger(\vec{b}_\perp) S_{\bar{n}}(\vec{b}_\perp) S_\perp(-\infty\bar{n};\vec{b}_\perp,\vec{0}_\perp) S_{\bar{n}}^\dagger(\vec{0}_\perp) S_n(\vec{0}_\perp) S_\perp^\dagger(-\infty n;\vec{b}_\perp,\vec{0}_\perp)\left|0\right\rangle$$

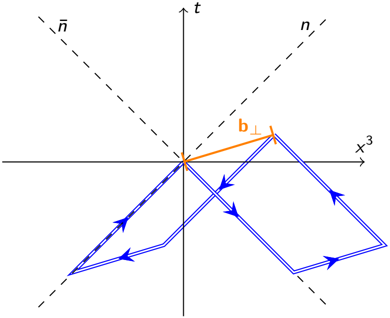
Soft Wilson line:

$$S_n\left(x\right) = P\exp\left\{-ig\int_{-\infty}^{0} \mathrm{d}s n^\mu A_\mu(x+sn)\right\}$$

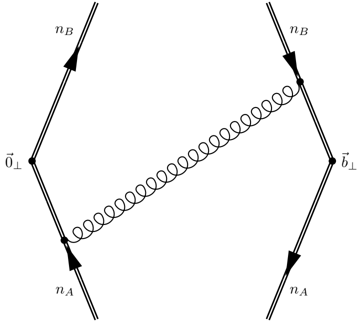
Lightlike vectors:

$$n=(1,0,0,1),\qquad \bar{n}=(1,0,0,-1)$$

$$n^2=0,\qquad \bar{n}^2=0,\qquad n\cdot\bar{n}=2$$



Rapidity divergence



Rapidity divergence:

$$\int \frac{\mathrm{d}k^+\mathrm{d}k^-}{(2\pi)^2} \frac{1}{k^+k^- - k_\perp^2 + i0} \frac{1}{n\cdot k - i0} \frac{1}{\bar{n}\cdot k + i0}$$
$$= \int \frac{\mathrm{d}\alpha}{(2\pi)^2} \frac{1}{\alpha - k_\perp^2 + i0} \frac{1}{\alpha - i0} \int_{-\infty}^{\infty} \mathrm{d}y$$

$$k^- = n\cdot k, \qquad k^+ = \bar{n}\cdot k, \qquad k^\pm = k^0 \pm k^3$$

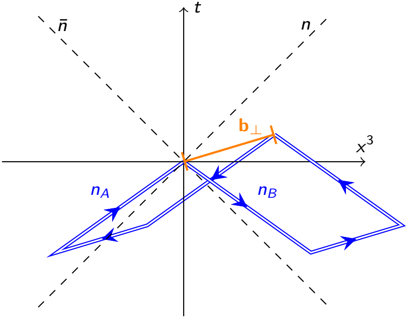
$$\alpha = k^+k^-, \quad y = \frac{1}{2}\ln\left(\frac{k^-}{k^+}\right)$$

Divergence associated with rapidity

$$y \to \pm\infty$$

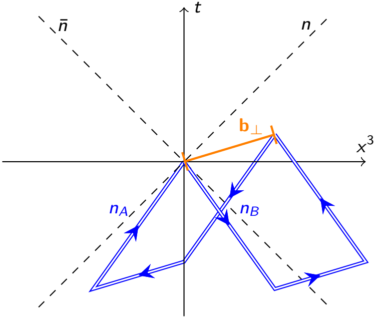
Spacelike Wilson lines:

$$n_A \equiv n - e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B}n$$

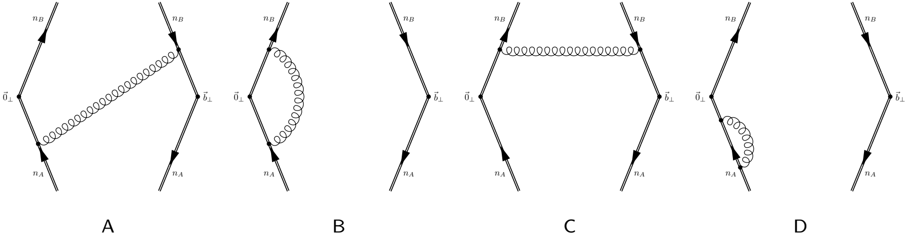


Timelike Wilson lines:

$$n_A \equiv n + e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} + e^{+y_B}n$$



One loop result in Minkowski space



With space-like regulator, Collins scheme [Collins, 2011]

$$S_{\mathrm{A}}(b_\perp, \epsilon, y_A, y_B) = g^2 C_F (n_A \cdot n_B) \mu_0^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-i\vec{b}_\perp \cdot \vec{k}_\perp}}{k^2 + i0} \frac{1}{n_A \cdot k - i0} \frac{1}{n_B \cdot k + i0}$$

$$= \frac{\alpha_s C_F}{2\pi} (y_A - y_B) \frac{1 + e^{2(y_B - y_A)}}{1 - e^{2(y_B - y_A)}} \left(-\frac{1}{\epsilon} - \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right) \right)$$

One loop result:

$$S(b_\perp, \epsilon, y_A, y_B)$$

$$= 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right) \right) \left\{ 2 - 2|y_A - y_B| \frac{1 + e^{2(y_B - y_A)}}{1 - e^{2(y_B - y_A)}} \right\} + \mathcal{O}(\alpha_s^2)$$

TMDPDF

$$f_i(x, \vec{b}_\perp, \mu, \zeta) = \lim_{\epsilon \to 0} Z^i_{\mathrm{UV}}(\mu, \epsilon, \zeta) \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} B_i\left(x, \vec{b}_\perp, \epsilon, y_B, xP^+\right) \times \sqrt{\frac{S_i(b_\perp, \epsilon, y_A - y_n)}{S_i(b_\perp, \epsilon, y_A - y_B)\, S_i(b_\perp, \epsilon, y_n - y_B)}}$$

- Collins-Soper scale: $\zeta = 2(xP^+)^2 e^{-2y_n}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q, \mu) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp \cdot \vec{q}_\perp} f_i\left(x_a, \vec{b}_\perp, \mu, \zeta_a\right) f_j\left(x_b, \vec{b}_\perp, \mu, \zeta_b\right) \times \left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right)\right]$$

- $\zeta_a \zeta_b = 2(x_a P^+)^2 e^{-2y_n}\, 2(x_b P^-)^2 e^{2y_n} = Q^4$

Evolution equations

$$\gamma_\mu^q(\mu,\zeta) = \frac{\mathrm{d} f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)}{\mathrm{d}\log\mu}$$
$$= \frac{\mathrm{d}\log B_q\left(x, \vec{b}_\perp, \mu, y_P - y_B\right)}{\mathrm{d}\log\mu} - \frac{1}{2}\frac{\mathrm{d}\log S_q\left(b_\perp, \mu, y_n - y_B\right)}{\mathrm{d}\log\mu}$$

$$\gamma_\zeta^q(\mu, b_\perp) = 2\frac{\mathrm{d} f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)}{\mathrm{d}\log\zeta} = \frac{\mathrm{d}\log B_q\left(x, \vec{b}_\perp, \mu, y_P - y_B\right)}{\mathrm{d} y_p}$$
$$= \frac{\mathrm{d}\log S_q\left(b_\perp, \mu, y_n - y_B\right)}{\mathrm{d} y_n}$$

- γ_ζ^q is the Collins-Soper kernel
- Lattice extraction of soft function would allow for a calculation of the CS kernel

Lattice extraction of TMDPDFs

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = C_q\left(x\tilde{P}^z, \mu\right) \exp\left[\frac{1}{2}\gamma_\zeta^q\left(\mu, b_\perp\right) \log\frac{\tilde{\zeta}}{\zeta}\right] f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)$$
$$\times\left\{1 + \mathcal{O}\left(\frac{1}{(x\tilde{P}^z b_\perp)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(x\tilde{P}^z)^2}\right)\right\}$$

[Ebert, *et. al.*, 2019], [Ebert, *et. al*, 2022]

- C_q is a perturbatively calculable matching kernel
- $\tilde{\zeta} = x^2 m_h^2 e^{2(y_{\tilde{P}} + y_B - y_n)}$

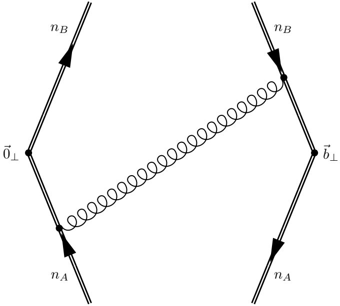
quasi-TMDPDF

$$\tilde{f}_q\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right) = \tilde{f}_q^{\mathrm{naive}}\left(x, \vec{b}_\perp, \mu\tilde{\zeta}, x\tilde{P}^z\right)\sqrt{\frac{\tilde{S}_q^{\mathrm{naive}}\left(b_\perp, \mu\right)}{S_q\left(b_\perp, \mu, 2y_n, 2y_B\right)}}$$

- $\tilde{f}_q^{\mathrm{naive}}$ and $\tilde{S}_q^{\mathrm{naive}}$ are lattice calculable objects
- S_q is the Collins soft function

An extraction of the TMDPDF was performed by the Lattice Parton Collaboration (LPC) using a different method to obtain the soft function. [He, *et. al.*, 2022]

Euclidean space calculation



Define Euclidean space Wilson line directions as:

$$\tilde{n}_A = (in_A^0, \vec{0}_\perp, n_A^3), \qquad \tilde{n}_B = (in_B^0, \vec{0}_\perp, -n_B^3)$$

$$r_a \equiv \frac{n_A^3}{n_A^0}, \qquad r_b \equiv \frac{n_B^3}{n_B^0}$$

$$S_A^E(b_\perp, \epsilon, r_a, r_b) = g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int_{-\infty}^{0} \mathrm{d}s \int_{-\infty}^{0} \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2}$$

Naively try to write down Wilson line propagators:

$$\int_{-\infty}^{0} \mathrm{d}s e^{sn_A^0 k_4 - isn_A^3 k_3} = \left. \frac{e^{sn_A^0 k_4 - isn_A^3 k_3}}{n_A^0 k_4 - in_A^3 k_3} \right|_{s=-\infty}^{0} \xrightarrow{k_4<0} \infty$$

Coordinate space

Perform integration in coordinate space:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2} = \int_0^\infty \mathrm{d}u \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-uk^2} e^{-(b+s\tilde{n}_A - t\tilde{n}_B)^2/4u}$$

$$= \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}} \frac{1}{((b+s\tilde{n}_A - t\tilde{n}_B)^2/4)^{d/2-1}}$$

'u' integral only valid for

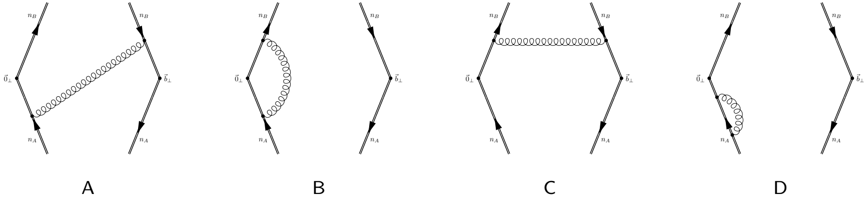
$$(s\tilde{n}_A - t\tilde{n}_B)^2 = s^2((n_A^3)^2 - (n_A^0)^2) + t^2((n_B^3)^2 - (n_B^0)^2) + st(n_A^3 n_B^3 + n_A^0 n_B^0) > 0$$

Euclidean space integral only finite when:

$$|n_A^3| > |n_A^0|, \quad |n_B^3| > |n_B^0|, \quad n_A^3 n_B^3 + n_A^0 n_B^0 > 0$$

$$\rightarrow \quad |r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$$

Soft function in Euclidean space at one loop



Calculation in coordinate space at one loop ($|r_a| > 1, |r_b| > 1$):

$$S^{(1)}(b_\perp, \epsilon, r_a, r_b)$$
$$= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 + \log\left|\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right|\frac{r_a r_b + 1}{r_a + r_b}\right\}$$

Rapidity regulator scheme

Time-like: $n_A = \left(1 + e^{-2y_A}, \vec{0}_\perp, 1 - e^{-2y_A}\right), \quad n_B = \left(1 + e^{2y_B}, \vec{0}_\perp, -1 + e^{2y_B}\right)$

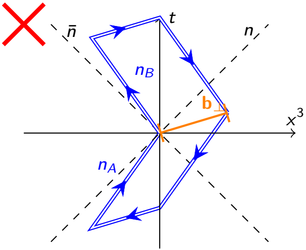
$$r_a = \frac{1 - e^{-2y_A}}{1 + e^{-2y_A}}, \quad r_b = \frac{1 - e^{2y_B}}{1 + e^{2y_B}}, \qquad |r_a|, |r_b| < 1 \qquad \text{fails}$$

Space-like: $n_A = \left(1 - e^{-2y_A}, \vec{0}_\perp, 1 + e^{-2y_A}\right), \quad n_B = \left(1 - e^{2y_B}, \vec{0}_\perp, -1 - e^{2y_B}\right)$

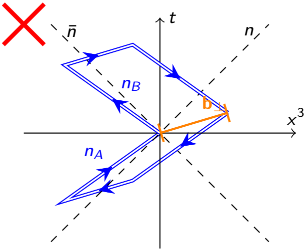
$$r_a = \frac{1 + e^{-2y_A}}{1 - e^{-2y_A}}, \quad r_b = \frac{1 + e^{2y_B}}{1 - e^{2y_B}}, \qquad |r_a|, |r_b| > 1 \qquad \text{succeeds}$$

$$S^{(1)}\left(b_\perp, \epsilon, r_a, r_b\right) = \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 - 2|y_A - y_B|\frac{1 + e^{2(y_B - y_A)}}{1 - e^{2(y_B - y_A)}}\right\}$$

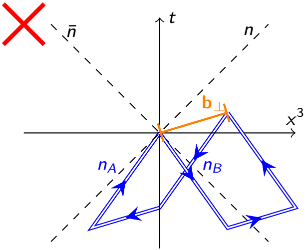
Wilson line directions



$|r_a| < 1, \quad |r_b| < 1, \quad n_A^0 n_B^0 (r_a r_b + 1) < 0$



$|r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) < 0$



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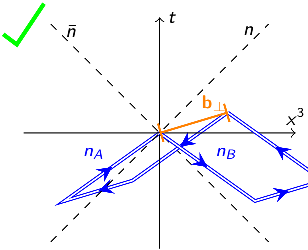

$|r_a| > 1, \quad |r_b| > 1, \quad n_A^0 n_B^0 (r_a r_b + 1) > 0$

For $L \to \infty$ and $r_a, r_b \to 1$:

$$S\left(b_\perp, a, r_a, r_b, L\right) = 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)}\log\left(\frac{(r_a - 1)(r_b - 1)}{(r_a + 1)(r_b + 1)}\right)\right)\log\left(\frac{b_\perp^2}{a^2}\right)$$
$$+ \frac{\alpha_s C_F}{2\pi}\left\{-4\log\left(\frac{b_\perp^2}{a^2}\right) + 2\frac{\pi b_\perp}{a} + 2\frac{\pi L}{b_\perp} - 2\frac{\pi L}{a}\right.$$
$$\left. + 2\frac{b_\perp^2}{L^2}\left(C_1 - \frac{1}{3}\right)\right\} + \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2\right)$$

$$C_1 = 1 - \frac{1}{2}\frac{1}{b_0^2(r_b^2 - 1)} - \frac{1}{2}\frac{1}{a_0^2(r_a^2 - 1)} \implies \frac{b_\perp^2}{L^2} \ll r_{a,b} - 1$$

- Incorrect $b_\perp$ dependence
- Linear divergence in L
- Power corrections are limited by $r_{a,b}$

Ratio

$$S_{\text{ratio}}\left(b_\perp, a, r_a, r_b, L\right)$$
$$= \frac{S\left(b_\perp, a, r_a, r_b, L\right)}{\sqrt{S\left(b_\perp, a, r_a, -r_a, L\right) S\left(b_\perp, a, -r_b, r_b, L\right)}} \qquad \text{[Ji, Liu, Liu 2020]}$$
$$= 1 + \frac{\alpha_s C_F}{2\pi}\left(2 + \frac{(r_a r_b + 1)}{(r_a + r_b)} \log\left(\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right)\right) \log\left(\frac{b_\perp^2}{a^2}\right)$$
$$+ \mathcal{O}\left(\frac{b_\perp^4}{L^4}, \alpha_s^2\right)$$

- Recover correct $b_\perp$ dependence
- Linear divergence removed
- Power corrections now start at $b_\perp^4/L^4$
- Approaches the soft function at $L \to \infty$

Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional 'fermions' that live along the path:

$$
\begin{aligned}
& P \exp\left\{ -ig \int_{s_i}^{s_f} \mathrm{d}s n^{\mu} A_{\mu}(y(s)) \right\} \\
&\quad = Z_{\psi}^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \psi\bar{\psi} \exp\left\{ ig \int_{s_i}^{s_f} \mathrm{d}s \bar{\psi} i\partial_s \psi - \bar{\psi} n \cdot A \psi \right\}
\end{aligned}
$$

[Gervais, Nevau 1980], [Aref'eva 1980]

Auxiliary field propagator:

$$
in \cdot D H_n(y) = i\delta^{(4)}(y) \xrightarrow{\text{Euclidean space}} i\tilde{n} \cdot D_E H_{\tilde{n}}(y_E) = \delta^{(4)}(y_E), \qquad \tilde{n} = (in_0, \vec{n})
$$

Auxiliary field propagator in Euclidean space

$$i\tilde{n}\cdot D_E H_{\tilde{n}}(y_E) = \delta^{(4)}(y_E)$$

- Meaningful solution only obtained with a UV cutoff [Aglietti, *et. al.* 1992], [Agglietti, 1994]
- Use lattice as UV cutoff and construct discretized solution to equation of motions [Mandula, Ogilvie, 1992]

$$n_0\left[U(x,x+\hat{t})G(x+\hat{t},y) - G(x,y)\right] + \sum_{\mu=1}^{3}\frac{-in_\mu}{2}\left[U(x,x+\hat{\mu})G(x+\hat{\mu},y) - U(x,x-\hat{\mu})G(x-\hat{\mu},y)\right] = \delta(x,y)$$

- Time independent object should be finite after removing the UV cutoff, i.e. zero lattice spacing
- Ratio with vanishing L dependence will approach Euclidean time independence on the lattice for large Euclidean time

- Euclidean space calculation of soft function can be analytically continued to Minkowski space result.
- Continuation only valid for space-like Wilson lines that are both future pointing or both past pointing.
- $S_{\rm ratio}$ should give correct rapidity and $b_\perp$ dependence and cancel linear divergence in L
- $S_{\rm ratio}$ on the lattice should give a time independent object and cancel the UV divergence associated with the Euclidean auxiliary propagator
- Numerical implementation on the lattice in progress

Thank you!

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